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ON THE NUMBER OF SIGN CHANGES IN THE REMAINDER-TERM OF THE PRIME-IDEAL THEOREM

BY

J. KACZOROWSKI AND W. STAŚ (POZNAŃ)

DEDICATED TO PROFESSOR S. HARTMAN
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1. Denote by K an algebraic number field, by n and Δ the degree and discriminant of K, respectively, and by $\zeta_K(s)$, $s = \sigma + it$, the Dedekind zeta-function. Moreover, we write $d = 2|\Delta|$.

It is well known that the properties of $\zeta_K(s)$, and especially the distribution of its roots, are strictly connected with the distribution of prime ideals (see [6]).

Write

$$\psi_K(x) = \sum_{N(\mathfrak{p}^r) \leqslant x} \log N(\mathfrak{p}) \quad \text{and} \quad \Delta_K(x) = \psi_K(x) - x.$$

The prime-ideal theorem in its simplest form can be written as

$$\psi_K(x) \sim x, \quad x \to \infty.$$

The function $\Delta_K(x)$ is called the *remainder-term of the prime-ideal theorem*.

The oscillatory character of $\Delta_K(x)$, $K \neq Q$, was first investigated by Landau [5], who, using the methods of Littlewood, proved that

$$\overline{\lim_{x \to \infty}} \frac{\Delta_K(x) + \sum_{\gamma = 0} x^{\varrho}/\varrho}{\sqrt{x} \log\log\log x} > 0 \tag{1.1}$$

and

$$\underline{\lim_{x \to \infty}} \frac{\Delta_K(x) + \sum_{\gamma = 0} x^{\varrho}/\varrho}{\sqrt{x} \log\log\log x} < 0, \tag{1.2}$$

where the sum is to be taken over the real roots of $\zeta_K(s)$, lying on the positive real axis (if there are any).

It is well known that $\zeta_K(s)$, $s=\sigma+it$, has infinitely many zeros $\varrho=\beta+i\gamma$ in the strip $0<\sigma<1$, $-\infty<t<+\infty$ of the complex plane.

We denote by $\gamma_K\geqslant 0$ the imaginary part of the "lowest" ζ_K-zero in the half plane $\sigma\geqslant 1/2$:

$$\gamma_K=\inf_{\substack{\varrho=\beta+i\gamma\\ \beta\geqslant 1/2,\gamma\geqslant 0}}\gamma.$$

Denote by $V_K(T)$ the number of sign changes of $\Delta_K(x)$ for $2<x\leqslant T$.

From a theorem of Pólya [8] it follows that if $\zeta_K(s)\neq 0$ for $s\in(0,1)$, then

$$\overline{\lim_{T\to\infty}}\frac{V_K(T)}{\log T}\geqslant\frac{\gamma_K}{\pi}.$$

Under the assumption that the natural extension of the Riemann hypothesis to the Dedekind zeta-function is true it can be proved by a method of Ingham [2] that

$$\underline{\lim_{T\to\infty}}\frac{V_K(T)}{\log T}>0.$$

In the following we shall restrict ourselves to algebraic number fields K such that $\zeta_K(s)$ has no real zeros in the interval (0, 1).

Subject to the above assumption, the second-named author proved that

$$\underline{\lim_{T\to\infty}}\frac{V_K(T)}{\log\log T}>0 \tag{1.3}$$

(see [1] and cf. [4]). In the proof of (1.3), Turán's one-sided theorems were used (see [13]).

2. Before stating our theorems we introduce some notation.

All the constants c_i, $i=1,2,\ldots$, are numerical and effectively computable. The same is true for the constants implied by the symbols O and $\ll$.

For the sake of convenience, we introduce the following notation: if f is a complex-valued function and g is a positive-valued function, both defined on a set Ω of complex numbers and satisfying the inequality $|f(x)|\leqslant g(x)$, $x\in\Omega$, then we write

$$f(x)=\bar{O}(g(x)),\qquad x\in\Omega.$$

It can be proved that there exists a numerical constant c_1 such that $0\leqslant\gamma_K<c_1$ for every K (see [10] and cf. [12], Lemma 5). Therefore

$$\varkappa\overset{\text{df}}{=}\sup_K\gamma_K<\infty.$$

Neugebauer [7] has recently proved using Siegel's methods, that $\varkappa < 60$. Obviously,

$$\varkappa \geqslant \gamma_Q = 14.13\ldots \tag{2.1}$$

E. Artin stated the conjecture that if K and L are given algebraic number fields and $K \subset L$, then the quotient $\zeta_K(s)/\zeta_L(s)$ is an entire function. From this conjecture and (2.1) it would then follow that

$$\varkappa = \gamma_Q = 14.13\ldots$$

It is interesting to observe that this Artin's conjecture is proved to be correct in many cases (see [14]).

Denote by $\varrho = \beta + i\gamma$ and $\varrho' = \beta' + i\gamma'$ any different zeros of $\zeta_K(s)$.

For an arbitrary ε, $0 < \varepsilon < 10$, we define the following constants connected with ζ_K-zeros:

$$\Gamma_K(\varepsilon) = \min\{|\gamma - \gamma'|:\ \gamma \neq \gamma',\ |\gamma|,\ |\gamma'| \leqslant \varkappa^{10/\varepsilon}\},$$

$$B_K(\varepsilon) = \min\{|\beta - \beta'|:\ \beta,\ \beta' \in (0, 1),\ \beta \neq \beta',\ |\gamma|,\ |\gamma'| \leqslant \varkappa^{10/\varepsilon}\}.$$

If all zeros $\varrho = \beta + i\gamma$ with $|\gamma| \leqslant \varkappa^{10/\varepsilon}$ are situated on the line $\sigma = 1/2$, then we put $B_K(\varepsilon) = 1$.

If $\gamma_K \neq 0$, then we have the following obvious inequality:

$$\tfrac{1}{2}\Gamma_K(\varepsilon) \leqslant \gamma_K \leqslant \varkappa, \quad 0 < \varepsilon < 10. \tag{2.2}$$

3. Now we can state the following

THEOREM 1. *If K is such that $\zeta_K(s)$ has no real roots in* $(0, 1)$, *then for any ε, $0 < \varepsilon < 1$, there exists an effective constant $c_0(\varepsilon) > 0$, depending only on ε, such that for*

$$T \geqslant \exp\left\{c_0(\varepsilon)\left(\frac{1}{B_K(\varepsilon)} + \frac{1}{\Gamma_K^2(\varepsilon)}\right)\log\left(2 + \frac{\log d}{\gamma_K}\right)\log^2 d\right\},$$

we have the estimate

$$V_K(T) \geqslant (1-\varepsilon)\frac{\gamma_K}{\pi}\log T.$$

Introducing as the modified remainder term

$$\Delta_K^*(x) \stackrel{\text{df}}{=} \Delta_K(x) + \sum_{\gamma=0} x^\varrho/\varrho$$

(cf. (1.2)) and denoting by $V_K^*(T)$ the number of sign changes of $\Delta_K^*(x)$ for $2 < x \leqslant T$, we get the following unconditional result:

THEOREM 2. *For any ε, $0 < \varepsilon < 1$, there exists an effective constant $c_1(\varepsilon)$, depending only on ε, such that for*

$$T \geqslant \exp\left\{c_1(\varepsilon)\left(\frac{1}{B_K(\varepsilon)} + \frac{1}{\Gamma_K^2(\varepsilon)}\right)\log\left(2 + \frac{\log d}{\gamma_K}\right)\log^2 d\right\}$$

we have the estimate

$$V_K^*(T) \geqslant (1-\varepsilon)\frac{\gamma_K}{\pi}\log T.$$

In Theorem 1 we assume that $\zeta_K(s)$ is free of zeros in the interval (0, 1). It is easily seen that an assumption of this kind is, in a way, necessary for the existence of sign changes of $\Delta_K(x)$, since the terms $\sum_{\gamma=0} x^\varrho/\varrho$ in the Riemann–Mangoldt formula generalized by Landau (see [6], Satz 195) can decide on the oscillatory character of $\Delta_K(x)$. For instance, if K is such that $\zeta_K(s)$ has zeros on the positive real axis and if

$$\sup_{\gamma=0}\operatorname{Re}\varrho > \sup_{\gamma\neq 0}\operatorname{Re}\varrho,$$

then $\Delta_K(x) = \psi_K(x) - x$ is of one sign for $x \geqslant x_0$.

A number of ζ_K-functions which have no zeros in (0, 1) is well known. For ζ_K-functions which do not have this property see [1].

The method of proof of Theorems 1 and 2 was introduced by the first-named author to the investigation of the number of sign changes in the remainder term of the prime-number theorem (see [3]).

4. For the proof we need some lemmas.

LEMMA 1 ([12], Lemma 6). *If $s = \sigma + it$, then there is a connected path L in the vertical strip $1/22 \leqslant \sigma \leqslant 1/20$ say, symmetrical to the real axis, consisting alternatively of horizontal and vertical segments and increasing monotonically from $-\infty$ to $+\infty$, on which for all ζ_K-functions we have*

$$\left|\frac{\zeta_K'}{\zeta_K}(s)\right| \leqslant c_2 \log^2\big(d(|t|+2)^n\big). \tag{4.1}$$

LEMMA 2 ([12], Lemma 4). *Let $N_K(T)$ be the number of zeros of $\zeta_K(s)$ in the rectangle $0 \leqslant \sigma \leqslant 1$, $0 \leqslant t \leqslant T$. Then for $T \geqslant 0$ we have*

$$N_K(T+1) - N_K(T) \leqslant c_3 \log\big(d(T+2)^n\big). \tag{4.2}$$

The following is the well-known fact:

LEMMA 3. *For any K we have*

$$n \ll \log d. \tag{4.3}$$

LEMMA 4 (see [9], pp. 40–41). *Let f be a real-valued, piecewise continuous function, non-constant in every interval (a, b), $0 < a < b < \infty$, and such that the integral*

$$\int_0^A |f(x)|\,dx$$

is finite for every $A > 0$. If $v(T)$ denotes the number of sign changes of f in the interval $(0, T]$ and $V(T)$ denotes the number of sign changes of the function

$$F(x) = \int_0^x f(t)\,dt$$

in the interval $(0, T]$, then for $T > 0$ we have $V(T) \leqslant v(T)$.

5. Proof of Theorem 1. Denote by m a sufficiently large positive integer which will be precisely determined later. For a function f defined over the set of positive numbers we determine

$$\delta_m(f;\ x) = \int_0^x \int_0^{t_{m-1}} \dots \int_0^{t_1} f(t)\frac{d(t)}{t}\dots\frac{dt_{m-2}}{t_{m-2}}\frac{dt_{m-1}}{t_{m-1}} \tag{5.1}$$

if the integrals on the right-hand side do exist.

From (5.1) and (1.1) we get

$$\delta_{m-1}(\Delta_K;\ x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left\{-\frac{\zeta_K}{\zeta_K}(s)\right\}\frac{x^s}{s^m}\,ds - x$$

$$= -\sum_{\varrho \succ L} x^{\varrho}/\varrho^m + \frac{1}{2\pi i}\int_L \left\{-\frac{\zeta_K}{\zeta_K}(s)\right\}\frac{x^s}{s^m}\,ds,$$

where L is the path from Lemma 1 and the sum runs over all ζ_K-roots lying to the right of L.

By (4.1) we have

$$\left|\frac{1}{2\pi i}\int_L \left\{-\frac{\zeta_K}{\zeta_K}(s)\right\}\frac{x^s}{s^m}\,ds\right| \ll \int_L \log^2(d(|t|+2)^n)\frac{x^{\sigma}}{|s|^m}|ds| \tag{5.2}$$

$$\ll \int_L \frac{\log^2(d(|t|+2)^n)}{|s|^2}\frac{x^{\sigma}}{\sigma^m}\sigma^2\,|ds|$$

$$\ll \max_{1/22 \leqslant \sigma \leqslant 1/20} \frac{x^{\sigma}}{\sigma^m}\log^2 d \ll 20^m x^{1/20}\log^2 d,$$

and the last inequality is true for $m \leqslant (1/22)\log x$.

Moreover, by (4.2) and (4.3) we obtain

$$\Big|\sum_{\substack{\varrho \succ L \\ |\gamma| > \varkappa^{10/\varepsilon}}} x^{\varrho}/\varrho^m\Big| \ll \frac{x}{\varkappa^{(m-2)10/\varepsilon}}\log d.$$

Denote by $\beta_1 < \beta_2 < \dots < \beta_l < 1$ the real parts of the ζ_K-roots in the region

$$\{s\colon\ s \succ L,\ |\mathrm{Im}\, s| \leqslant \varkappa^{10/\varepsilon}\}.$$

Then

$$\delta_{m-1}(\Delta_K;\ x) = -\sum_{\nu=1}^{l} \Phi_\nu(x) + O_\varepsilon\left(20^m x^{1/20}\log^2 d + \frac{x}{\varkappa^{10m/\varepsilon}}\log^2 d\right),$$

where

$$\Phi_\nu(x) = \sum_{\substack{\varrho \succ L \\ \beta = \beta_\nu}} x^\varrho/\varrho^m.$$

Now, let us denote by

$$\varrho_\nu = \beta_\nu + i\gamma_\nu = |\varrho_\nu| \exp\{i\varphi_\nu\}, \qquad \gamma_\nu > 0,\ \nu = 1, 2, \ldots, l,$$

the "lowest" zero lying to the right of L, such that $\operatorname{Re}\varrho_\nu = \beta_\nu$. Then

$$\Phi_\nu(x) = \frac{2e_\nu x^{\beta_\nu}}{|\varrho_\nu|^m}\{\cos(\gamma_\nu \log x - m\varphi_\nu) + r_\nu(x)\}, \tag{5.3}$$

where $e_\nu := \operatorname{ord}\varrho_\nu$ and

$$r_\nu(x) = \frac{|\varrho_\nu|^m}{2e_\nu x^{\beta_\nu}} \sum_{\varrho \in E_\nu(\varepsilon)} x^\varrho/\varrho^m$$

and the sum is over zeros ϱ satisfying the following conditions: $\varrho \succ L$, $\operatorname{Re}\varrho = \beta_\nu$, $\gamma_\nu < |\gamma| \leqslant \varkappa^{10/\varepsilon}$. Thus we see that

$$|r_\nu(x)| \leqslant \sum_{\varrho \in E_\nu(\varepsilon)} |\varrho_\nu/\varrho|^m \leqslant |\varrho_\nu|^2 \max_{\varrho \in E_\nu(\varepsilon)} |\varrho_\nu/\varrho|^{m-2} \sum_{\varrho} |\varrho|^{-2}$$

and

$$\sum_{\varrho} \frac{1}{|\varrho|^2} = \sum_{k=0}^{\infty} \sum_{\substack{\varrho \\ k < |\gamma| \leqslant k+1}} \frac{1}{|\varrho|^2} \ll \sum_{\substack{\varrho \\ |\gamma| < 1}} \frac{1}{|\varrho|^2} + \sum_{k=1}^{\infty} \frac{1}{k^2}\log(d(k+2)^n)$$

$$\ll \frac{1}{\gamma_K^2}\log d + \log d \ll \frac{\log d}{\gamma_K^2}.$$

Now

$$\left|\frac{\varrho}{\varrho_\nu}\right| = \sqrt{\frac{\beta_\nu^2+\gamma^2}{\beta_\nu^2+\gamma_\nu^2}} = \sqrt{1 + \frac{\gamma^2 - \gamma_\nu^2}{\beta_\nu^2+\gamma_\nu^2}}$$

$$= \sqrt{1 + \frac{(|\gamma|-|\gamma_\nu|)(|\gamma|+|\gamma_\nu|)}{\beta_\nu^2+\gamma_\nu^2}}.$$

We have

$$|\gamma| - |\gamma_\nu| = \big||\gamma| - |\gamma_\nu|\big| \geqslant \Gamma_K(\varepsilon),$$

$$|\gamma| + |\gamma_\nu| \geqslant \tfrac{1}{2}\Gamma_K(\varepsilon) + \tfrac{1}{2}\Gamma_K(\varepsilon) = \Gamma_K(\varepsilon),$$

$$\beta_\nu^2 + \gamma_\nu^2 \leqslant 1 + \varkappa^{20/\varepsilon}.$$

Consequently,

$$\left|\frac{\varrho}{\varrho_\nu}\right| \geqslant \sqrt{1+\Gamma_K^2(\varepsilon)/(1+\varkappa^{20/\varepsilon})} = 1+\frac{\Gamma_K^2/(1+\varkappa^{20/\varepsilon})}{1+\sqrt{1+\Gamma_K^2/(1+\varkappa^{20/\varepsilon})}}$$
$$\geqslant 1+\frac{\Gamma_K^2(\varepsilon)}{(1+\varkappa^{20/\varepsilon})(1+\sqrt{1+4\varkappa^2/(1+\varkappa^{20/\varepsilon})})} \geqslant 1+\frac{\Gamma_K^2(\varepsilon)}{3(1+\varkappa^{20/\varepsilon})}$$

and

$$|\varrho_\nu/\varrho| \leqslant [1+\Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))]^{-1}.$$

Finally, we have

$$(5.4) \qquad |r_\nu(x)| \ll (1+\varkappa^{10/\varepsilon})^2 \left(\frac{1}{1+\Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))}\right)^{m-2} \frac{\log d}{\gamma_K^2}.$$

For the sensibleness of (5.3) it is necessary that $|r_\nu(x)| < 1$. Obviously, the stronger inequality $|r_\nu(x)| \leqslant 1/2$ takes place if

$$(1+\varkappa^{10/\varepsilon})^2 \left(\frac{1}{1+\Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))}\right)^{m-2} \leqslant c_4 \frac{\gamma_K^2}{\log d},$$

i.e.,

$$c_5 \frac{\log d}{\gamma_K^2}(1+\varkappa^{10/\varepsilon})^2 \leqslant [1+\Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))]^{m-2}$$

for a sufficiently large $c_5 > 0$.

The last inequality holds if

$$c_5 \left(\frac{\log d}{\gamma_K^2}+2\right)^2 (1+\varkappa^{10/\varepsilon})^2 \leqslant [1+\Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))]^{m-2},$$

$$m-2 \geqslant c_6 \frac{\log((\log d)/\gamma_K+2)+\log(1+\varkappa^{10/\varepsilon})}{\log[1+\Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))]}$$

for a sufficiently large $c_6 > 0$.

Finally, we have the following condition for the parameter m:

$$(5.5) \qquad m \geqslant 2+c_6 \frac{\log((\log d)/\gamma_K+2)+\log(1+\varkappa^{10/\varepsilon})}{\log[1+\Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))]}.$$

Now, we restrict the range for x as follows:

$$T^{\varepsilon/2} \leqslant x \leqslant T.$$

The parameter m will be determined later (see (5.8)) in such a way that m will tend to infinity together with T.

Hence, for $T \geqslant T_1(K, \varepsilon)$ we have $|r_\nu(x)| \leqslant 1/2$ for $\nu = 1, 2, \ldots, l$. Therefore, each term Φ_ν has an oscillation absolutely greater than or equal to $e_\nu x^{\beta_\nu}/|\varrho_\nu|^m$. Suppose that for a certain ν_0, $1 \leqslant \nu_0 \leqslant l$, the oscillation of the ν_0-th term is maximal. Then, by (2.2),

$$\frac{e_{\nu_0} x^{\beta_{\nu_0}}}{|\varrho_{\nu_0}|^m} \geqslant \frac{\operatorname{ord} \varrho_K \, x^{\beta_K}}{|\varrho_K|^m} \geqslant \frac{x^{1/2}}{(1+\varkappa)^m},$$

where $\varrho_K = \beta_K + i\gamma_K$, $\beta_K \geqslant 1/2$, denotes the "lowest" zero of $\zeta_K(s)$.

Now, we have to determine the conditions under which one of the terms in the sum $\sum_{\nu=1}^{l} \Phi_\nu$ is dominating over the other terms.

Putting

$$u_\nu(t) = \beta_\nu t - m \log|\varrho_\nu|$$

and supposing that $\eta > 0$, we consider the set

$$A(T, \eta) = \{t\colon (\varepsilon/2)\log T \leqslant t \leqslant \log T,\ |u_\nu(t) - u_{\nu'}(t)| \geqslant \eta \text{ for all } \nu \neq \nu',\ 1 \leqslant \nu,\ \nu' \leqslant l\}.$$

If $l = 1$, then, obviously,

$$A(T, \eta) = [(\varepsilon/2)\log T, \log T].$$

If $t \notin A(T, \eta)$, then

$$\big|t(\beta_\nu - \beta_{\nu'}) - m \log|\varrho_\nu/\varrho_{\nu'}|\big| < \eta$$

for some $\nu \neq \nu'$. Hence

$$\left|t - m\frac{\log|\varrho_\nu/\varrho_{\nu'}|}{\beta_\nu - \beta_{\nu'}}\right| < \frac{\eta}{|\beta_\nu - \beta_{\nu'}|},$$

which proves that

$$|[(\varepsilon/2)\log T, \log T] \setminus A(T, \eta)| \leqslant 2\eta \sum_{\nu \neq \nu'}\sum \frac{1}{|\beta_\nu - \beta_{\nu'}|}$$
$$\ll \frac{\varkappa^{20/\varepsilon}}{B_K(\varepsilon)} \eta \log^2(d\varkappa^{10n/\varepsilon}) \leqslant \frac{c_7(\varepsilon)}{B_K(\varepsilon)} \eta \log^2 d.$$

Therefore

$$|A(T, \eta)| \geqslant (1 - \varepsilon/2)\log T - \frac{c_7(\varepsilon)}{B_K(\varepsilon)} \eta \log^2 d,$$

which implies that for a sufficiently large $c_8(\varepsilon)$ we have

$$\big|A\big(T, L_K(\varepsilon)\log T\big)\big| \geqslant (1 - 3\varepsilon/4)\log T,$$

where

$$L_K(\varepsilon) = B_K(\varepsilon)/(c_8(\varepsilon)\log^2 d).$$

We have further

$$A(T, L_K(\varepsilon)\log T) \subset \bigcup_{\nu=1}^{l} I_\nu,$$

where I_ν are intervals determined as follows:

$$I_\nu = \{t \in [(\varepsilon/2)\log T, \log T]:\ u_\nu(t) \geqslant \max_{\mu \neq \nu} u_\mu(t) + L_K(\varepsilon)\log T\}.$$

Moreover, it is easily seen that $I_\nu \cap I_{\nu'} = \emptyset$ for $\nu \neq \nu'$ and that at least one of the intervals I_ν is not empty.

For $\log x \in I_{\nu_0}$, by (5.3) and (5.4) we have (only if (5.5) is satisfied)

$$\sum_{\nu=1}^{l} \Phi_\nu(x) = \Phi_{\nu_0}(x) + O\big(l \max_{\nu \neq \nu_0} |\Phi_\nu(x)|\big)$$
$$= \frac{2e_{\nu_0}}{|\varrho_{\nu_0}|^m}\{\cos(\gamma_{\nu_0}\log x - m\varphi_{\nu_0}) + \bar{O}(1/2) + O_\varepsilon\big((\log d)\, T^{-L_K(\varepsilon)}\big)\}.$$

Finally, for $\log x \in I_{\nu_0}$ we have

(5.6) $\delta_{m-1}(\Delta_K;\ x)$

$$= \frac{2e_{\nu_0} x^{\beta_{\nu_0}}}{|\varrho_{\nu_0}|^m}\Bigg\{\cos(\gamma_{\nu_0}\log x - m\varphi_{\nu_0}) + \bar{O}(1/2)$$
$$+ O_\varepsilon(T^{-L_K(\varepsilon)}\log d) + O_\varepsilon\bigg(\frac{(1+\varkappa)^m}{x^{1/2}} 20^m x^{1/20}\log^2 d$$
$$+ \frac{(1+\varkappa)^m}{x^{1/2}}\frac{x}{\varkappa^{10m/\varepsilon}}\log^2 d\bigg)\Bigg\}$$
$$\equiv \frac{2e_{\nu_0} x^{\beta_{\nu_0}}}{|\varrho_{\nu_0}|^m}\Bigg\{\cos(\gamma_{\nu_0}\log x - m\varphi_{\nu_0}) + \bar{O}(1/2) + O_\varepsilon(T^{-L_K(\varepsilon)}\log d)$$
$$+ O_\varepsilon\bigg((20(1+\varkappa))^m T^{-9\varepsilon/40}\log^2 d + T^{1/2}\Big(\frac{1+\varkappa}{\varkappa^{10/\varepsilon}}\Big)^m \log^2 d\bigg)\Bigg\}.$$

Like before, the remainders in the last formula have to be made "small" enough, so that the cosine-term could prevail. Thus we have the following conditions which restrict the range for m:

$$c_9(\varepsilon)(20(1+\varkappa))^m \log^2 d \leqslant T^{9\varepsilon/40}$$

and

$$c_{10}(\varepsilon)\, T^{1/2} \log^2 d \leqslant \left(\frac{\varkappa^{10/\varepsilon}}{1+\varkappa}\right)^m.$$

The first condition is satisfied if

$$m \log(20(1+\varkappa)) + 2 \log\log(d+2) \leqslant \frac{9\varepsilon}{40} \log T - c_{11}(\varepsilon).$$

Thus

$$m \leqslant \frac{(9\varepsilon/40)\log T - c_{11}(\varepsilon) - 2\log\log(d+2)}{\log(20(1+\varkappa))}$$

$$= \frac{9\varepsilon}{40\log(20(1+\varkappa))} \log T + O_\varepsilon(\log\log(d+2)).$$

The second condition is satisfied if

$$c_{12}(\varepsilon) + (1/2)\log T + 2\log\log(d+2) \leqslant (10m/\varepsilon)\log\varkappa - m\log(1+\varkappa).$$

This means that

$$m \geqslant \frac{(1/2)\log T + 2\log\log(d+2) + c_{12}(\varepsilon)}{(10/\varepsilon)\log\varkappa - \log(1+\varkappa)}$$

and, finally,

$$m \geqslant \frac{\varepsilon}{16\log\varkappa} \log T + O_\varepsilon(\log\log(d+2)),$$

since

$$(10/\varepsilon)\log\varkappa - \log(1+\varkappa) > (8/\varepsilon)\log\varkappa.$$

For the existence of an integer m having the above properties it is sufficient that

$$\frac{\varepsilon}{16\log\varkappa}\log T < \frac{9\varepsilon}{40\log(20(1+\varkappa))}\log T + O_\varepsilon(\log\log(d+2)),$$

which is obviously equivalent to the condition

$$\log T < \frac{9\log\varkappa}{2.5\log(20(1+\varkappa))}\log T + O_\varepsilon(\log\log(d+2)).$$

Since the function

$$f(u) = \frac{9\log u}{2.5\log(20(1+u))}$$

is increasing for $u > 1$, the last condition is satisfied only if

$$\log T < \frac{9 \log 14}{2.5 \log(20 \cdot 15)} \log T + O_\varepsilon(\log\log(d+2)). \tag{5.7}$$

However,

$$\frac{9 \log 14}{2.5 \log(20 \cdot 15)} > \frac{9 \log 10}{2.5 \log 10^3} = \frac{9}{7.5} > 1.$$

Therefore, (5.7) is satisfied if

$$T \geqslant \exp(c_{13}(\varepsilon) \log\log(d+2)).$$

We can determine the integer m by putting

$$m = \left[\frac{(1/2)\log T + 2\log\log(d+2) + c_{12}(\varepsilon)}{(10/\varepsilon)\log \varkappa - \log(1+\varkappa)}\right] + 2. \tag{5.8}$$

Obviously, $m \geqslant 2$. It remains to prove that

$$m \leqslant (1/22)\log x,$$

which is necessary for the estimate (5.2) to be true.

From (5.8) it follows that

$$m = \frac{(\varepsilon/2)\log T}{10\log\varkappa - \varepsilon\log(1+\varkappa)} + O_\varepsilon(\log\log(d+2)).$$

Further we have

$$10\log\varkappa - \varepsilon\log(1+\varkappa) \geqslant 10\log\varkappa - \log(1+\varkappa) \geqslant 10\log 14 - \log 15 > 23.$$

Hence for $T \geqslant \exp(c_{14}(\varepsilon)\log\log(d+2))$ we get

$$\begin{aligned} m &< \frac{(\varepsilon/2)\log T}{23} + O_\varepsilon(\log\log(d+2)) \\ &< \frac{(\varepsilon/2)\log T}{22} - (\varepsilon/10^5)\log T + O_\varepsilon(\log\log(d+2)) \\ &< (1/22)\log x. \end{aligned}$$

Also, the remainder $O_\varepsilon(T^{-L_K(\varepsilon)}\log d)$ in (5.6) has to be chosen small enough, e.g., equal to $\bar{O}(1/8)$. It is really so if

$$O_\varepsilon(\exp\{-L_K(\varepsilon)\log T + \log\log(d+2)\}) = \bar{O}(1/8),$$

$$L_K(\varepsilon)\log T > c_{15}(\varepsilon)\log\log(d+2),$$

$$T \geqslant \exp\{c_{16}(\varepsilon) B_K^{-1}(\varepsilon)\log^2 d \log\log(d+2)\}.$$

Now we have to determine the size of $T_1(K, \varepsilon)$.

For $T \geqslant T_1(K, \varepsilon)$. condition (5.5) has to be satisfied, i.e.,

$$\frac{(1/2)\log T + 2\log\log(d+2) + c_{12}(\varepsilon)}{(10/\varepsilon)\log \varkappa - \log\log(1+\varkappa)} \geqslant 2 + c_6 \frac{\log((\log d)/\gamma_K + 2) + \log(1+\varkappa^{10/\varepsilon})}{\log\left(1 + \Gamma_K^2(\varepsilon)/(3(1+\varkappa^{20/\varepsilon}))\right)}.$$

For this it is sufficient that

$$\log T \geqslant \frac{c_{17}(\varepsilon)}{\Gamma_K^2(\varepsilon)} \log\left(\frac{\log d}{\gamma_K} + 2\right)$$

and finally that

$$T \geqslant \exp\left\{\frac{c_{17}(\varepsilon)}{\Gamma_K^2(\varepsilon)} \log\left(\frac{\log d}{\gamma_K} + 2\right)\right\}.$$

Thus we can take

$$T_1(K, \varepsilon) = \exp\left\{\frac{c_{17}(\varepsilon)}{\Gamma_K^2(\varepsilon)} \log\left(\frac{\log d}{\gamma_K} + 2\right)\right\}.$$

All the conditions together are satisfied if

$$T \geqslant T_0(K, \varepsilon) = \exp\left\{c_0(\varepsilon)\left(\frac{1}{B_K(\varepsilon)} + \frac{1}{\Gamma_K^2(\varepsilon)}\right)\log\left(\frac{\log d}{\gamma_K} + 2\right)\log^2 d\right\}.$$

For such T and $\log x \in I_{\nu_0}$ we have

$$(5.9) \qquad \delta_{m-1}(\Delta_K;\ x) = \frac{2e_{\nu_0} x^{\beta_{\nu_0}}}{|\varrho_{\nu_0}|^m} \{\cos(\gamma_{\nu_0}\log x - m\varphi_{\nu_0}) + \bar{O}(3/4)\}.$$

From (5.9) it follows that if t runs over the interval I_{ν_0}, then the function $\delta_{m-1}(\Delta_K;\ e^t)$ has at least

$$\frac{\gamma_{\nu_0}|I_{\nu_0}|}{\pi} - 2$$

sign changes. Denote by $V^{(m-1)}(T)$ the number of sign changes of the function $\delta_{m-1}(\Delta_K;\ x)$ in the interval $(2, T]$. Now we can estimate $V^{(m-1)}T$ from below as follows:

$$\begin{aligned} V^{(m-1)}(T) &\geqslant V^{(m-1)}(T) - V^{(m-1)}(T^{\varepsilon/2}) \\ &\geqslant \sum_{\nu=1}^{l}\left(\frac{\gamma_K}{\pi}|I_\nu| - 2\right) \geqslant \frac{\gamma_K}{\pi}|A(T, L_K(\varepsilon))| - O_\varepsilon(\log d) \\ &\geqslant \frac{\gamma_K}{\pi}\left(1 - \frac{3\varepsilon}{4}\right)\log T - O_\varepsilon(\log d) \geqslant (1-\varepsilon)\frac{\gamma_K}{\pi}\log T. \end{aligned}$$

To complete the proof of Theorem 1 it remains to notice that by Lemma 4 we have

$$V_K(T) \geqslant V^{(m-1)}(T)$$

for any natural $m \geqslant 2$ and real $T \geqslant 2$.

The proof of Theorem 2 is similar.

REFERENCES

[1] I. V. Armitage, *Zeta-functions with zero at s = 1/2*, Invent. Math. 15 (1972), pp. 199–205.
[2] A. E. Ingham, *A note on the distribution of primes*, Acta Arith. 1 (1936), pp. 201–211.
[3] J. Kaczorowski, *On sign-changes in the remainder-term of the prime-number formula. I, II*, ibidem 44 (1984), pp. 365–377; ibidem 45 (1985), pp. 65–74.
[4] S. Knapowski und P. Turán, *Über einige Fragen der vergleichenden Primzahltheorie*, Abhandlungen aus Zahlentheorie und Analysis 1968, pp. 159–177.
[5] E. Landau, *Über Ideale und Primideale in Idealklassen*, Math. Z. 2 (1918), pp. 52–154.
[6] — *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Leipzig 1927.
[7] A. Neugebauer, *Every Dedekind zeta-function has a zero in the rectangle* $1/2 \leqslant \sigma \leqslant 1$, $0 < t < 60$, Discussiones Math. 7 (1985), pp. 141–144.
[8] G. Pólya, *Über das Vorzeichen des Restgliedes im Primzahlsatz*, Gött. Nachrichten 1930, pp. 19–27.
[9] — und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, 2. Band, Berlin 1925.
[10] C. L. Siegel, *Wurzeln Heckescher Zetafunktionen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, 2 (1972), pp. 11–20.
[11] W. Staś, *On sign-changes in the remainder-term of the prime-ideal formula*, Funct. Approx. Comment. Math. 13 (1982), pp. 159–166.
[12] — and K. Wiertelak, *A comparison of certain remainders connected with prime ideals in ideal classes* (mod $\mathfrak{f}$), ibidem 13 (1982), pp. 173–174.
[13] P. Turán, *On some further one-sided theorems of new type in the theory of diophantine approximations*, Acta Math. Acad. Sci. Hungar. 12 (1961), pp. 455–468.
[14] R. W. van der Waall, *Holomorphy of quotients of zeta-functions*, pp. 649–662 in: *Algebraic Number Fields*, Academic Press 1977.

INSTITUTE OF MATHEMATICS
ADAM MICKIEWICZ UNIVERSITY
POZNAŃ

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