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LATTICE OF BOREL STRUCTURES

BY

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1. Summary. In this paper we show that the lattice of Borel structures on a set is complemented iff the set is countable. This incidentally answers that the minimal weak complements in the sense of D. Basu need not exist. Though some special sub-algebras of the Borel algebra of I, the unit interval, are shown to possess minimal weak complements the author could not characterize all such subalgebras. Finally we conclude with characterizing all fixed maximal ideals in this lattice. It should be noted that this paper is only a start of the study of the deep properties of this lattice and consequently much is yet to be done.

2. Preliminaries. For any non-empty set X, L_X denotes the lattice of all σ-algebras on X. A σ-algebra of subsets of X is also referred to as a Borel structure on X. A σ-algebra is said to be *separable* if it is countably generated and contains all singleton sets. To avoid certain trivialities we make the blanket assumption that X has more than two elements. For $\sigma, \sigma' \in L_X$ put $\sigma \leqslant \sigma'$ iff $\sigma \subset \sigma'$. Then L_X is a lattice and one can see that $\sigma \vee \sigma'$ is the smallest σ-algebra generated by σ and σ', whereas $\sigma \wedge \sigma'$ is the set-theoretic intersection of σ and σ'. In fact, L_X is a complete lattice possessing null element and unit element, viz. $(\emptyset, X)$ and (Power set of X) $= C_X$, respectively. These will be denoted by 0 and 1 respectively. One can also observe that L_X is not distributive. We recall a few definitions from lattice theory.

Let L be a lattice with 0 and 1. Let $a, b \in L$. Say that a' is a *complement of a relative to b* if $a \vee a' = b$ and $a \wedge a' = 0$. Say that a' is *weak complement of a relative to b* if $a \vee a' = b$. If $b = 1$, complement relative to b will be written as complement. Similar remark applies for weak complements, abbreviated by w.c. Clearly a complement is a w.c. though not conversely. Of course, neither complements nor w.c. need be unique. A w.c. is said to be *minimal* if no element smaller than it is again a w.c. A subset A of L is said to be an *ideal* if $a, b \in A$ implies $a \vee b \in A$ and $a \in A$, $b \leqslant a$ imply $b \in A$. An ideal is *proper* if it is not L itself. A proper ideal is *maximal* if no ideal bigger than it is proper. An ideal A is *principal* or *fixed* if there is an element $a_0 \in L$ such that $a \in A$ iff $a \leqslant a_0$. Maximal ideals which are not fixed are said to be *free*.

3. Complementation in L_X.

THEOREM 1. *L_X is complemented iff X is countable. In case X is countable complement of no element other than* 0 *and* 1 *is unique.*

Proof. Let us first assume that X is countable and $\sigma \in L_X$. It is clear that σ has atoms $(A_n, n \geqslant 1)$, either finitely many or infinitely many; and that a subset of X is in σ iff it is the union of some atoms. Let A_0 be any choice set for $(A_n; n \geqslant 1)$. Then the σ-algebra generated by (A_0, singleton sets of A_0^c) will work as a complement of σ. Observe that if σ is neither 0 nor 1 then there is more than one such choice set.

We now turn to the proof of the other part of the theorem. First note that if $X \subset Y$ and L_X is not complemented then neither is L_Y. For, let σ be an element of L_X without a complement. Put σ_1 as the structure on Y generated by σ and all subsets of $Y - X$. If σ_1 has a complement in L_Y, then it follows that its restriction to X will be a complement of σ in L_X. So it suffices to show that L_X is not complemented when the cardinality of X is $\aleph_1$, the first uncountable cardinal.

We now assume that the cardinality of X is $\aleph_1$. Let σ^* be any separable structure to X. Let σ_0 be the countable-cocountable structure to X contained in σ^*. We first note that σ_0 has no complement relative to σ^*. For, if it did have, say σ_0', then by standard techniques one can assume that σ_0' is countably generated and then take any atom A of σ_0'. Since A is not in σ_0, A will be uncountable and the easily verifiable identity $\sigma^*|A = \sigma_0|A$ (here $|$ denotes restriction) gives us a contradiction.

We now claim that σ_0 has no complement in L_X. For, if it did have, say σ_1, then by straightforward calculation one observes that $\sigma_0' = \sigma_1 \cap \sigma^*$ will be a complement of σ_0 relative to σ^*, which cannot exist by the conclusion of the previous part. The only fact that one needs in this verification is that for any σ in L_X we have $\sigma \vee \sigma_0 = [Z; Z \Delta A$ is countable for some A in $\sigma]$. This completes the proof of the theorem.

We now state a theorem which enables us to conclude a statement stronger than the above theorem.

THEOREM 2. *Let $\sigma < \sigma^*$ be in L_X. If σ' is a minimal w.c. of σ relative to σ^*, then σ' is also a complement relative to σ^*. The converse need not be true.*

Proof. For the proof of the first sentence we have only to show that $\sigma \wedge \sigma' = 0$. On the contrary suppose that there is a non-empty proper subset, say A, in $\sigma \cap \sigma'$. Fix two points x and y in A and A^c respectively and put

$$\sigma'' = \big[B \in \sigma';\ B \supset [x, y] \text{ or } B^c \supset [x, y]\big].$$

Then $A \notin \sigma'' \subset \sigma'$. We show that σ'' is a w.c. of σ relative to σ^* to contradict the minimality of σ', which then proves our assertion. Since $\sigma \vee \sigma'$

$= \sigma^*$, to do this, it suffices to show that $\sigma \vee \sigma'' \supset \sigma'$. Let $B \epsilon \sigma'$. If both x and y are in B or B^c, then B is in σ'' by the definition. Assume that $x \epsilon B$, $y \epsilon B^c$. Observe that both $B \cap A^c$ and $(B \cap A) \cup A^c$ are in σ'' and since $A \epsilon \sigma$ it follows that $[(B \cap A) \cup A^c] \cap A$ is in $\sigma \vee \sigma''$ and $B \cap A^c$ is also in $\sigma \vee \sigma''$. Since the union of these two latter sets is B, it follows that $B \epsilon \sigma \vee \sigma''$ as desired.

To prove the last sentence of the theorem we give an example. Take X as the unit square $I \times I$, σ^* as its Borel algebra, σ and σ'' to be the structures of vertical Borel cylinders and horizontal Borel cylinders respectively. Put σ' to be σ'' enlarged by adding a singleton set. It is clear that both σ' and σ'' are complements of σ relative to σ^* and $\sigma'' \subset \sigma'$, so that σ' cannot be a minimal w.c.

From theorem 2 combined with theorem 1 one concludes that in L_X even minimal w.c. need not exist. This answers a question raised by Basu [1]. Though our theorem theoretically answers the complementation problem in L_X, still many interesting problems remain. To mention one, what are those countably generated substructures of $\boldsymbol{B}$ on I that have complements relative to $\boldsymbol{B}$ (**P 741**). Trivially any countably generated $\boldsymbol{B}_0$ contained in $\boldsymbol{B}$ that has only countably many atoms has a relative complement. We do not have any satisfactory answer to the above question. We exhibit in the theorem below a class of such structures. To do this we feel it convenient to work with 2^ω rather than I. We denote by $\boldsymbol{B}$ the Borel structure on 2^ω and since we do not bring in I now there is no fear of confusion.

THEOREM 3. *Let g be a continuous function on 2^ω into a Polish space such that for all v in V ($=$ the range of g), $g^{-1}(v)$ is homeomorphic to 2^ω. Let $\boldsymbol{B}_g \subset \boldsymbol{B}$ be the subalgebra induced by g. Then $\boldsymbol{B}_g$ has a minimal w.c. relative to $\boldsymbol{B}$.*

Proof. By a trivial modification of a lemma of Purves [2] there exist homeomorphisms Ψ_v on $g^{-1}(v)$ to 2^ω such that the map $s\colon 2^\omega \to 2^\omega$ defined by $s(x) = \Psi_v(x)$ if x is in $g^{-1}(v)$ is a Borel map. Consequently, the map $f\colon 2^\omega \to V \times 2^\omega$ given by $f(x) = (g(x), s(x))$ is a Borel isomorphism which transports g to the projection to the V-axis in $V \times 2^\omega$. In other words, $\boldsymbol{B}_g$ is transported to the σ-algebra of vertical cylinders in $V \times 2^\omega$ for which the σ-algebra of horizontal cylinders will work as a minimal weak w.c.

4. Ideals in L_X. Let us say that a Borel structure σ on X is an ultrastructure if $\sigma \neq C_X$ and $\sigma' > \sigma$ implies $\sigma' = C_X$. With each σ in L_X associate the ideal A_σ generated by it. Observe that A_σ is maximal iff σ is an ultrastructure. Consequently, characterization of fixed maximal ideals of L_X reduces to finding all ultrastructures on X. This is done in the following theorem. Recall that a σ-ideal on X means a σ-ideal in the lattice of all subsets of X. A σ-ideal is ultra if it is proper and maximal.

THEOREM 4. *Let $\boldsymbol{I}$ and $\boldsymbol{J}$ be distinct maximal σ-ideals on X. Put*

$$\sigma(\boldsymbol{I}, \boldsymbol{J}) = [A; A \text{ or } A^c \text{ is in } \boldsymbol{I} \cap \boldsymbol{J}].$$

Then $\sigma(\boldsymbol{I}, \boldsymbol{J})$ is an ultrastructure on X. Conversely, every ultrastructure on X is of this form.

Proof. Clearly, $\sigma(\boldsymbol{I}, \boldsymbol{J})$ is a σ-algebra on X and since $\boldsymbol{I}$ and $\boldsymbol{J}$ are distinct, this is different from $\boldsymbol{C}_X$. Let σ' be any structure properly containing $\sigma(\boldsymbol{I}, \boldsymbol{J})$. Let A belong to σ' but not to $\sigma(\boldsymbol{I}, \boldsymbol{J})$. For specificness, let $A \in \boldsymbol{I}$ and $A \notin \boldsymbol{J}$, so that $A^c \in \boldsymbol{J}$. To show that $\sigma' = \boldsymbol{C}_X$, it suffices to show that $\boldsymbol{I} \subset \sigma'$. If $B \in \boldsymbol{I}$, then $A^c \cap B$ is in both $\boldsymbol{I}$ and $\boldsymbol{J}$ and so is in σ'. Moreover, $A \cap B = [(A \cap B) \cup A^c] \cap A$ is also in σ'. These two statements show that $B \in \sigma'$.

For the converse, let σ_0 be an ultrastructure on X. Put $\boldsymbol{H} = (E \subset X;$ all subsets of E are in $\sigma_0)$. Then $\boldsymbol{H}$ is a σ-ideal of subsets of X which is not maximal since $\sigma_0 \neq \boldsymbol{C}_X$. Consequently we can find $A \subset X$ such that $\boldsymbol{I} =$ Ideal generated by $\boldsymbol{H}$ and A, and $\boldsymbol{J} =$ Ideal generated by $\boldsymbol{H}$ and A^c are both proper. By the maximality of σ_0 it follows that $\sigma_0 = \sigma(\boldsymbol{H}, \boldsymbol{H})$, which is contained in both $\sigma(\boldsymbol{I}, \boldsymbol{I})$ and $\sigma(\boldsymbol{J}, \boldsymbol{J})$. This contradicts the maximality of σ_0 unless both $\boldsymbol{I}$ and $\boldsymbol{J}$ are maximal. Clearly, these two are distinct and, being finite extensions of $\boldsymbol{H}$, they are also σ-ideals. Finally, note that $\sigma(\boldsymbol{I}, \boldsymbol{J})$ is an ultrastructure from the previous part and σ_0 is an ultrastructure contained in it. This proves that $\sigma_0 = \sigma(\boldsymbol{I}, \boldsymbol{J})$. This completes the proof of the theorem.

The author had originally obtained a theorem weaker than Theorem 4 by similar arguments, and the present formulation of Theorem 4 is due to Prof. C. Ryll-Nardzewski.

Observe that if X is such that countably additive 0-1 measure cannot be defined on its power set, then every maximal σ-ideal is fixed and, consequently, every ultrastructure is of the form $[B; B$ or B^c contains both x and $y]$ for some two distinct points x and y. If X is finite, then L_X is finite and hence has no free maximal ideals, while if X is infinite there always exist free maximals ideals. We do not yet have any way of characterizing them.

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