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REMARKS ON LATTICES OF CONGRUENCE RELATIONS OF QUASI-ALGEBRAS

BY

B. WOJDYŁO (TORUŃ)

Quasi-algebra of type G is a system $\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G)\rangle$, where A is a set and, for each $g \in G$, $g_{\boldsymbol{A}}$ denotes an $n(g)$-ary partial operation (quasi-operation) on the set A, i.e. $g_{\boldsymbol{A}}$: $D(g_{\boldsymbol{A}}) \to A$, where $D(g_{\boldsymbol{A}}) \subseteq A^{n(g)}$, $g \in G$.

If $D(g_{\boldsymbol{A}}) = A^{n(g)}$ for any $g \in G$, then the quasi-algebra $\boldsymbol{A}$ of type G is an *algebra of type* G. A quasi-algebra $\boldsymbol{B} = \langle B, (g_{\boldsymbol{B}}, g \in G)\rangle$ of type G is called a *weak subquasi-algebra* (1) of $\boldsymbol{A}$ if $B \subseteq A$ and $g_{\boldsymbol{B}} \subseteq g_{\boldsymbol{A}}$ for each $g \in G$. A weak subquasi-algebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is said to be an *ordinary subquasi-algebra* of $\boldsymbol{A}$ if the set B is closed for any partial operation $g_{\boldsymbol{A}}$, $g \in G$, i.e. for each $g \in G$ and for each sequence $\boldsymbol{b} \in B^{n(g)}$, if $g_{\boldsymbol{A}}(\boldsymbol{b})$ exists, then $g_{\boldsymbol{A}}(\boldsymbol{b}) \in B$ and $g_{\boldsymbol{B}}(\boldsymbol{b}) = g_{\boldsymbol{A}}(\boldsymbol{b})$.

For homomorphisms of algebras we have in quasi-algebras a few kinds of mappings. Any ordinary subquasi-algebra h of the direct product $\boldsymbol{A} \times \boldsymbol{B}$ of quasi-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of type G such that for any $a \in A$ there exists at most one element $b \in B$ with $\langle a, b\rangle \in h$ is called a *partial homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$. Any partial homomorphism h of $\boldsymbol{A}$ into $\boldsymbol{B}$ such that for all $a \in A$ there exists one and only one element $b \in B$, with $\langle a, b\rangle \in h$, is called a *full homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$. A mapping h: $A \to B$ ($\boldsymbol{A}$ and $\boldsymbol{B}$ are quasi-algebras of type G) such that, for any $g \in G$,

1° if $(a_\sigma, \sigma < n(g)) \in D(g_{\boldsymbol{A}})$, then $(h(a_\sigma), \sigma < n(g)) \in D(g_{\boldsymbol{B}})$, and

2° $h\big(g_{\boldsymbol{A}}(a_\sigma, \sigma < n(g))\big) = g_{\boldsymbol{B}}\big(h(a_\sigma), \sigma < n(g)\big)$ for $(a_\sigma, \sigma < n(g)) \in D(g_{\boldsymbol{A}})$

is called an *ordinary homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$.

An ordinary homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ such that, for each $g \in G$ and each sequence $\boldsymbol{a} = (a_\sigma, \sigma < n(g)) \in A^{n(g)}$, if $(h(a_\sigma), \sigma < n(g)) \in D(g_{\boldsymbol{B}})$, then there exists a sequence $\boldsymbol{a}' = (a'_\sigma, \sigma < n(g)) \in A^{n(g)}$ such that $\boldsymbol{a}' \in D(g_{\boldsymbol{A}})$ and $h(a'_\sigma) = h(a_\sigma)$ for all $\sigma < n(g)$, is called a *strong homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$. If $\boldsymbol{a}' = \boldsymbol{a}$, a strong homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ is called an *exact homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$.

(1) Definitions of several kinds of subquasi-algebras and homomorphisms coincide in general with conceptions of Grätzer [2] and Słomiński [3].

Concepts of full, ordinary, strong and exact homomorphisms of quasi-algebras are all equivalent in the case of algebras.

For congruence relations of algebras we have two distinct kinds of relations in the case of quasi-algebras. Let A be a quasi-algebra of type G. A relation $\sim \subseteq A\times A$ is said to be a *weak congruence relation* of A if

1° $\sim$ is an equivalence relation on the set A,

2° for each $g\in G$ and each pair of sequences $(a_\sigma, \sigma<n(g)), (b_\sigma, \sigma<n(g)) \in A^{n(g)}$, if $a_\sigma\sim b_\sigma$ for all $\sigma<n(g)$ and $g_A(a_\sigma, \sigma<n(g))$, $g_A(b_\sigma, \sigma<n(g))$ are defined, then $g_A(a_\sigma, \sigma<n(g))\sim g_A(b_\sigma, \sigma<n(g))$.

A relation $\approx\subseteq A\times A$ is called a *strong congruence relation* of A if

1° $\approx$ is an equivalence relation on the set A,

2° for each $g\in G$ and each pair of sequences $(a_\sigma, \sigma<n(g)), (b_\sigma, \sigma<n(g)) \in A^{n(g)}$, if $a_\sigma\approx b_\sigma$ for all $\sigma<n(g)$, then $g_A(a_\sigma, \sigma<n(g))$ is defined if and only if $g_A(b_\sigma, \sigma<n(g))$ is, and then $g_A(a_\sigma, \sigma<n(g))\approx g_A(b_\sigma, \sigma<n(g))$.

Obviously, each strong congruence relation is a weak congruence relation, but not conversely. In the case of algebras strong and weak congruence relations are equivalent.

We say that two ordinary homomorphisms $h\colon A\to B$ and $h'\colon A\to B'$ are *equivalent* if there exists a one-to-one function $s\colon h(A)\to h'(A)$ such that $h'=sh$. If s is an isomorphism, then we say that h and h' are *isomorphic*.

In the case of algebras there is a one-to-one correspondence (up to an isomorphism) between homomorphisms and congruence relations. In the case of quasi-algebras we have the following theorems (²):

THEOREM 1. *Weak congruence relations of a quasi-algebra A of type G are in a one-to-one correspondence with equivalence classes of ordinary homomorphisms of A.*

THEOREM 2. *Strong congruence relations of a quasi-algebra A of type G are in a one-to-one correspondence with isomorphic classes of exact homomorphisms of A.*

A relation induced by a full homomorphism need not be even a weak congruence relation. For instance, if A is an algebra of type G and B is a discrete (³) quasi-algebra of type G, then any mapping $h\colon A\to B$ is a full homomorphism, but the relation induced by h on A need not be even a weak congruence relation of A.

(²) Theorems 1 and 2 are a bit more precise than Theorem 2, II, Section 13, of Grätzer [2], where it is proved that there exists a one-to-one correspondence between ordinary homomorphisms and weak congruence relations and a one-to-one correspondence between exact homomorphisms and strong congruence relations in quasi-algebras.

(³) A quasi-algebra B is called *discrete* if $D(g_B)=\emptyset$ for each $g\in G$.

Relations induced by strong homomorphisms are in general only weak congruence relations. (For example, relations induced by strong homomorphisms which are not exact homomorphisms.)

Any quasi-algebra A of type G has the following two trivial congruence relations:

1° zero-congruence relation ι: $a\iota b$ if and only if $a = b$ $(a, b \in A)$,

2° unit-congruence relation θ: $a\theta b$ for all $a, b \in A$.

It follows from the definition that ι is a strong congruence relation and θ is, in general, only a weak congruence relation. Note that the intersection of weak (strong) congruence relations is again a weak (strong) congruence relation. Hence we have the following theorem:

THEOREM 3. *Weak congruence relations of a quasi-algebra A of type G form a complete lattice under the inclusion as the partial order.*

Proof. Infimum for any set Z of weak congruence relations of quasi-algebra A of type G is the intersection of all elements of the set Z. Supremum for the set Z is the intersection of all elements of a set $Z' = \{\varrho\colon \varrho$ is a weak congruence relation of a quasi-algebra A such that $a \subseteq \varrho$ holds for any $a \in Z\}$.

The set Z' is not empty because $\theta \in Z'$.

Note that so defined supremum is not, in general, equal to supremum in the lattice of all equivalence relations defined on A (see an example, p. 190).

THEOREM 4. *Let ϱ_1 and ϱ_2 be strong congruence relations of a quasi-algebra A and let $\varrho_1 \circ \varrho_2$ denote the relative product of ϱ_1 and ϱ_2. Then the relation $\varrho_1 \circ \varrho_2$ is a strong congruence relation if and only if $\varrho_1 \circ \varrho_2 = \varrho_2 \circ \varrho_1$. Moreover, we have then $\varrho_1 \cup \varrho_2 = \varrho_1 \circ \varrho_2$, where $\varrho_1 \cup \varrho_2$ denotes supremum of ϱ_1 and ϱ_2.*

Proof. Let ϱ_1 and ϱ_2 be strong congruence relations of A. As is known, $\varrho_1 \circ \varrho_2$ is an equivalence relation if and only if $\varrho_1 \circ \varrho_2 = \varrho_2 \circ \varrho_1$, and hence the condition is necessary. To prove sufficiency, let us assume that $\varrho_1 \circ \varrho_2 = \varrho_2 \circ \varrho_1$ and let $(a_\sigma, \sigma < n(g))$, $(b_\sigma, \sigma < n(g)) \in A^{n(g)}$, where $g \in G$, be sequences such that $a_\sigma \varrho_1 \circ \varrho_2 b_\sigma$ for all $\sigma < n(g)$. Let $(c_\sigma, \sigma < n(g)) \in A^{n(g)}$ be a sequence such that $a_\sigma \varrho_1 c_\sigma$ and $c_\sigma \varrho_2 b_\sigma$, $\sigma < n(g)$.

If $(a_\sigma, \sigma < n(g)) \in D(g_A)$, then $(c_\sigma, \sigma < n(g)) \in D(g_A)$ because ϱ_1 is a strong congruence relation in A. Consequently, $(b_\sigma, \sigma < n(g)) \in D(g_A)$ because ϱ_2 is a strong congruence relation in A. Thus we have

$$g_A(a_\sigma, \sigma < n(g))\, \varrho_1 g_A(c_\sigma, \sigma < n(g))$$

and

$$g_A(c_\sigma, \sigma < n(g))\, \varrho_2 g_A(b_\sigma, \sigma < n(g)),$$

whence also

$$g_A(a_\sigma, \sigma < n(g))\, \varrho_1 \circ \varrho_2 g_A(b_\sigma, \sigma < n(g)).$$

Hence $\varrho_1 \circ \varrho_2$ is a strong congruence relation of $\boldsymbol{A}$. Now, by Cohn [1], II,6.6, we infer that $\varrho_1 \cup \varrho_2 = \varrho_1 \circ \varrho_2$. This completes the proof of Theorem 4.

Theorem 4 is a generalization of a well-known theorem for algebras (see, for example, Cohn [1], II,6.7). From Theorem 4 we have immediately the following corollary:

COROLLARY. *The set of strong congruence relations of a quasi-algebra $\boldsymbol{A}$ of type G forms a lattice with zero under the inclusion as the partial order.*

If ϱ_1 and ϱ_2 are weak congruence relations, then the condition $\varrho_1 \circ \varrho_2 = \varrho_2 \circ \varrho_1$ does not suffices for $\varrho_1 \circ \varrho_2$ to be a weak congruence relation.

For instance, let $\boldsymbol{A} = \langle A, + \rangle$ be a quasi-algebra such that $A = \{a, b, c, d, e, f\}$ and $+$ is a 2-ary partial operation defined by the following table:

$+$	a	b	c	d	e	f
a	—	—	e	—	—	—
b	—	—	—	f	—	—
c	—	—	—	—	—	—
d	—	—	—	—	—	—
e	—	—	—	—	—	—
f	—	—	—	—	—	—

Let ϱ_1 and ϱ_2 be two weak congruence relations described by the following tables ($\times$ means that a pair belongs to ϱ_1 or ϱ_2, respectively):

ϱ_1	a	b	c	d	e	f
a	×	×	—	—	—	—
b	×	×	—	—	—	—
c	—	—	×	—	—	—
d	—	—	—	×	—	—
e	—	—	—	—	×	—
f	—	—	—	—	—	×

ϱ_2	a	b	c	d	e	f
a	×	—	—	—	—	—
b	—	×	—	—	—	—
c	—	—	×	×	—	—
d	—	—	×	×	—	—
e	—	—	—	—	×	—
f	—	—	—	—	—	×

The relation $\varrho_1 \circ \varrho_2$ has the following table:

$\varrho_1 \circ \varrho_2$	a	b	c	d	e	f
a	×	×	—	—	—	—
b	×	×	—	—	—	—
c	—	—	×	×	—	—
d	—	—	×	×	—	—
e	—	—	—	—	×	—
f	—	—	—	—	—	×

Obviously, $\varrho_1 \circ \varrho_2 = \varrho_2 \circ \varrho_1$, but $\varrho_1 \circ \varrho_2$ is not a weak congruence relation because $a \varrho_1 \circ \varrho_2 b$, $c \varrho_1 \circ \varrho_2 d$, $a + c = e$ and $b + d = f$, and it is not true that $e \varrho_1 \circ \varrho_2 f$.

REFERENCES

[1] P. M. Cohn, *Universal algebra*, New York 1965.
[2] G. Grätzer, *Universal algebra*, 1968.
[3] J. Słomiński, *Peano-algebras and quasi-algebras*, Dissertationes Mathematicae 57 (1968).

INSTITUTE OF MATHEMATICS
N. COPERNICUS UNIVERSITY, TORUŃ

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