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SOME PL INVOLUTIONS OF PRODUCT 3-MANIFOLDS WHICH PRESERVE THE CENTER OF THE FUNDAMENTAL GROUP*

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1. Introduction. Let h be a PL involution of a closed 3-manifold X with fixed point set $F \neq \emptyset$ and $x_0 \in F$. The PL involution h is said to be *fiberwise* (see [2]) if there exists a fibering of X over S^1 such that each fiber is invariant under h.

It is a consequence of a result of Kim and Tollefson [3] that if $F \neq \emptyset$ and h is fiberwise, then each component of F is a simple closed curve, a torus or a Kleinbottle whose fundamental group is mapped nontrivially into $\pi_1(S^1)$ under the homomorphism induced by the fibering.

The following theorem is proved in [1], [2], [4].

THEOREM 1. *Let X be a P^2 irreducible compact connected 3-manifold and h a PL involution of X with fixed point set $F \neq \emptyset$ and $x_0 \in F$. Then h is fiberwise if and only if there exists an epimorphism $\varepsilon\colon \pi_1(X, x_0) \to Z$ such that $\operatorname{Ker}\varepsilon$ is finitely generated, $h_\#(\operatorname{Ker}\varepsilon) = \operatorname{Ker}\varepsilon$ and $h_\#$ fixes an element of $\pi_1(X, x_0)$ not in $\operatorname{Ker}\varepsilon$. (The fiber M' of the fibering can be chosen so that $\pi_1 M' \approx \operatorname{Ker}\varepsilon$.)*

The following result is also given in [2] and [4].

COROLLARY. *Let M be a compact connected orientable surface of genus greater than 0. Let h be a PL involution of $M \times S^1$ with fixed point set $F \neq \emptyset$ and $(m_0, s_0) \in F$ such that $m_0 \in M$, $s_0 \in S^1$. Then h is equivalent to $\beta \times 1_{S^1}$, β being an involution of M, if and only if under a suitable product structure of $M \times S^1$ and the canonical identification of $\pi_1(M \times S^1, (m_0, s_0))$ with $\pi_1(M, m_0) \times \pi_1(S^1, s_0)$ we have $h_\#(\pi_1(M, m_0)) = \pi_1(M, m_0)$ and $h_\# | \pi_1(S^1, s_0)$ is the identity.*

In [2], it is shown by an example that the conclusion in the corollary is false if it is merely assumed that $h_\# | \pi_1(S^1, s_0)$ is the identity. However, it is

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pointed out by H. Kim and J. Kim [1] that the example cited is nevertheless fiberwise. It is the purpose of this note to show that this conclusion is true in a complete generality.

THEOREM 2. *Let M be a compact connected (orientable or nonorientable) surface not homeomorphic to S^2 or P^2. Let h be a PL involution of $M \times S^1$ with fixed point set $F \neq \emptyset$ and $(m_0, s_0) \in F$ such that $m_0 \in M$ and $s_0 \in S^1$. If $h_\# | \pi_1(S^1, s_0)$ is the identity, then h is fiberwise.*

Remark. The fiber M' of the fibering of the conclusion of Theorem 2 can be chosen such that there exists a covering $M' \to M$ of multiplicity 1 or 2.

2. The proof of Theorem 2.

2.1. Canonically identify $\pi_1(M \times S^1, (m_0, s_l))$ with the internal direct product $A \times H$, where A and H are images of the composites

$$\pi_1(M, m_0) \approx \pi_1(M \times \{s_0\}, (m_0, s_0)) \xrightarrow{i_\#} \pi_1(M \times S^1, (m_0, s_0))$$

and

$$\pi_1(S^1, s_0) \approx \pi_1(\{m_0\} \times M, (m_0, s_0)) \xrightarrow{i'_\#} \pi_1(M \times S^1, (m_0, s_0)),$$

where unlabeled isomorphisms are obvious ones and i, i' are inclusions.

Consider the diagram

$$A \times H \xrightarrow{h_\#} A \times H \begin{array}{l} \xrightarrow{p_1} A \\ \xrightarrow{p_2} H \end{array}$$

where p_i $(i = 1, 2)$ are projections. Identify H with Z. Let $A \xrightarrow{\eta} A$ and $A \xrightarrow{\zeta} H$ be defined by

$$\eta(a) = p_1 h_\#(a, 0) \quad \text{and} \quad \zeta(a) = p_2 h_\#(a, 0).$$

Since $h_\# | H$ is the identity, $h_\#(a, t) = (\eta(a), \zeta(a) + t)$. Since $h_\#^2$ is the identity, it follows that

$$\eta^2 = 1_A \quad \text{and} \quad \zeta(\eta(a)) = -\zeta(a).$$

2.2. If $h_\#(A) = A$, Theorem 1 proves Theorem 2.

2.3. If $h_\#(A) \neq A$, then $h_\#(A) \not\subset A$ and ζ is nontrivial. Let $\zeta(A) = kH$, where k is a positive integer.

Let $N = \{(a, t) \in A \times H \mid \zeta(a) = -2t\}$. It follows from 2.1 that $h_\#(N) = N$.

Define α: $A \times H \to H$ by $\alpha(a, t) = \zeta(a) + 2t$. Then $\alpha(H) = 2H$ and $\alpha(A \times H) = qH$, where $q = 1$ or 2.

Thus $N = \operatorname{Ker}\alpha$ and there is a short exact sequence

$$1 \to N \to A \times H \xrightarrow{\alpha} qH \to 1.$$

Furthermore, N is isomorphic to $N' \subset A$, where $N' = \{a \in A \mid \zeta(a) \text{ is even}\}$. Hence N' is a subgroup of A of index 1 or 2 according as k is even or not. Hence N', and therefore N, is finitely generated. Since $N \cap H = 1$, Theorem 1 and the above short exact sequence prove Theorem 2.

The remark following Theorem 2 is clear from the proof, since M is closed.

REFERENCES

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