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ON THE ORBITS OF HAT-FUNCTIONS

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The object of this note is to study the orbits of some hat-functions defined in the interval $\langle 0, 1\rangle$ whose values escape from this interval. This problem has been considered in papers [1] and [3].

Suppose that a function f is defined in a set X and $X \subset f(X)$. The iterates f^n of the function f are defined as follows:

$$f^0(x) = x, \quad f^{n+1}(x) = f\big(f^n(x)\big) \quad \text{if } f^n(x) \in X,\ n = 0, 1, 2, \ldots$$

Two points $x, y \in X$ are said to be *equivalent* under f if there exist non-negative integers m and n such that both $f^m(x)$ and $f^n(y)$ exist and $f^m(x) = f^n(y)$. The set of all points which are equivalent to a given x will be called the *orbit* of x under f. The orbit of an x will be denoted by $C(x)$ (see [2], p. 14).

We prove the following result:

THEOREM. *Let a real-valued function f be defined and absolutely continuous in the interval $I = \langle 0, 1\rangle$, where $f(0) = f(1) = 0$. Moreover, let f be strictly increasing in an interval $\langle 0, a\rangle$, strictly decreasing in $\langle a, 1\rangle$, $f(a) > 1$, and* $\operatorname{ess\,inf}\{|f'(x)| : x \in f^{-1}[I]\} > 1$. *Then the set $C = \{x \in I : f^n(x) \leqslant 1,\ n = 1, 2, \ldots\}$ is nowhere-dense and perfect. If $x \in C$, then $\overline{C(x)} = C$ (in particular, $\overline{C(0)} = C$). If $y \in I \setminus C$, then $\overline{C(y)} \setminus C(y) = C$.*

Proof. Let the function f satisfy the assumptions of our theorem. We introduce the following notation:

$$I_1 = \langle 0, a\rangle, \quad I_2 = \langle a, 1\rangle \quad \text{and} \quad f_i = f|_{I_i} \text{ for } i = 1, 2.$$

Let $T = \{x \in I : f(x) > 1\}$. T is an open interval.

First we show that C is nowhere dense. We have the relation

$$\bigcup_{n=0}^{\infty} f^{-n}[T] = I \setminus C, \tag{1}$$

where $f^0[T] = T$ and $f^{-n-1}[T] = f^{-1}[f^{-n}[T]]$.

Since the function f is continuous, the sets $f^{-n}[T]$ are open, and hence the set C is closed.

Let us suppose that there exists an interval U contained in C. Then $f^n[U] \subset C$ for $n = 0, 1, 2, \ldots$ In view of the continuity of the function f, $f^n[U]$ are intervals. Since $f(a) > 1$, we infer that $a \notin C$ and, consequently, $a \notin U$. Therefore, $U \subset I_1$ or $U \subset I_2$. Moreover, $C \subset I \setminus T$, whence $U \subset I \setminus T$. Let

$$\alpha = \operatorname{ess\,inf}\{|f'(x)|: x \in f^{-1}[I]\}.$$

Then $|f'(x)| \geqslant \alpha > 1$ almost everywhere in $I \setminus T$.

Since the functions f_i are monotonic and absolutely continuous in I_i, for an i we have

$$m(f[U]) = m(f_i[U]) = \int_U |f_i'(x)|\,dx \geqslant \int_U \alpha\,dx = \alpha m(U),$$

where m denotes the Lebesgue measure.

Further, by induction, we get

$$m(f^n[U]) \geqslant \alpha^n m(U) \quad \text{for } n = 1, 2, \ldots$$

Hence

$$1 \geqslant m(C) \geqslant m(f^n[U]) \geqslant \alpha^n m(U).$$

Since $\alpha > 1$, the set U is of measure zero, but this contradicts our supposition.

This shows that the set C has no interior points. And since C is closed, it is nowhere dense.

We have the relation

$$f^{-1}[B] = f_1^{-1}[B] \cup f_2^{-1}[B]. \tag{2}$$

The functions f_i are homeomorphisms of the sets I_i onto the interval $\langle 0, f(a)\rangle$. If B is an open interval contained in I, then $f^{-1}[B]$ is a sum of two disjoint open intervals.

The set $T = \{x \in I: f(x) > 1\}$ yields an open interval $(c, d) \subset I$ such that $f(c) = f(d) = 1$. Then the set $f^{-n}[T]$ is a sum of 2^n disjoint open intervals $U_{n,k}$ $(k = 1, 2, \ldots, 2^n)$ whose ends belong to the orbit $C(1) = C(0)$. We may write

$$f^{-n}[T] = \bigcup_{k=1}^{2^n} U_{n,k}, \quad n = 0, 1, 2, \ldots \tag{3}$$

It is easy to verify the following property:

(i) If $f^{-1}[A] = A$ and $x \in A$, then $C(x) \subset A$.

From the definition of the set C it follows that $f^{-1}[C] = C$. Hence, for each x in C, $C(x) \subset C$. And since C is closed, we have $\overline{C(x)} \subset C$. In particular, $\overline{C(0)} \subset C$.

Let $x \in C$. The set C is nowhere dense which implies that, for any open interval U such that $x \in U$, $U \cap (I \setminus C) \neq \emptyset$. Hence relation (1) implies that there exists a non-negative integer n such that $U \cap f^{-n}[T] \neq \emptyset$. From relation (3) it follows that there exists an integer k such that $U \cap U_{n,k} \neq \emptyset$. We know that U and $U_{n,k}$ are intervals as well as $x \in U$ and $x \notin U_{n,k}$. Hence $U \cap \mathrm{Fr}(U_{n,k}) \neq \emptyset$, and so we have $U \cap C(0) \neq \emptyset$, since $\mathrm{Fr}(U_{n,k}) \subset C(0)$. This shows that $x \in \overline{C(0)}$. Thus $\overline{C(0)} = C$.

We show that $0 \in \overline{C(x)}$.

Put $g(x) = f_1(x) - x$ for $x \in I_1$. Since $|f'(x)| \geqslant a$ a.e. in $I_1 \setminus T$, we have $g'(x) \geqslant a-1$ a.e. in $I_1 \setminus T$. Hence g is strictly increasing in $\langle 0, c \rangle = I_1 \setminus T$. The equality $g(0) = 0$ implies $f_1(x) > x$ for $x \in (0, c\rangle$. Then $x > f_1^{-1}(x) > 0$ for $x \in (0, 1\rangle$, and so $0 < f_1^{-n-1}(x) < f_1^{-n}(x)$ for $n = 0, 1, 2, \ldots$ Consequently, there exists

$$\lim_{n\to\infty} f_1^{-n}(x) = q,$$

while the continuity of f_1^{-1} implies that $f_1^{-1}(q) = q$, whence $q = 0$. For all positive integers n, $f_1^{-n}(x) \in C(x)$. This shows that $0 \in \overline{C(x)}$.

Let $x \in C$. It is easy to verify that $f_i^{-1}[C(x)] = C(x) \cap I_i$ for $i = 1, 2$. The functions f_i are homeomorphisms. Consequently,

$$f_i^{-1}[\overline{C(x)}] = \overline{C(x) \cap I_i} = \overline{C(x)} \cap I_i, \quad i = 1, 2.$$

Hence by (2) we get

$$f^{-1}[\overline{C(x)}] = \overline{C(x)} \cap I_1 \cup \overline{C(x)} \cap I_2 = \overline{C(x)}.$$

Since $0 \in \overline{C(x)}$, we have, by property (i), $C(0) \subset \overline{C(x)}$. Therefore $C = \overline{C(0)} \subset \overline{C(x)}$. Thus $\overline{C(x)} = C$.

To prove that C is perfect it suffices to show that $x \in \overline{C \setminus \{x\}}$ for $x \in C$.

The function f has two fixed points — 0 and $x_0 \in (d, 1)$. For these two points we have $C(0) \cap C(x_0) = \emptyset$. Evidently, 0 and x_0 belong to C. Hence the set C contains two disjoint orbits.

Let $x \in C$. Then there exists a $y \in C$ such that $x \notin C(y)$. We have proved that $\overline{C(y)} = C$. Therefore, for any neighbourhood U of x, $(U \setminus \{x\}) \cap C(y) \neq \emptyset$, whence

$$(U \setminus \{x\}) \cap C \neq \emptyset \quad \text{and} \quad U \cap (C \setminus \{x\}) \neq \emptyset.$$

This shows that $x \in \overline{C \setminus \{x\}}$.

Now we show that $\overline{C(y) \setminus C(y)} = C$ for any $y \in I \setminus C$.

We have the property $\overline{C(0) \setminus \{x\}} = C$. In fact, the equality $\overline{C(0)} = C$ implies

$$C \setminus \{x\} \subset \overline{C(0) \setminus \{x\}} \subset C.$$

Thus $C = \overline{C\setminus\{x\}} \subset \overline{C(0)\setminus\{x\}}$, since C is a perfect set. The sets $f^{-n}[0]$ are finite, so it is clear that for any positive integer N and for the set

$$C_N(0) = C(0)\setminus\bigcup_{n=0}^{N} f^{-n}[0]$$

we have $\overline{C_N(0)} = C$.

It is easy to verify that $f^{-n}[T]\cap f^{-m}[T] = \emptyset$ for $n \neq m$. Hence (3) implies that $U_{n,k}$ for $k = 1, \ldots, 2^n$ and $n = 0, 1, \ldots$ are disjoint intervals contained in I. Then

$$\sum_{n=0}^{\infty}\sum_{k=1}^{2^n} m(U_{n,k}) \leqslant 1. \tag{4}$$

For any $\delta > 0$ there exists an integer $N(\delta)$ such that $m(U_{n,k}) < \delta/2$ for $n > N(\delta)$ and $k = 1, \ldots, 2^n$.

For $y\in I\setminus C$, $C(y)\cap C = \emptyset$. Moreover, for any n and $k = 1, \ldots, 2^n$, $C(y)\cap U_{n,k} \neq \emptyset$.

Let $x\in C$ and $\delta > 0$. Then $x\in\overline{C_{N(\delta)}(0)}$. Therefore, there exists a $z\in C_{N(\delta)}(0)$, and $z\in(x-\delta/2, x+\delta/2)$. Since the ends of the intervals $U_{n,k}$ for $n > N(\delta)$ belong to $C_{N(\delta)}(0)$, and $C(0)$ is a set of all ends of the intervals $U_{n,k}$, there exist $n > N(\delta)$ and k such that $z\in \overline{U_{n,k}}$. On the other hand, $m(U_{n,k}) < \delta/2$. Thus $U_{n,k} \subset (x-\delta, x+\delta)$ and, consequently,

$$C(y)\cap(x-\delta, x+\delta) \neq \emptyset.$$

This shows that $x\in\overline{C(y)}$. Thus $C \subset \overline{C(y)}\setminus C(y)$.

Now we prove the opposite inclusion. Let $x\in\overline{C(y)}\setminus C(y)$. There exists a sequence $y_n\in C(y)$ such that

$$y_n \neq x, \quad y_i \neq y_j \text{ for } i \neq j, \quad \text{and} \quad \lim_{n\to\infty} y_n = x.$$

The orbit $C(y)$ is contained in $\bigcup_{n=0}^{\infty} f^{-n}[I]$, since $y\in I\setminus C$. From equality (3) it follows that for any $n \geqslant 0$ there exists an interval U_{m_n,k_n} such that $y_n\in U_{m_n,k_n}$. For $n \neq n'$ we have $U_{m_n,k_n}\cap U_{m_{n'},k_{n'}} = \emptyset$, since the intersection $C(y)\cap U_{n,k}$ for any n and k is a single point. Hence it follows that

$$\lim_{n\to\infty} m_n = \infty$$

and further, by (4), that

$$\lim_{n\to\infty} m(U_{m_n,k_n}) = 0.$$

We may write $U_{m_n,k_n} = (a_n, \beta_n)$, where $a_n, \beta_n\in C(0)$. Then

$$\lim_{n\to\infty} (a_n - \beta_n) = 0.$$

Since

$$y_n \in (\alpha_n, \beta_n) \text{ for } n = 1, 2, \ldots \quad \text{and} \quad \lim_{n\to\infty} y_n = x,$$

it follows that

$$\lim_{n\to\infty} \alpha_n = x.$$

Therefore $x \in \overline{C(0)} = C$.

COROLLARY. *Assume that a function f satisfies the hypothesis of the Theorem and that a real-valued function φ defined in the interval $f[I]$ satisfies the functional equation*

$$\varphi\big(f(x)\big) = \varphi(x), \qquad x \in I. \tag{5}$$

If φ is continuous at an $x_0 \in C$, then φ is constant in I.

Proof. Let φ satisfy equation (5) and be continuous at $x_0 \in C$. Relation (5) implies that φ is constant on every orbit $C(x)$.

Let $x \in C$. Since $\overline{C(x)} = C$, there exists a sequence $x_n \in C(x)$ such that

$$\lim_{n\to\infty} x_n = x_0.$$

Similarly, if $x \in I \setminus C$, then from the equality $\overline{C(x)} \setminus C(x) = C$ it follows that there exists a sequence $x_n \in C(x)$ such that

$$\lim_{n\to\infty} x_n = x_0.$$

Hence, by the continuity of φ at x_0,

$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x_0).$$

Moreover, $\varphi(x_n) = \varphi(x)$ for all n, whence $\varphi(x) = \varphi(x_0)$. Thus $\varphi(x) = \varphi(x_0)$ for $x \in I$.

If the function f satisfies assumptions of the Theorem except for

$$\operatorname{ess\,inf}\{|f'(x)|:\ x \in f^{-1}[I]\} = 1,$$

then the Theorem is not true. This is shown by the example given in [1].

It is easy to see that, for $f(x) = 3/2 - |3x - 3/2|$ and $x \in I$, C is the Cantor set.

The measure characterization of the set C for some functions has been given in [1] and [3].

The set C for some functions plays a role in biological research works (see [1]).

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