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CONCERNING TWO METHODS OF DEFINING THE CENTER OF A DYNAMICAL SYSTEM, II

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Birkhoff and Smith [2] and Maier [6] defined two processes which describe the center of a dynamical system F defined on a compact metric space S (a continuous mapping from $(-\infty, \infty)\times S$ onto S such that if x is a number, y is a number, and p is a point of S, then $F(x+y, p) = F\big(x, F(y, p)\big)$ and $F(0, p) = p$). In each of these processes, a monotonic (perhaps transfinite) sequence W of subsets of S is defined so that the center of F is the closure of the common part of all of the members of W.

Suppose that F is a dynamical system defined on a compact metric space S. If I is a subset of S and I is invariant under F, then let $\Omega(I) = \{p: p \text{ is an } \omega\text{-limit point of a movement of } F \text{ lying in } I\}$. The sequence $B_1, B_2, \ldots$ used in the process of Birkhoff and Smith is defined as follows.

Let $B_1 = S$. Notice that $\mathrm{cl}\big(\Omega(B_1)\big)$ is a compact invariant subset of B_1 (cf. [8], pp. 311 and 338). If $\mathrm{cl}\big(\Omega(B_1)\big) = B_1$, then B_1 will be the only member of the sequence. If $\mathrm{cl}\big(\Omega(B_1)\big) \neq B_1$, then the set $B_2 = \mathrm{cl}\big(\Omega(B_1)\big)$. Continue this process inductively as follows. If u is an ordinal number and there is a term of the sequence with subscript u, and $\mathrm{cl}\big(\Omega(B_u)\big) = B_u$, then B_u will be the last term of the sequence. If $\mathrm{cl}\big(\Omega(B_u)\big) \neq B_u$, then the set $B_{u+1} = \mathrm{cl}\big(\Omega(B_u)\big)$. Notice that B_{u+1} is a closed invariant subset of B_u. If v is an ordinal number with no immediate predecessor and, for each ordinal number $w < v$, there is a term of the sequence with a subscript w, then the set $B_v = \bigcap_{z<v} B_z$. Notice that B_v is closed and invariant. $B_1, B_2, \ldots$ is a monotonically decreasing sequence of compact subsets of the compact metric (and, therefore, separable) space S. Therefore, by Baire's theorem, $B_1, B_2, \ldots$ has only countably many members. The common part C of all of the members of $B_1, B_2, \ldots$ is the closed invariant set defined to be the center of F.

The sequence $M_1, M_2, \ldots$ used in the process of Maier is defined as follows. Set $M_1 = S$. Notice that $\Omega(M_1)$ is an invariant subset of M_1. If $\Omega(M_1) = M_1$, then M_1 will be the only member of the sequence. If

$\Omega(M_1) \neq M_1$, then the set $M_2 = \Omega(M_1)$. Notice that M_2 is an invariant subset of M_1. Continue this process as follows. If u is an ordinal number and there is a member of $M_1, M_2, \ldots$ with a subscript u, and $\Omega(M_u) = M_u$, then M_u will be the last term of the sequence. If $\Omega(M_u) \neq M_u$, then set $M_{u+1} = \Omega(M_u)$. Notice that M_{u+1} is an invariant subset of M_u. If v is an ordinal number with no immediate predecessor and, for each ordinal number $w < v$, there is a term of the sequence with a subscript w, then the set $M_v = \bigcap_{z<v} M_z$.

For each dynamical system F, let $M(F)$ denote the number of terms of the sequence $M_1, M_2, \ldots$ associated with F and let $B(F)$ denote the number of terms of $B_1, B_2, \ldots$ In [7], Nemyckii shows that $M(F) \leqslant B(F)$ and that the closure of the common part of all of the members of M_1, $M_2, \ldots$ is the center of F. Nemyckii [7] asked if there is an example where $M(F) < B(F)$. In [9], an example is given such that $B(F) = 4$ and $M(F) = 3$. The purpose of this paper is to describe an example of a dynamical system F defined on a compact subspace S of E^3 such that $B(F) = \omega$ and $M(F) = 3$.

LEMMA (1). *If for each cube C in E^3 and each positive integer $n > 2$ there is a dynamical system F defined on a closed subset of C such that $B(F) = n$ and $M(F) = 3$, then there is a dynamical system G defined on a closed and bounded subset of E^3 such that $B(G) = \omega$ and $M(G) = 3$.*

Proof. For each positive integer $n > 2$, let C_n denote a cube which contains $(1/n, 0, 0)$ and has edge length less than $1/4n$ and let F_n denote a dynamical system defined on a closed subset of C_n such that $B(F_n) = n$ and $M(F_n) = 3$. Let G denote the dynamical system such that the set of motions of G consists of the motions of $F_3, F_4, \ldots$ along with the rest point $(0, 0, 0)$. Notice that G is continuous, $B(G) = \omega$, and $M(G) = 3$.

Definitions and notation. If p and q are points of E^3 and ε is a positive number, then let $\varrho(p, q)$ denote the distance from p to q and let $R(p, \varepsilon) = \{s: \varrho(s, p) < \varepsilon\}$. If H is a number set, F is a dynamical system defined on a subspace S of E^3 and p is a point of S, then let $F(H, p) = \{F(x, p): x \in H\}$. $F\big((-\infty, \infty), p\big)$ will be denoted by $\Psi(p)$. p is a *type* (1) *point* means that p is a rest point of F. p is a *type* (2) *point* means that there is a type (1) point q such that q is the only α-limit point of the movement of p and q is the only ω-limit point of the movement of p. p is a *type* (3) *point* means that there is a type (1) point u and a type (2) point v such that u is the α-limit point and ω-limit point of the movement of v and such that u is the only α-limit point of the movement of p and $\{u\} \cup \Psi(v)$ is the set of ω-limit points of the movement of p. A building block is a dynamical system consisting of the three motions through such a triple (u, v, p), see Fig. (1). Let S_1, S_2 and S_3 denote, respectively, the sets consisting of the type (1), type (2) and type (3) points of S.

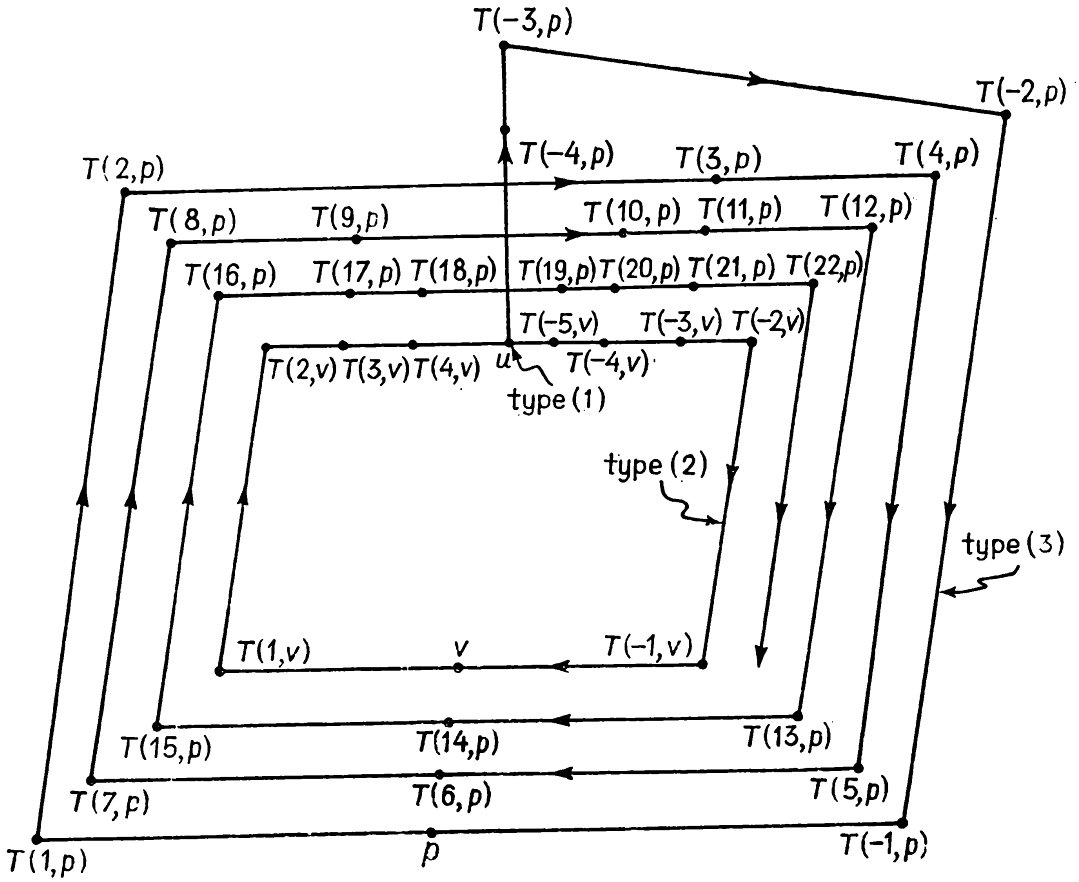

Fig.(1)

LEMMA (2). *If C is a cube in E^3, d is a positive integer and, for each non-negative integer $n \leqslant d$, Z_n is a collection of building blocks whose union is defined on the subset X_n of C,*

$$\Big(\mathrm{cl}\Big(\bigcup_{i=0}^{n-1} X_i\Big)\Big) \cap X_n = \emptyset,$$

$$\bigcup_{i=0}^{n-1} X_i \subset \mathrm{cl}(\{q\colon\ q \text{ is a type (2) point of } X_n\}),$$

and $\bigcup_{i=0}^{d} Z_i$ forms a dynamical system F, then $B(F) = d+3$ and $M(F) = 3$.

Proof. Assume the hypothesis of the lemma and let $S = \bigcup_{i=0}^{d} X_i$.

$$M_1 = S.$$

$$M_2 = S_1 \cup S_2.$$

$$M_3 = S_1.$$

$$B_1 = \bigcup_{i=0}^{d} X_i.$$

$$B_2 = \Big(\bigcup_{i=0}^{d-1} X_i\Big) \cup \big((S_1 \cup S_2) \cap X_d\big).$$

$$B_3 = \Big(\bigcup_{i=0}^{d-2} X_i\Big) \cup \big((S_1 \cup S_2) \cap X_{d-1}\big) \cup (S_1 \cap X_d).$$

$$B_4 = \Big(\bigcup_{i=0}^{d-3} X_i\Big) \cup \big((S_1 \cup S_2) \cap X_{d-2}\big) \cup \Big(S_1 \cap \Big(\bigcup_{i=d-1}^{d} X_d\Big)\Big).$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$B_d = \Big(\bigcup_{i=0}^{1} X_i\Big) \cup \big((S_1 \cup S_2) \cap X_2\big) \cup \Big(S_1 \cap \Big(\bigcup_{i=3}^{d} X_d\Big)\Big).$$

$$B_{d+1} = X_0 \cup \big((S_1 \cup S_2) \cap X_1\big) \cup \Big(S_1 \cap \Big(\bigcup_{i=2}^{d} X_d\Big)\Big).$$

$$B_{d+2} = \big((S_1 \cup S_2) \cap X_0\big) \cup \Big(S_1 \cap \Big(\bigcup_{i=1}^{d} X_d\Big)\Big).$$

$$B_{d+3} = S_1 \cap \Big(\bigcup_{i=0}^{d} X_d\Big) = S_1.$$

Example. Suppose that C is an open cube in E^3 with edge length less than $1/2$ and d is a positive integer. We will now establish the existence of a dynamical system G defined on a closed subspace S of C such that $B(G) = \omega$ and $M(G) = 3$ by defining a sequence $Z_0, Z_1, \ldots, Z_d$ of collections of building blocks in C satisfying the hypothesis of Lemma (2). Notice that $Z_0, Z_1, \ldots, Z_d$ must be defined in such a way as to insure that (1) for each non-negative integer $i < d$, Z_i is such that there is "still room to fit in Z_{i+1}" and (2) $\bigcup_{i=1}^{d} Z_i$ constitutes a continuous flow F.

Each trajectory of F will contain a point q with the property that if k is an integer and x is a number in $(0, 1)$, then $F(k+x, q) = (1-x)F(k, q)+xF(k+1, q)$. For each trajectory of F, exactly one such point q will be designated as a determining point of F. The movement through q will, therefore, be described by defining $F(k, q)$ for each integer k. The example will be constructed (as in Fig. (1)) so that if B is a building block of F with type (1) point u, type (2) determining point v, and type (3) determining point p, then there is a positive integer n such that $F\big((-\infty, -n], v\big) \cup \{u\} \cup F([n, \infty), v)$ is the straight line interval $[F(-n, v), F(n, v)]$ and there exists a positive integer m such that $F\big((-\infty, -m], p\big) \cup \{u\}$ is the interval $[F(-m, p), u]$.

Z_0 will contain only one building block B. B will contain a type (1) point u, a type (2) point v, and a type (3) point p. Let u denote a point of C and define u to be a rest point of F. Let v denote a point of C distinct from u. Extend F to include a type (2) trajectory through v in such a way that v is a determining point of F, $\Psi(v) \subset C$, and u is the α-limit point and the ω-limit point of the movement of v. There will be associated with each type (2) determining point w of F, a positive number $\varepsilon(w)$ and a sequence of positive integers $N_1(w), N_2(w), \ldots$ which will be needed to complete the description of F. Let $\varepsilon(v)$ denote a positive number such that $R(\Psi(v), \varepsilon(v)) \subset C$. Let $N_1(v)$ denote a positive integer such that

$$F([N_1(v), \infty), v) \cup \{u\} \cup F((-\infty, -N_1(v)], v)$$

is a straight line interval which is a subset of $R(u, \varepsilon(v))$. In Fig. (1), $N_1(v) = 2$. For each integer $i > 1$, set

$$N_i(v) = (2i-1)N_1(v) + \sum_{j=1}^{i-1}(2j+1).$$

In Fig. (1), $N_2(v) = 9$ and $N_3(v) = 18$. Let p denote a point of $R(v, \varepsilon(v)) - (\{u\} \cup \Psi(v))$. Extend F to include a movement of p in such a way that p is a determining point of F, $\Psi(p) \subset R(\Psi(v), \varepsilon(v)) - (\{u\} \cup \Psi(v))$ and, for each integer k, the following three statements are true:

(1) if $-N_1(v) \leqslant k \leqslant N_1(v)$, then $\varrho(F(k, p), F(k, v)) < \varepsilon(v)$;

(2) if $k < -N_1(v)$, then $F(k, p) = (F(k+1, p) + u)/2$;

(3) for each positive integer j, if $N_j(v) < k \leqslant N_{j+1}(v)$, then we have $\varrho(F(k, p), F(k - N_j(v) - N_1(v) - j - 1, v)) < \varepsilon(v)/j$.

We will now define a sequence $W_1(B), W_2(B), \ldots$ of building blocks (called a $(u, v, p, \varepsilon(v), N_1(v), N_2(v), \ldots)$ *sequence*) so that $Z_1 = \{W_1(B), W_2(B), \ldots\}$. For each positive integer n, $W_n(B)$ will contain a type (1) point u_n, a type (2) point v_n, and a type (3) point p_n, see Fig. (2). Let u_1 denote a point of

$$R(u, \varepsilon(v)) - (\{u\} \cup \Psi(v) \cup \Psi(p)).$$

Extend F in such a way that u_1 is a rest point of F. Let v_1 denote a point of

$$(R(\Psi(v), \varepsilon(v)) \cap R(p, \varepsilon(v))) - (\{u\} \cup \Psi(v) \cup \Psi(p) \cup \{u_1\}).$$

Extend F to include a type (2) movement through v_1 so that v_1 is a determining point of F,

$$\Psi(v_1) \subset R(\Psi(v), \varepsilon(v)) - (\{u\} \cup \Psi(v) \cup \Psi(p) \cup \{u_1\})$$

and, for each integer k, the following three statements are true:

(1) if $-N_1(v) \leqslant k \leqslant N_1(v)$, then $\varrho(F(k, v_1), F(k, p)) < \varepsilon(v)$;

(2) if $k < -N_1(v)$, then $F(k, v_1) = (F(k+1, v_1) + u_1)/2$;

(3) if $N_1(v) < k$, then $F(k, v_1) = (F(k-1, v_1) + u_1)/2$.

Let $\varepsilon(v_1)$ denote a positive number less than $\varepsilon(v)$ such that

$$R(\Psi(v_1), \varepsilon(v_1)) \subset R(\Psi(v), \varepsilon(v))$$

and

$$R(\Psi(v_1), \varepsilon(v_1)) \cap R(\Psi(p), \varepsilon(v_1)) = \emptyset.$$

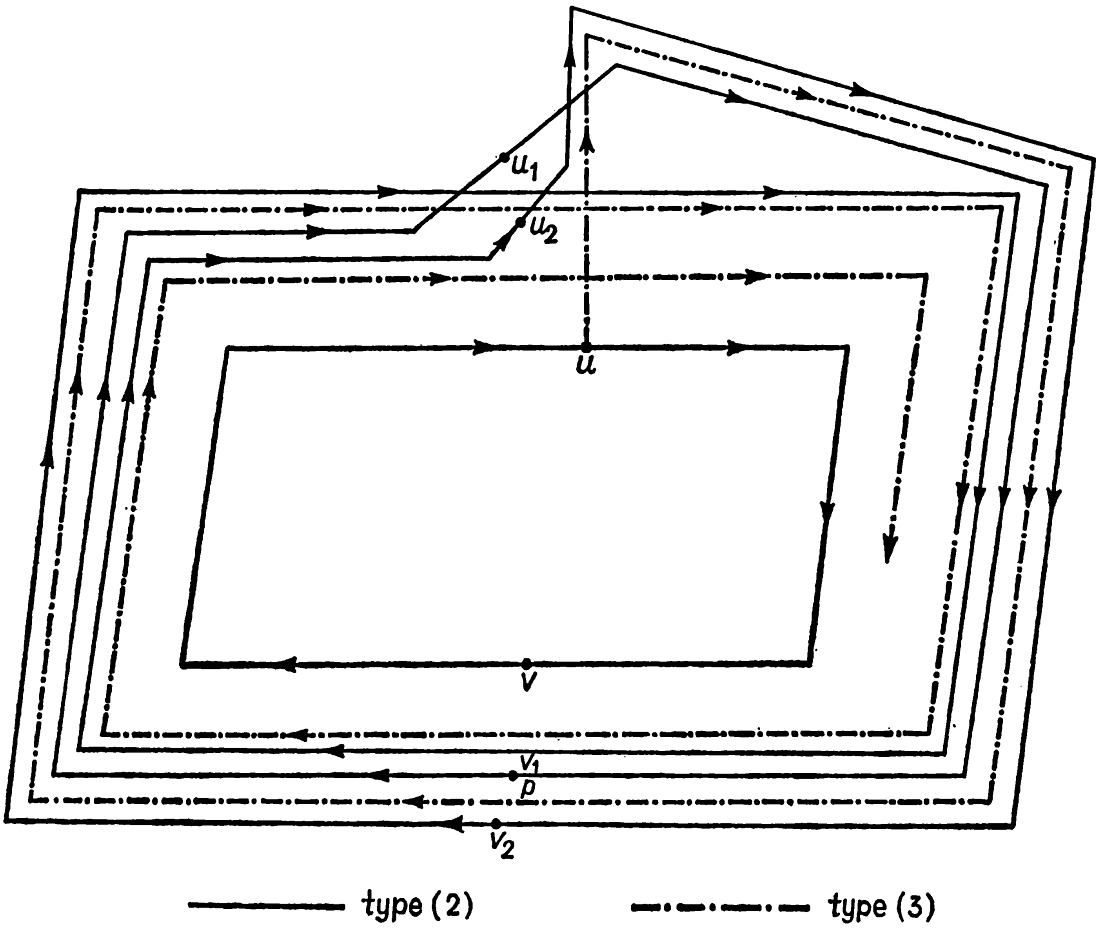

Fig. (2)

Let $N_1(v_1)$ denote a positive integer such that $N_1(v_1) > N_1(v)$ and

$$F\big([N_1(v_1),\ \infty),\ v_1\big) \cup \{u_1\} \cup F\big((-\infty,\ -N_1(v_1)],\ v_1\big)$$

is a straight line interval which is a subset of $R\big(u_1,\ \varepsilon(v_1)\big)$. For each integer $i > 1$, set

$$N_i(v_1) = (2i-1)N_1(v_1) + \sum_{j=1}^{i-1}(2j+1).$$

Let p_1 denote a point of $R\big(v_1,\ \varepsilon(v_1)\big) - \big(\{u_1\} \cup \Psi(v_1)\big)$. Extend F to include a type (3) movement through p_1 in such a way that p_1 is a determining point of F,

$$\Psi(p_1) \subset R\big(\Psi(v_1),\ \varepsilon(v_1)\big) - \big(\{u_1\} \cup \Psi(v_1)\big)$$

and, for each integer k, the following three statements are true:

(1) if $-N_1(v_1) \leqslant k \leqslant N_1(v_1)$, then $\varrho\big(F(k, p_1),\ F(k, v_1)\big) < \varepsilon(v_1)$;

(2) if $k < -N_1(v_1)$, then $F(k, p_1) = \big(F(k+1, p_1) + u_1\big)/2$;

(3) for each positive integer j, if $N_j(v_1) < k \leqslant N_{j+1}(v_1)$, then $\varrho\big(F(k, p_1),\ F\big(k - N_j(v_1) - N_1(v_1) - j - 1,\ v_1\big)\big) < \varepsilon(v_1)/j$.

Continue this construction inductively for each positive integer $n > 1$ as follows. Let u_n denote a point of

$$R\big(u, \varepsilon(v_{n-1})\big) - \big(\{u\} \cup \Psi(v) \cup \Psi(p)\big).$$

Extend F in such a way that u_n is a rest point of F. Let v_n denote a point of

$$\big(R\big(\Psi(v), \varepsilon(v)\big) \cap R\big(p, \varepsilon(v_{n-1})\big)\big) - \big(\{u\} \cup \Psi(v) \cup \Psi(p) \cup \{u_n\}\big).$$

Extend F to include a type (2) movement through v_n in such a way that v_n is a determining point of F,

$$\Psi(v_n) \subset \Big(\big(R\big(\Psi(p), \varepsilon(v_{n-1})\big) \cap R\big(\Psi(v), \varepsilon(v)\big)\big) - \big(\{u\} \cup \Psi(v) \cup \Psi(p)\big)\Big)$$

and, for each integer k, the following three statements are true:

(1) if $-N_n(v) \leqslant k \leqslant N_n(v)$, then $\varrho\big(F(k, v_n), F(k, p)\big) < \varepsilon(v_{n-1})$;

(2) if $k < -N_n(v)$, then $F(k, v_n) = \big(F(k+1, v_n) + u_n\big)/2$;

(3) if $N_n(v) < k$, then $F(k, v_n) = \big(F(k-1, v_n) + u_n\big)/2$.

Let $\varepsilon(v_n)$ denote a positive number less than $\varepsilon(v)/n$ such that

$$R\big(\Psi(v_n), \varepsilon(v_n)\big) \subset \big(R\big(\Psi(p), \varepsilon(v_{n-1})\big) \cap R\big(\Psi(v), \varepsilon(v)\big)\big)$$

and

$$R\big(\Psi(v_n), \varepsilon(v_n)\big) \cap R\big(\Psi(p), \varepsilon(v_n)\big) = \emptyset.$$

Let $N_1(v_n)$ denote a positive integer greater than $N_n(v)$ such that

$$F\big([N_1(v_n), \infty), v_n\big) \cup \{u_n\} \cup F\big((-\infty, -N_1(v_n)], v_n\big)$$

is a straight line interval which is a subset of $R(u_n, \varepsilon(v_n))$. For each integer $i > 1$, set

$$N_i(v_n) = (2i-1)N_1(v_n) + \sum_{j=1}^{i-1} (2j+1).$$

Let p_n denote a point of $R\big(v_n, \varepsilon(v_n)\big) - \big(\{u_n\} \cup \Psi(v_n)\big)$. Extend F to include a type (3) movement through p_n in such a way that p_n is a determining point of F,

$$\Psi(p_n) \subset \big(R\big(\Psi(v_n), \varepsilon(v_n)\big) - \big(\{u_n\} \cup \Psi(v_n)\big)\big)$$

and, for each integer k, the following three statements are true:

(1) if $-N_1(v_n) \leqslant k \leqslant N_1(v_n)$, then $\varrho\big(F(k, P_n), F(k, v_n)\big) < \varepsilon(v_n)$;

(2) if $k < -N_1(v_n)$, then $F(k, p_n) = \big(F(k+1, p_n) + u_n\big)/2$;

(3) for each positive integer j, if $N_j(v_n) < k \leqslant N_{j+1}(v_n)$, then $\varrho\big(F(k, p_n), F\big(k - N_j(v_n) - N_1(v_n) - j - 1, v_n\big)\big) < \varepsilon(v_n)/j$, see Fig. (2).

This concludes the construction of the $\big(u, v, p, \varepsilon(v), N_1(v), N_2(v), \ldots\big)$ sequence $\big(W_1(B), W_2(B), \ldots\big)$. Let $Z_1 = \{W_1(B), W_2(B), \ldots\}$.

Continue the construction of $Z_0, Z_1, \ldots, Z_d$ inductively as follows. Suppose that m is a positive integer less than d for which Z_m is defined. For each building block D of Z_m (where D has type (1) point a, type (2) determining point b, and type (3) determining point c), define the $(a, b, c, \varepsilon(b), N_1(b), N_2(b), \ldots)$ sequence of building blocks $W_1(D), W_2(D), \ldots$ Now define Z_{m+1} to be $\bigcup_{D \in Z_m} \{W_1(D), W_2(D), \ldots\}$. This completes the definition of the sequence $Z_0, Z_1, \ldots, Z_d$. Let

$$F = \bigcup_{n=0}^{d} Z_n.$$

It will now be demonstrated that F is a continuous flow.

For each integer n in $[0, d]$, let X_n denote the subset of C upon which Z_n is defined. Let

$$S = \bigcup_{n=1}^{d} X_n.$$

Notice that S is closed. Define S_1, S_2 and S_3 as before. If r is a type (2) or type (3) determining point and z is a point of $\Psi(r)$, then let t_z be the number such that $F(t_z, r) = z$, let k_z denote the largest integer not exceeding t_z, and let $y_z = t_z - k_z$. Suppose that w is a point of S and x is a number. We will now show that F is continuous at (x, w). There are five cases to consider: (1) $w \in S_2$ and $y_w \neq 0$; (2) $w \in S_2$ and $y_w = 0$; (3) $w \in S_3$ and $y_w \neq 0$; (4) $w \in S_3$ and $y_w = 0$; and (5) $w \in S_1$.

Let e denote a positive number. Suppose (case 1) that $w \in S_2$ and $y_w \neq 0$. There is a non-negative integer h and a building block E of Z_h (where E has type (1) point α, type (2) determining point β, and type (3) determining point γ) such that $w = F(t_w, \beta)$. We will now demonstrate that F is continuous at (x, w) by showing, inductively, that for each non-negative integer N such that $h + N \leqslant d$, the following statement (statement (A)) is true:

There are positive numbers Δ_N and Δ'_N such that

(1) if i is a non-negative integer less than h, then $X_i \cap R(w, \Delta_N) = \emptyset$;

(2) $F\big((x-\Delta_N, x+\Delta_N), (R(w, \Delta_N) \cap S_2 \cap X_{h+N}) \cup (R(w, \Delta'_N) \cap S_3 \cap \cap X_{h+N})\big) \subset R\big(F(x, w), (4N+4)e/(4d+4)\big)$.

There is a positive number δ_0 such that

(1) $\delta_0 < \min\big(1/2, e/(4d+4)\big)$;

(2) if i is a number and $\varrho\big(F(i, \beta), w\big) < \delta_0$, then $|i - t_w| < e/(4d+4)$;

(3) $\varrho(w, a) > (4d+4)\delta_0$.

There is a positive number $\delta_1 < \delta_0$ such that if i is an integer distinct from k_w, then $\varrho\big(F(i, \beta), F(k_w, \beta)\big) > \delta_1$.

There is a positive number $\delta_2 < \delta_1/2$ such that if θ is a point of $R\big(F(k_w, \beta), \delta_2\big)$ and θ' is a point of $R\big(F(k_w+1, \beta), \delta_2\big)$ and z is a number such that $(1-z)\theta+z\theta'$ is in $R(w, \delta_2)$, then $0<z<1$ and $|y_w-z| < e/(4d+4)$.

Let m_0 denote an integer such that

(1) $N_1(\beta)+|k_w|+|x|+2 < m_0$;

(2) $\varepsilon(\beta)/m_0 < \delta_2/2$;

(3) $\varrho(F(m_0, \beta), a) < \delta_0$.

There is a positive number $\delta_3 < \delta_2$ such that

$$R\big(F([-m_0, m_0], \beta), \delta_3\big) \cap R\Big(F\big((-\infty, N_{m_0}(\beta)], \gamma\big), \delta_3\Big) = \emptyset.$$

There is a positive number δ_4 such that if j is a number with $F(j, \gamma) \in R(w, \delta_4)$, then there is an integer i such that

(1) $i<j<i+1$;

(2) $F(i, \gamma) \in R\big(F(k_w, \beta), \delta_3/2\big)$;

(3) $F(i+1, \gamma) \in R\big(F(k_w+1, \beta), \delta_3/2\big)$.

If $h<d$, then let $U_1, U_2, \ldots$ denote the $\big(a, \beta, \gamma, \varepsilon(\beta), N_1(\beta), N_2(\beta) \ldots\big)$ sequence of Z_{h+1} and, for each integer i, let a_i denote the type (1) point of U_i, let β_i denote the type (2) determining point of U_i, and let γ_i denote the type (3) determining point of U_i, and let δ_5 denote a positive number less than $\varepsilon(\beta_{m_0})$.

If $h=d$, then let $\delta_5 = \varepsilon(\beta)/m_0$. There is a positive number δ_6 such there are no type (1) points in $R(w, \delta_6)$.

Let $\Delta_0 = \min(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6)$. Since $\Delta_0 < \varepsilon(\beta)$, we infer that if i is a non-negative integer less than h, then $X_i \cap R(w, \Delta_0) = \emptyset$.

Let t denote a number in $(x-\Delta_0, x+\Delta_0)$. Suppose that q is a point of $S_2 \cap X_h \cap R(w, \Delta_0)$. Since $\Delta_0 < \delta_5 < \varepsilon(\beta)$, we infer that $q \in \Psi(\beta)$. Since $\Delta_0 < \delta_0$, we have $|t_w - t_q| < e/(4d+4)$. Since $\tau \in (x-\Delta_0, x+\Delta_0)$, we infer that

$$|(x-\tau)+(t_w-t_q)| < 2e/(4d+4)$$

and, therefore, since the edge length of C is less than $1/2$, we have

$$\varrho\Big(F(\tau+t_q, \beta), F\big(x-\tau+t_w-t_q, F(\tau+t_q, \beta)\big)\Big) < 2e/(4d+4)$$

which says that

$$\varrho\big(F(\tau, q), F(x, w)\big) < 2e/(4d+4).$$

Let $\Delta_0' = \Delta_0$. Suppose that r is a point of $X_h \cap S_3 \cap R(w, \Delta_0')$. Since $\Delta_0' \leqslant \delta_5 < \varepsilon(\beta)$, we infer that $r \in \Psi(\gamma)$. Since $\Delta_0' < \delta_4$, it follows that $F(t_r, \gamma) \in R(w, \delta_4)$ and, therefore,

$$F(k_r, \gamma) \in R\big(F(k_w, \beta), \delta_3/2\big) \quad \text{and} \quad F(k_r+1, \gamma) \in R\big(F(k_w+1, \beta), \delta_3/2\big).$$

But $(1-y_r)F(k_r, \gamma)+y_rF(k_r+1, \gamma) = r$ and $r \in R(w, \delta_2)$, therefore, $|y_w - y_r| < e/(4d+4)$.

Since $|k_w|+|x|+2 < m_0$, we have $F(k_r, \gamma) \in R\big(F([-m_0, m_0], \beta), \delta_3\big)$ and, therefore, $k_r > N_{m_0}(\beta)$. There is a positive integer $j \geqslant m_0$ such that $N_j(\beta) < k_r \leqslant N_{j+1}(\beta)$.

$$\varrho\Big(F(k_r, \gamma), F\big(k_r - N_j(\beta) - N_1(\beta) - j - 1, \beta\big)\Big) < \varepsilon(\beta)/j < \varepsilon(\beta)/m_0 < \delta_1/2$$

and

$$\varrho\big(F(k_r, \gamma), F(k_w, \beta)\big) < \delta_1/2,$$

therefore,

$$\varrho\Big(F(k_w, \beta), F\big(k_r - N_j(\beta) - N_1(\beta) - j - 1, \beta\big)\Big) < \delta_1$$

which implies that

$$k_r - N_j(\beta) - N_1(\beta) - j - 1 = k_w.$$

By combining this equality with the equality $N_{j+1}(\beta) = N_j(\beta) + 2N_1(\beta) + 2j + 1$ and the inequality $|k_w| + |x| + 2 < m_0 < j$, we obtain the following

$$N_j(\beta) < k_r - |x| - 2 < k_r + |x| + 2 < N_{j+1}(\beta).$$

Let i denote the integer such that $i \leqslant x + y_w < i+1$. Both $|i|$ and $|i+1|$ are less than $|x|+2$ and, therefore, both $k_r + i$ and $k_r + i + 1$ are between $N_j(\beta)$ and $N_{j+1}(\beta)$. This gives us the inequality

$$\varrho\Big(F(k_r + i, \gamma), F\big(k_r + i - N_j(\beta) - N_1(\beta) - j - 1, \beta\big)\Big) < \varepsilon(\beta)/j$$

which says that

$$\varrho\Big(F\big(i, F(k_r, \gamma)\big), F\big(i, F(k_w, \beta)\big)\Big) < \varepsilon(\beta)/j < e/(4d+4).$$

Similarily,

$$\varrho\Big(F\big(i+1, F(k_r, \gamma)\big), F\big(i+1, F(k_w, \beta)\big)\Big) < e/(4d+4).$$

Therefore,

$$\varrho\Big(F\big(x+y_w, F(k_r, \gamma)\big), F\big(x+y_w, F(k_w, \beta)\big)\Big) < e/(4d+4).$$

But this says that

$$\varrho\big(F(x+y_w+k_r, \gamma), F(x, w)\big) < e/(4d+4).$$

But since $|y_w - y_r| < e/(4d+4)$ and $|x - t| < e/(4d+4)$ it follows that

$$|(x+y_w+k_r) - (t+y_r+k_r)| < 2e/(4d+4)$$

and since the edge length of C is less than $1/2$, we infer that

$$\varrho\big(F(x+y_w+k_r, \gamma), F(t+y_r+k_r, \gamma)\big) < 2e/(4d+4).$$

Combining this with $\varrho\big(F(x+y_w+k_r, \gamma), F(x, w)\big) < e/(4d+4)$ we infer that

$$\varrho\big(F(t, r), F(x, w)\big) < 3e/(4d+4).$$

This completes the proof that statement (A) is true for $N = 0$.

There is a positive integer $m_1 > m_0$ such that $\varepsilon(\beta)/m_1 < \Delta_0/(4d+4)$. There is a positive number $\delta_7 < \delta_0$ such that

$$R\big(F([-m_1, m_1], \beta), \delta_7\big) \cap R\big(F\big((-\infty, N_{m_1}(\beta)], \gamma\big), \delta_7\big) = \emptyset.$$

There is a positive integer $m_2 > m_1$ such that $\varepsilon(\beta)/m_2 < \delta_7$. There is a positive number δ_8 such that

$$R\big(\Psi(\beta), \delta_8\big) \cap \Big(\bigcup_{i=1}^{m_2} R\big(\Psi(\beta_i), \varepsilon(\beta_i)\big)\Big) = \emptyset.$$

Let $\Delta_1 = \min\big(\Delta_0/(4d+4), \delta_7, \delta_8\big)$. Suppose that q' is a point of $S_2 \cap X_{h+1} \cap R(w, \Delta_1)$ and τ' is a number in $(x-\Delta_1, x+\Delta_1)$. There is a positive integer n such that $q' \in \Psi(\beta_n)$. Since $\Delta_1 < \delta_8$, we infer that $n > m_2$ and, therefore,

$$\varrho(a, a_n) < \delta_7 \quad \text{and} \quad \varrho\big(F\big(-N_n(\beta), \beta_n\big), F\big(-N_n(\beta), \gamma\big)\big) < \delta_7.$$

Therefore,

$$F\big((-\infty, -N_n(\beta)], \beta_n\big) \subset R\big(F\big((-\infty, N_{m_1}(\beta)], \gamma\big), \delta_7\big).$$

But $R(w, \Delta_1) \subset R\big(F([-m_1, m_1], \beta), \delta_7\big)$. These two facts along with the definition of δ_7 imply that

$$F\big((-\infty, -N_n(\beta)], \beta_n\big) \cap R(w, \Delta_1) = \emptyset.$$

Therefore, $t_{q'} > -N_n(\beta)$.

Suppose that $t_{q'} \geqslant N_n(\beta)$. Then q' is in $\big[F\big(N_n(\beta), \beta_n\big), a_n\big]$. Notice from the definition of the trajectory of γ that

$$\varrho\big(F\big(N_n(\beta), \gamma\big), F\big(N_n(\beta) - N_{n-1}(\beta) - N_1(\beta) - (n-1) - 1, \beta\big)\big) < \varepsilon(\beta)/(n-1).$$

So,

$$\varrho\big(F\big(N_n(\beta), \gamma\big), F\big(N_1(\beta) + n - 1, \beta\big)\big) < \varepsilon(\beta)/(n-1) < \varepsilon(\beta)/m_0 < \delta_0.$$

Combining this inequality with the inequalities

$$\varrho\big(F\big(N_1(\beta) + n - 1, \beta\big), a\big) < \varrho\big(F(m_0, \beta), a\big) < \delta_0$$

and

$$\varrho\big(F\big(N_n(\beta), \gamma\big), F\big(N_n(\beta), \beta_n\big)\big) < \varepsilon(\beta_{n-1}) < \delta_0,$$

we get that

$$\varrho\big(a, F\big(N_n(\beta), \beta_n\big)\big) < 3\delta_0.$$

But $\varrho(a_n, a) < \delta_0$, therefore, both $F\big(N_n(\beta), \beta_n\big)$ and a_n are in $R(a, 3\delta_0)$ and q' is in $\big[F\big(N_n(\beta), \beta_n\big), a_n\big]$. Therefore, $\varrho(q', a) < 3\delta_0$. But $\varrho(q', w) < \delta_0$, so $\varrho(a, w) < 4\delta_0$ which contradicts the definition of δ_0. Therefore, $-N_n(\beta) < t_{q'} < N_n(\beta)$, and so

$$\varrho\big(F(t_{q'}, \beta_n), F(t_{q'}, \gamma)\big) < \varepsilon(\beta_{n-1}) \leqslant \varepsilon(\beta_{m_1}) < \Delta_0/(4d+4).$$

But $\varrho(F(t_{q'}, \beta_n), w) < \Delta_1 \leqslant \Delta_0/(4d+4)$ and, therefore, $F(t_{q'}, \gamma) \in R(w, \Delta_0)$.

As we have seen, there is an integer $j' > m_0$ such that $N_{j'}(\beta) < k_{q'} - |x| - 2 < k_{q'} + |x| + 2 < N_{j'+1}(\beta)$. Therefore, there is no integer i such that $N_i(\beta)$ is between $t_{q'} - |x| - 1$ and $t_{q'} + |x| + 1$. Therefore, $t_{q'} + \tau' < N_n(\beta)$.

Let μ be the integer such that $\mu \leqslant t_{q'} + \tau' < \mu + 1$. Since $-N_n(\beta) < \mu < \mu + 1 \leqslant N_n(\beta)$, we infer that

$$\varrho(F(\mu, \beta_n), F(\mu, \gamma)) < \varepsilon(\beta_{n-1}) < \varepsilon(\beta)/m_1 < \Delta_0/(4d+4)$$

and

$$\varrho(F(\mu+1, \beta_n), F(\mu+1, \gamma)) < \Delta_0/(4d+4).$$

Therefore,

$$\varrho(F(t_{q'} + \tau', \beta_n), F(t_{q'} + \tau', \gamma)) < \Delta_0/(4d+4) < e/(4d+4).$$

Since $F(t_{q'}, \gamma) \in R(w, \Delta_0)$, we infer that

$$\varrho(F(t_{q'} + \tau', \gamma), F(x, w)) < 3e/(4d+4).$$

Therefore,

$$\varrho(F(\tau', q'), F(x, w)) < 4e/(4d+4).$$

There is a positive integer $m_3 > m_2$ such that $\varepsilon(\beta)/m_3 < \Delta_1/4$. There is a positive number $\delta_9 < \Delta_1/2$ such that

$$R(\Psi(\beta), \delta_9) \cap \Big(\bigcup_{i=1}^{m_3} \big(R(\Psi(\beta_i), \varepsilon(\beta_i))\big)\Big) = \emptyset.$$

Let $\Delta_1' = \delta_9$. Suppose that r' is a point of $X_{h+1} \cap S_3 \cap R(w, \Delta_1')$. There is a positive integer j such that $r' = F(t_{r'}, \gamma_j)$. Since $r' \in R(w, \delta_9)$, it follows that $j > m_3$. Suppose that $t_{r'} \leqslant N_1(\beta_j)$. The inequalities

$$\varrho(F(-N_1(\beta_j), \beta_j), a_j) < \varepsilon(\beta_j) < \varepsilon(\beta)/m_2 < \delta_7 < \delta_0$$

and

$$\varrho(F(-N_1(\beta_j), \beta_j), F(-N_1(\beta_j), \gamma_j)) < \varepsilon(\beta_j) < \delta_0$$

imply that

$$\varrho(F(-N_1(\beta_j), \gamma_j), a_j) < 2\delta_0.$$

But since $t_{r'} \leqslant -N_1(\beta_j)$, we infer that $r' \in [a_j, F(-N_1(\beta_j), \gamma_j)]$ and, therefore, $\varrho(a_j, r') < 2\delta_0$. But $\varrho(a_j, a) < \varepsilon(\beta_{j-1}) < \delta_0$ and $\varrho(r', w) < \Delta_1' < \delta_0$, therefore, $\varrho(w, a) < 4\delta_0$. But this contradicts the definition of δ_0, so we infer that $t_{r'} > -N_1(\beta_j)$.

Either $t_{r'} \leqslant N_1(\beta_j)$ or else there is a positive integer i such that $N_i(\beta_j) < t_{r'} \leqslant N_{i+1}(\beta_j)$. In the first case,

$$-N_1(\beta_j) < k_{r'} - |x| - 2 < k_{r'} + |x| + 2 < N_1(\beta_j).$$

In the second case, there is a positive integer i such that

$$N_i(\beta_j) < k_{r'} - |x| - 2 < k_{r'} + |x| + 2 < N_{i+1}(\beta_j).$$

In the first case, let

$$z = F(t_{r'}, \beta_j).$$

In the second case, let

$$z = F\big(t_{r'} - N_i(\beta_j) - N_1(\beta_j) - i - 1, \beta_j\big).$$

In either case, notice that

$$\varrho(z, r') < \varepsilon(\beta_j) < \varepsilon(\beta)/m_3 < \Delta_1/2$$

and

$$\varrho\big(F(\tau', z), F(\tau', r')\big) < e/(4d+4).$$

But since $\varrho(r', w) < \Delta'_1 < \Delta_1/2$, it follows that $\varrho(z, w) < \Delta_1$ and, therefore, $\varrho\big(F(\tau', z), F(x, w)\big) < 4e/(4d+4)$. Therefore,

$$\varrho\big(F(\tau', r'), F(x, w)\big) < 5e/(4d+4).$$

This completes the proof that statement (A) holds for $N = 1$.

Suppose that N is a positive integer such that $h + N \leqslant d$ and statement (A) holds for each non-negative integer n less than N. There is a positive integer $m_4 > m_3$ such that $\varepsilon(\beta)/m_4 < \Delta'_{N-1}/2$. There is a positive number $\delta_{10} < \delta_9/2$ such that

$$R\big(\Psi(\beta), \delta_{10}\big) \cap \Big(\bigcup_{i=1}^{m_4} \big(R\big(\Psi(\beta_i), \varepsilon(\beta_i)\big)\big)\Big) = \emptyset.$$

Let $\Delta_N = \min(\delta_{10}, \Delta'_{N-1}/2)$. Suppose that q'' is a point of $S_2 \cap X_{h+N} \cap R(w, \Delta_N)$ and τ'' is a number in $(x - \Delta_N, x + \Delta_N)$. There is a building block E' of Z_{h+N-1} (where α' denotes the type (1) point of E', β' — the type (2) determining point of E', γ' — the type (3) determining point of E', $U'_1, U'_2, U'_3, \ldots$ — the $\big(\alpha', \beta', \gamma', \varepsilon(\beta'), N_1(\beta'), N_2(\beta'), \ldots\big)$ sequence of Z_{h+N} and, for each positive integer i, α'_i — the type (1) point of U'_i, β'_i — the type (2) determining point of U'_i, and γ'_i — the type (3) determining point of U'_i) and a positive integer n such that $q'' \in \Psi(\beta'_n)$.

Notice that $-N_n(\beta') < k_{q''} - |x| - 2 < k_{q''} + |x| + 2 < N_n(\beta')$. Therefore,

$$\varrho\big(F(t_{q''}, \gamma'), F(T_{q''}, \beta'_n)\big) < \varepsilon(\beta') < \varepsilon(\beta)/m_4 < \Delta'_{N-1}/2$$

and

$$\varrho\big(F(t_{q''} + \tau'', \gamma'), F(t_{q''} + \tau'', \beta'_n)\big) < \Delta'_{N-1}/2 < e/(4d+4).$$

But since $\varrho\big(F(t_{q''}, \beta'_n), w\big) < \Delta'_{N-1}/2$, it follows that $\varrho\big(F(t_{q''}, \gamma'), w\big) < \Delta'_{N-1}$ and, therefore,

$$\varrho\big(F(t_{q''} + \tau'', \gamma'), F(x, w)\big) < 4Ne/(4d+4).$$

Therefore, we infer that

$$\varrho\big(F(t_{q''}+\tau'', \beta'_n), F(x, w)\big) < (4Ne+1)/(4d+4)$$

which says that

$$\varrho\big(F(\tau'', q''), F(x, w)\big) < (4Ne+1)/(4d+4).$$

There is a positive integer m_5 such that $\varepsilon(\beta)/m_5 < \Delta_N/2$. There is a positive number δ_{11} such that

$$R\big(\Psi(\beta), \delta_{11}\big) \cap \Big(\bigcup_{i=1}^{m_5}\big(R(\Psi(\beta_i), \varepsilon(\beta_i))\big)\Big) = \emptyset.$$

Let $\Delta'_N = \min(\delta_{11}, \Delta_N/2)$. Suppose that r'' is a point of $S_3 \cap X_{h+N} \cap \cap R(w, \Delta'_N)$. There is a building block E'' of Z_{h+N} (where a'' denotes the type (1) point of E'', β'' — the type (2) determining point of E'', and γ'' — the type (3) determining point of E'') such that $r'' \in \Psi(\gamma'')$.

Either $-N_1(\beta'') < k_{r''} - |x| - 2 < k_{r''} + |x| + 2 < N_1(\beta'')$ or else there is a positive integer i such that $N_i(\beta'') < k_{r''} - |x| - 2 < k_{r''} + |x| + 2 < N_{i+1}(\beta'')$.

In the first case, let

$$z' = F(t_{r''}, \beta'').$$

In the second case, let

$$z' = F\big(t_{r''} - N_i(\beta'') - N_1(\beta'') - i - 1, \beta''\big).$$

In either case,

$$\varrho(z', r'') < \varepsilon(\beta'') < \varepsilon(\beta)/m_5 < \Delta_N/2$$

and

$$\varrho\big(F(\tau'', z'), F(\tau'', r'')\big) < e/(4d+4).$$

But since $\varrho(r'', w) < \Delta_N/2$, we infer that $\varrho(z', w) < \Delta_N$. Therefore,

$$\varrho\big(F(\tau'', z'), F(x, w)\big) < (4Ne+1)/(4d+4)$$

and

$$\varrho\big(F(\tau'', r''), F(x, w)\big) < (4Ne+2)/(4d+4).$$

This completes the proof of continuity for case (1). Notice that in cases (2)-(5), F can be shown to be continuous at (x, w) by using arguments which very closely parallel the argument for continuity in case (1).

Consider the following related questions which are still open:

Is there a dynamical system G defined on all of E^3 such that $B(G) = \omega$ and $M(G) = 3$? In particular, can the F of this paper be extended to such a G? Could one find a differential equation corresponding to such a G? (See [10].) Is there a dynamical system H defined on a closed subset of E^3 such that $B(H) > \omega$ and $M(H) = 3$? **(P 825)**

REFERENCES

[1] G. D. Birkhoff, *Über gewisse Zentralbewegungen dynamischer Systeme*, Nachrichten der Akademie der Wissenschaften in Göttingen (1926), p. 81-92.
[2] — and P. A. Smith, *Structure analysis of surface transformations*, Journal de Mathématiques Pures et Appliquées (9) 7 (1928), p. 345-379.
[3] A. G. Maier, *Sur un problème de Birkhoff*, Comptes Rendus de l'Académie des Sciences de l'URSS (NS) 55 (1947), p. 473-475.
[4] — *Sur les trajectoires dans l'espace à trois dimensions*, ibidem 55 (1947), p. 579--581.
[5] А. Г. Майер, *О порядковом числе центральных траекторий*, Доклады Академии Наук СССР 59 (1948), p. 1395-1396.
[6] — *О центральных траекториях и проблеме Биркгоффа*, Математический Сборник 26 (68) (1950), p. 266-290.
[7] V. V. Nemyckii, *Topological problems of the theory of dynamical systems*, American Mathematical Society Translation 103 (1954).
[8] — and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Mathematical Series 22 (1960).
[9] C. S. Reed, *Concerning two methods of defining the center of a dynamical system*, *Topological dynamics* edited by Joseph Auslander, Walter H. Gottschalk and W. A. Benjamin, 1968, p. 391-399.
[10] G. Stephen Jones and James A. Yorke, *The existence and nonexistence of critical points in bounded flows*, Journal of Differential Equations 6 (1969), p. 238-246.

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