COLLOQUIUM MATHEMATICUM

VOL. XLVIII 1984 FASC. 2

PRE-ADJUNCTIONS AND LAMBDA-ALGEBRAIC THEORIES

BY

ANTONI WIWEGER (WARSZAWA)

0. Introduction. Various generalizations of the notion of adjointness were considered by many authors. A detailed discussion of these generalizations is given by Börger and Tholen [2]. The main aim of this paper is to point out that generalized adjunctions may serve as a convenient language for describing lambda-algebraic theories in the sense of Obtułowicz (cf. [7], [9]). However, the variants of the notion of generalized adjointness listed in [2] are not suitable for this purpose, and our considerations will be based on the concept of a "pre-adjunction" defined in Section 2. Each functor which is pseudo-left-adjoint in the sense of [2] has a pre-adjunction but not conversely, because a functor having a pre-adjunction need not be weakly left adjoint. It is shown in Section 3 that the Cartesian product of sets in the category of sets and partial functions provides an example of a pre-adjunction with no "weak universality" property. The main example of a pre-adjunction which is not a weak adjunction is described in Section 4; in that section the notion of an algebraic λ-pre-adjunction of an endofunctor in an algebraic theory is introduced, and the connection of this notion with the categorical approach to the syntax of the type free lambda calculus is discussed.

The author is obliged to Dr. Adam Obtułowicz and Professor Manfred B. Wischnewsky for valuable comments.

1. Preliminaries.

1.1. We shall use the following notation:

$?$, $?'$, $?_1$, $?_2$ are symbols of variables.

N is the set of all non-negative integers $\{0, 1, 2, \ldots\}$; by *non-negative integers* we mean the finite numbers in the sense of von Neumann, i.e. $0 = \emptyset$, $n+1 = \{0, 1, \ldots, n\}$.

N^+ is the set of all positive integers $\{1, 2, \ldots\}$.

If f is a function with the domain X and $Y \subset X$, then $f|Y$ denotes the restriction of f to Y. If S and A are sets, then by a *family* $(a_s : s \in S)$ of elements of A we mean the function $s \mapsto a_s$ from S into A.

1.2. For all unexplained terms concerning category theory we refer the reader to [6]. If $\mathfrak{X}$ is a category, then Ob $\mathfrak{X}$ denotes the class of all objects of $\mathfrak{X}$. If X, $Y \in \mathrm{Ob}\,\mathfrak{X}$, then $\mathfrak{X}(X, Y)$ is the set of all arrows of $\mathfrak{X}$ with domain X and codomain Y. The composition of arrows $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $g \circ f\colon X \to Z$. The opposite category of $\mathfrak{X}$ is denoted by $\mathfrak{X}^{\mathrm{op}}$, and the canonical contravariant functor from $\mathfrak{X}$ into $\mathfrak{X}^{\mathrm{op}}$ is denoted by $\mathrm{Op}\colon \mathfrak{X} \to \mathfrak{X}^{\mathrm{op}}$. By a *functor* we always mean a covariant functor if not explicitly stated otherwise.

Set denotes the category of sets, and **N** denotes the full subcategory of Set whose objects are all non-negative integers.

If $(f_i\colon X \to Y_i \,\vdots\, i \in n)$ is a family of arrows in a category $\mathfrak{X}$ and Y is a product $Y_0 \times \ldots \times Y_{n-1}$ with product projections $\mathrm{pr}^i\colon Y \to Y_i$, then $\langle f_0, f_1, \ldots, f_{n-1}\rangle$ will denote the unique arrow $f\colon X \to Y$ of $\mathfrak{X}$ such that $\mathrm{pr}^i \circ f = f_i$ for all $i \in n$. If, in particular, $(f_i\colon X \to Y_i \,\vdots\, i \in n)$ is a family of arrows in Set, then $\langle f_0, f_1, \ldots, f_{n-1}\rangle$ denotes the function from X into $Y_0 \times \ldots \times Y_{n-1}$ defined by

$$\langle f_0, f_1, \ldots, f_{n-1}\rangle(x) = \big(f_0(x), f_1(x), \ldots, f_{n-1}(x)\big)$$

for all $x \in X$.

2. Pre-adjunctions.

2.1. Let $\mathfrak{X}$ and $\mathfrak{A}$ be categories and let F be a functor from $\mathfrak{X}$ into $\mathfrak{A}$. By a *right pre-adjunction* of F (shortly, a *pre-adjunction* of F) we mean any triple (G_0, φ, ψ), where G_0 is a function from Ob $\mathfrak{A}$ to Ob $\mathfrak{X}$, and φ, ψ are functions which assign to each pair of objects $X \in \mathrm{Ob}\,\mathfrak{X}$, $A \in \mathrm{Ob}\,\mathfrak{A}$ functions

$$\mathfrak{A}(FX, A) \underset{\psi_{X,A}}{\overset{\varphi_{X,A}}{\rightleftarrows}} \mathfrak{X}(X, G_0 A)$$

such that the following condition is satisfied:

(α_0) $\psi_{X,A}$ is *natural* in X, i.e. for each $A \in \mathrm{Ob}\,\mathfrak{A}$ the components $\psi_{X,A}$ for all X define a natural transformation of contravariant functors:

$$\mathfrak{X}(?, G_0 A) \xrightarrow{\;\cdot\;} \mathfrak{A}(F(?), A).$$

If (G_0, φ, ψ) is a pre-adjunction of F, then we say that G_0 is a *right pre-adjoint* (shortly, a *pre-adjoint*) of F. Note that a pre-adjoint of a functor is not a functor but a function assigning objects to objects. Given a functor F, there may exist many pre-adjoints of F. In Section 3.3 an example will be given which shows that a functor may have at the same time a pre-adjoint defined in a very natural way and an adjoint whose construction is more complicated.

It follows from the Yoneda lemma that the condition (α_0) is satisfied if and only if there is a function ε which assigns to each $A \in \mathrm{Ob}\,\mathfrak{A}$ an arrow

$\varepsilon_A\colon FG_0A \to A$ of $\mathfrak{A}$ such that

(1) $$\psi_{X,A}(g) = \varepsilon_A \circ F(g) \quad \text{for any arrow } g\colon X \to G_0A \text{ of } \mathfrak{X}.$$

The function ε is determined by ψ uniquely; in fact,

(2) $$\varepsilon_A = \psi_{G_0A,A}(\mathrm{id}_{G_0A}) \quad \text{for any } A \in \mathrm{Ob}\,\mathfrak{A}.$$

In other words, a pre-adjunction (G_0, φ, ψ) may be identified with a triple $(G_0, \varphi, \varepsilon)$, where ε and ψ satisfy (1) and (2).

2.2. We shall consider pre-adjunctions (G_0, φ, ψ) of F satisfying some of the following additional conditions:

(α_1) $\varphi_{X,A}$ is natural in X;

(α_2) $\psi_{X,A} \circ \varphi_{X,A} = \mathrm{id}_{\mathfrak{A}(FX,A)}$ for all $X \in \mathrm{Ob}\,\mathfrak{X}$, $A \in \mathrm{Ob}\,\mathfrak{A}$;

(α_3) $\varphi_{X,A} \circ \psi_{X,A} = \mathrm{id}_{\mathfrak{X}(X,G_0A)}$ for all $X \in \mathrm{Ob}\,\mathfrak{X}$, $A \in \mathrm{Ob}\,\mathfrak{A}$.

Let $i, j \in \{1, 2, 3\}$. We say that (G_0, φ, ψ) is an *α_i-pre-adjunction* if (G_0, φ, ψ) is a pre-adjunction satisfying the condition (α_i). We say that (G_0, φ, ψ) is an *α_{ij}-pre-adjunction* if (G_0, φ, ψ) is both an α_i-pre-adjunction and an α_j-pre-adjunction.

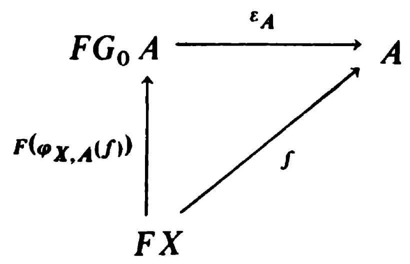
The condition (α_1) means that for each arrow $f\colon FX \to A$ of $\mathfrak{A}$ and each arrow $h\colon X' \to X$ of $\mathfrak{X}$ we have

$$\varphi_{X,A}(f) \circ h = \varphi_{X',A}(f \circ F(h)).$$

By (1) the condition (α_2) means that for each arrow $f\colon FX \to A$ of $\mathfrak{A}$ we have

(3) $$\varepsilon_A \circ F(\varphi_{X,A}(f)) = f,$$

i.e. the diagram



is commutative. Therefore, a functor $F\colon \mathfrak{X} \to \mathfrak{A}$ having a right α_2-pre-adjoint is the same as the weak left adjoint functor in the terminology of [2]. A functor $F\colon \mathfrak{X} \to \mathfrak{A}$ having a right α_{12}-pre-adjoint is the same as the pseudo-left-adjoint functor in the sense of [2].

By (1) the condition (α_3) asserts that for each arrow $g\colon X \to G_0A$ of $\mathfrak{X}$ we have

(4) $$\varphi_{X,A}(\varepsilon_A \circ F(g)) = g.$$

In other words, the condition (α_3) means that if $f\colon FX \to A$ is an arrow of $\mathfrak{A}$ and $g\colon X \to G_0 A$ is an arrow of $\mathfrak{X}$ such that the diagram

$$\begin{array}{ccc} FG_0 A & \xrightarrow{\ \varepsilon_A\ } & A \\ {\scriptstyle F(g)}\uparrow & \nearrow{\scriptstyle f} & \\ FX & & \end{array} \tag{5}$$

is commutative, then $g = \varphi_{X,A}(f)$.

The condition (α_2) implies that for a given $f\colon FX \to A$ there is at least one $g\colon X \to G_0 A$ making the diagram (5) commutative, whereas the condition (α_3) asserts that there is at most one such g. Börger and Tholen [2] consider only such generalizations of adjointness where the existence of g in (5) for any f is ensured. However, this requirement is too restrictive in some cases. Examples of pre-adjunctions which are not α_2-pre-adjunctions will be given in Sections 3 and 4. It follows from Theorem 4.6 below that in the categorical approach to the syntax of the type free lambda calculus the conditions (α_2) and (α_3) correspond to the well-known axioms of β-reduction and η-reduction, respectively (cf. [1] and [9]).

If $F\colon \mathfrak{X} \to \mathfrak{A}$, $G\colon \mathfrak{A} \to \mathfrak{X}$ are functors and (F, G, φ) is an adjunction from $\mathfrak{X}$ to $\mathfrak{A}$, then the triple (G_0, φ, ψ), where G_0 is the object function of G and $\psi_{X,A} = \varphi^{-1}_{X,A}$, is a pre-adjunction of F satisfying the conditions (α_1)–(α_3). The converse is also true in the following sense:

2.3. Proposition. *If (G_0, φ, ψ) is an α_{23}-pre-adjunction of $F\colon \mathfrak{X} \to \mathfrak{A}$, then there exists a unique functor $G\colon \mathfrak{A} \to \mathfrak{X}$ such that G_0 is the object function of G and (F, G, φ) is an adjunction from $\mathfrak{X}$ to $\mathfrak{A}$. The functor G is defined by the formula*

$$G(k) = \varphi_{G_0A,A'}(k \circ \varepsilon_A) \quad \textit{for any arrow } k\colon A \to A' \textit{ of } \mathfrak{A}. \tag{6}$$

This proposition is a simple reformulation of Theorem 2 (iv) in [6], p. 81. It follows from this proposition that every α_{23}-pre-adjunction is an α_1-pre-adjunction (one may easily check this fact straightforwardly).

It should be noted that equations (3) and (4) were used by Lambek [4] to describe adjunctions.

Two successive pre-adjunctions can be composed in the following sense:

2.4. Proposition. *If $p = (G_0, \varphi, \psi)$ is a pre-adjunction of $F\colon \mathfrak{X} \to \mathfrak{A}$ and $p' = (G'_0, \varphi', \psi')$ is a pre-adjunction of $F'\colon \mathfrak{A} \to \mathfrak{M}$, then the triple $p'' = (G''_0, \varphi'', \psi'')$, where*

$$G''_0 = G_0 \circ G'_0, \qquad \varphi''_{X,M} = \varphi_{X,G'_0M} \circ \varphi'_{FX,M}, \qquad \psi''_{X,M} = \psi'_{FX,M} \circ \psi_{X,G'_0M}$$

for all $X \in \mathrm{Ob}\,\mathfrak{X}$, $M \in \mathrm{Ob}\,\mathfrak{M}$, *is a pre-adjunction of* $F' \circ F$. *If* p *and* p' *are* α_i*-pre-adjunctions, then* p'' *is an* α_i*-pre-adjunction.*

The proof is straightforward.

Using this composition we may form a category Pre-adj whose objects are all (small) categories $\mathfrak{X}$, $\mathfrak{A}$, $\mathfrak{M}$, ... and whose arrows from $\mathfrak{X}$ to $\mathfrak{A}$ are all quadruples $(p, F, \mathfrak{X}, \mathfrak{A})$, where p is a pre-adjunction of $F\colon \mathfrak{X} \to \mathfrak{A}$. The category Pre-adj has subcategories α_i-Pre-adj and α_{ij}-Pre-adj whose objects are all objects of Pre-adj and whose arrows are all quadruples $(p, F, \mathfrak{X}, \mathfrak{A})$ such that p is an α_i-pre-adjunction (an α_{ij}-pre-adjunction, respectively) of $F\colon \mathfrak{X} \to \mathfrak{A}$. The category α_{23}-Pre-adj may be identified with the category Adj of adjunctions (cf. [6], p. 102).

2.5. In a similar way we may consider a *left pre-adjunction* (F_0, φ, ψ) of a functor $G\colon \mathfrak{A} \to \mathfrak{X}$. Here F_0 is a function from $\mathrm{Ob}\,\mathfrak{X}$ into $\mathrm{Ob}\,\mathfrak{A}$ and φ, ψ are functions which assign to each pair of objects $X \in \mathrm{Ob}\,\mathfrak{X}$, $A \in \mathrm{Ob}\,\mathfrak{A}$ functions

$$\mathfrak{A}(F_0 X, A) \underset{\psi_{X,A}}{\overset{\varphi_{X,A}}{\rightleftarrows}} \mathfrak{X}(X, GA)$$

such that the following condition is satisfied:

(α'_0) $\varphi_{X,A}$ is natural in A.

It is obvious that (F_0, φ, ψ) is a left pre-adjunction of $G\colon \mathfrak{A} \to \mathfrak{X}$ if and only if (F_0, ψ, φ) is a right pre-adjunction of the functor

$$\mathrm{Op} \circ G \circ \mathrm{Op}\colon \mathfrak{A}^{\mathrm{op}} \to \mathfrak{X}^{\mathrm{op}}.$$

We say that (F_0, φ, ψ) is a *left* α'_i*-pre-adjunction* (a *left* α'_{ij}*-pre-adjunction*) of G if (F_0, ψ, φ) is a right α_i-pre-adjunction (a right α_{ij}-pre-adjunction) of $\mathrm{Op} \circ G \circ \mathrm{Op}$.

The condition (α'_0) is equivalent to the existence of a function η which assigns to each $X \in \mathrm{Ob}\,\mathfrak{X}$ an arrow $\eta_X\colon X \to GF_0 X$ of $\mathfrak{X}$ such that

$$\varphi_{X,A}(f) = G(f) \circ \eta_X \quad \text{for any arrow } f\colon F_0 X \to A \text{ of } \mathfrak{A}.$$

The function η is defined by

$$\eta_X = \varphi_{X,F_0 X}(\mathrm{id}_{F_0 X}) \quad \text{for any } X \in \mathrm{Ob}\,\mathfrak{X}.$$

Let us consider the following conditions:

(α'_1) $\psi_{X,A}$ is natural in A;

(α'_2) $\varphi_{X,A} \circ \psi_{X,A} = \mathrm{id}_{\mathfrak{X}(X,GA)}$ for all $X \in \mathrm{Ob}\,\mathfrak{X}$, $A \in \mathrm{Ob}\,\mathfrak{A}$;

(α'_3) $\psi_{X,A} \circ \varphi_{X,A} = \mathrm{id}_{\mathfrak{A}(F_0 X,A)}$ for all $X \in \mathrm{Ob}\,\mathfrak{X}$, $A \in \mathrm{Ob}\,\mathfrak{A}$.

It is easy to see that a left pre-adjunction (F_0, φ, ψ) of G is a left α'_i-pre-adjunction of G if and only if (F_0, φ, ψ) satisfies the condition (α'_i).

The condition (α_1') means that for each arrow $g\colon X \to GA$ of $\mathfrak{X}$ and each arrow $k\colon A \to A'$ of $\mathfrak{A}$ we have

$$k \circ \psi_{X,A}(g) = \psi_{X,A'}(G(k) \circ g).$$

The condition (α_2') means that for each arrow $g\colon X \to GA$ of $\mathfrak{X}$ we have

$$G(\psi_{X,A}(g)) \circ \eta_X = g.$$

The condition (α_3') means that for each arrow $f\colon F_0 X \to A$ of $\mathfrak{A}$ we have

$$\psi_{X,A}(G(f) \circ \eta_X) = f.$$

3. Examples of pre-adjunctions.

3.1. *Constant morphisms.* Let $\mathfrak{X}$ be a category and let $\mathbf{1}$ be the category with the unique object 0 and the unique arrow id_0. Consider the unique functor $J\colon \mathfrak{X} \to \mathbf{1}$. Any pre-adjunction (G_0, φ, ψ) of J can be described as follows: the function G_0 assigns to the unique object 0 of $\mathbf{1}$ an object $C = G_0(0)$ of $\mathfrak{X}$; the function $\varphi_{X,0}$ assigns to the unique arrow id_0 of $\mathbf{1}$ an arrow $\gamma_X\colon X \to C$ of $\mathfrak{X}$; the function $\psi_{X,0}$ assigns to each arrow $g\colon X \to C$ of $\mathfrak{X}$ the arrow id_0 of $\mathbf{1}$. In other words, each pre-adjunction (G_0, φ, ψ) of J is completely determined by a function γ which assigns to each object X of $\mathfrak{X}$ an arrow $\varphi_{X,0}(0) = \gamma_X\colon X \to C$ of $\mathfrak{X}$ with a fixed codomain $C = G_0(0)$. For any such function γ the conditions (α_0) and (α_2) are automatically satisfied. The pre-adjunction determined by γ satisfies (α_1) if and only if

(7) $$\gamma_X \circ h = \gamma_{X'} \quad \text{for any arrow } h\colon X' \to X \text{ of } \mathfrak{X}.$$

The condition (7) implies that for each $X = \mathrm{Ob}\,\mathfrak{X}$ the arrow $\gamma_X\colon X \to C$ is a constant morphism in the sense of Herrlich and Strecker [3]. (If, in particular, $\mathfrak{X} = \mathrm{Set}$ and $\gamma_X\colon X \to C$ are functions satisfying (7), then there is an element c in C such that $\gamma_X(X) = c$ for any set X and any element x in X.) The pre-adjunction determined by γ satisfies (α_3) if and only if for each object X of $\mathfrak{X}$ there is exactly one arrow from X to C, i.e. C is a terminal object of $\mathfrak{X}$.

Dually, each left pre-adjunction (F_0, φ, ψ) of J is completely determined by a function δ which assigns to each object X of $\mathfrak{X}$ an arrow $\psi_{0,X}(0) = \delta_X\colon D \to X$ of $\mathfrak{X}$ with a fixed domain $D = F_0(0)$. The conditions (α_0') and (α_2') are automatically satisfied. If the left pre-adjunction determined by δ satisfies (α_1'), then for each X in $\mathrm{Ob}\,\mathfrak{X}$ the arrow $\delta_X\colon D \to X$ is a coconstant morphism in the sense of [3]. If the left pre-adjunction determined by δ satisfies (α_3'), then D is an initial object in $\mathfrak{X}$.

3.2. *Generalized products.* Let $\mathfrak{X}$ be a category and let $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ be the diagonal functor. A pre-adjunction (G_0, φ, ψ) of the functor Δ can be described as follows: the function G_0 assigns to each pair of objects

$Y \in \mathrm{Ob}\,\mathfrak{X}$, $Z \in \mathrm{Ob}\,\mathfrak{X}$ an object

$$Y \diamondsuit Z = G_0(Y, Z) \in \mathrm{Ob}\,\mathfrak{X};$$

the function $\varphi_{X,(Y,Z)}$ assigns to each pair of arrows

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

of $\mathfrak{X}$ an arrow $[f, g] = \varphi_{X,(Y,Z)}(f, g)$: $X \to Y \diamondsuit Z$ of $\mathfrak{X}$; the function $\psi_{X,(Y,Z)}$ assigns to each arrow g: $X \to Y \diamondsuit Z$ of $\mathfrak{X}$ a pair of arrows

$$Y \xleftarrow{\psi^1_{X,(Y,Z)}(g)} X \xrightarrow{\psi^2_{X,(Y,Z)}(g)} Z$$

of $\mathfrak{X}$. Condition (α_0) implies that there is a function ε which assigns to each pair of objects $Y \in \mathrm{Ob}\,\mathfrak{X}$, $Z \in \mathrm{Ob}\,\mathfrak{X}$ a pair of arrows

$$Y \xleftarrow{\varepsilon^1_{Y,Z}} Y \diamondsuit Z \xrightarrow{\varepsilon^2_{Y,Z}} Z$$

such that $\psi^i_{X,(Y,Z)}(g) = \varepsilon^i_{Y,Z} \circ g$ for any arrow g: $X \to Y \diamondsuit Z$ and $i = 1, 2$.

By a *generalized product* in the category $\mathfrak{X}$ we mean a collection of the functions

(8) $$?_1 \diamondsuit ?_2, \quad \varepsilon^1_{?_1,?_2}, \quad \varepsilon^2_{?_1,?_2}, \quad [?, ?']$$

defined as above by a pre-adjunction (G_0, φ, ψ) of the diagonal functor Δ: $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. We say that (8) is an *α_i-product* (*α_{ij}-product*) in $\mathfrak{X}$ if the pre-adjunction (G_0, φ, ψ) is an α_i-pre-adjunction (α_{ij}-pre-adjunction) of Δ. It is easy to see that the generalized product (8) is

(i) an α_1-product if and only if, for each pair of arrows $Y \xleftarrow{f} X \xrightarrow{g} Z$ with the same domain X and each arrow $X' \xrightarrow{h} X$,

$$[f, g] \circ h = [f \circ h, g \circ h];$$

(ii) an α_2-product if and only if, for each pair of arrows $Y \xleftarrow{f} X \xrightarrow{g} Z$ with the same domain X,

$$\varepsilon^1_{Y,Z} \circ [f, g] = f, \quad \varepsilon^2_{Y,Z} \circ [f, g] = g;$$

(iii) an α_3-product if and only if, for each arrow $X \xrightarrow{k} Y \diamondsuit Z$,

$$[\varepsilon^1_{Y,Z} \circ k, \varepsilon^2_{Y,Z} \circ k] = k.$$

An α_{23}-product is a categorical product; in this case the arrows $\varepsilon^1_{Y,Z}$ and $\varepsilon^2_{Y,Z}$ are usual product projections, and $[f, g] = \langle f, g \rangle$, where $\langle \cdot, \cdot \rangle$ is the symbol defined in Section 1.2.

Some applications of generalized products are given in [8].

3.3. *Generalized products in the category of partial functions.* Let Pfn be the category whose objects are all sets X, Y, ... and whose arrows from X to Y are all partial maps f: $X \multimap Y$; by a *partial map* f: $X \multimap Y$ we mean a triple (X, f, Y), where f is a function defined on a subset $\mathrm{dom}\, f$ of X with values in Y; the composition of partial maps is defined as the composition of relations.

Let $Y \times Z$ be the Cartesian product of sets and let

$$\mathrm{pr}^1_{Y,Z}\colon Y \times Z \to Y, \quad \mathrm{pr}^2_{Y,Z}\colon Y \times Z \to Z$$

be the canonical projections. Given a pair of partial maps

$$Y \overset{f}{\leftarrow\!\circ\!-} X \overset{g}{-\!\circ\!\rightarrow} Z,$$

we define the partial map

$$[f, g]\colon X -\!\circ\!\rightarrow Y \times Z$$

in the following way:

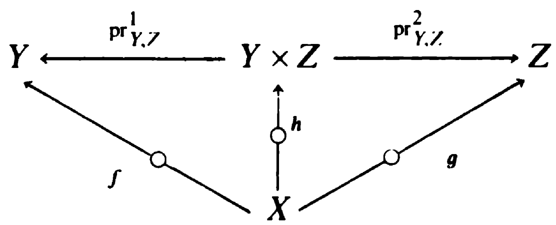
$$\operatorname{dom}[f, g] = \operatorname{dom} f \cap \operatorname{dom} g,$$

$$[f, g](x) = \big(f(x), g(x)\big) \quad \text{for } x \in \operatorname{dom}[f, g].$$

It is easy to verify that the functions

(9) $$?_1 \times ?_2, \quad \mathrm{pr}^1_{?_1, ?_2}, \quad \mathrm{pr}^2_{?_1, ?_2}, \quad [?, ?']$$

constitute an α_{13}-product in **Pfn**. This generalized product is not a product in **Pfn**; moreover, it is not a weak product in the sense of Mac Lane [6], p. 231. In fact, consider the diagram



If f and g are such that $\operatorname{dom} f \neq \operatorname{dom} g$, then there is no partial map h making this diagram commutative; if $\operatorname{dom} f = \operatorname{dom} g$, then such an h exists and is unique.

It is well known that there are products in the category **Pfn**. A categorical product $Y \sqcap Z$ of two sets in **Pfn** may be described as follows: Let $Y \sqcap Z = (Y \times Z) + Y + Z$, where $+$ denotes a disjoint union of sets and let

$$\tau^1_{Y,Z}\colon Y \sqcap Z -\!\circ\!\rightarrow Y, \quad \tau^2_{Y,Z}\colon Y \sqcap Z -\!\circ\!\rightarrow Z$$

be partial maps defined in the following way (to simplify the notation we identify sets with their images in a disjoint union):

$$\operatorname{dom} \tau^1_{Y,Z} = (Y \times Z) + Y,$$

$$\tau^1_{Y,Z} | (Y \times Z) = \mathrm{pr}^1_{Y,Z}, \quad \tau^1_{Y,Z} | Y = \mathrm{id}_Y,$$

$$\operatorname{dom} \tau^2_{Y,Z} = (Y \times Z) + Z,$$

$$\tau^2_{Y,Z} | (Y \times Z) = \mathrm{pr}^2_{Y,Z}, \quad \tau^2_{Y,Z} | Z = \mathrm{id}_Z.$$

Given a pair of partial maps $Y \overset{f}{\leftarrow\!\circ\!-} X \overset{g}{-\!\circ\!\rightarrow} Z$, we define the partial map

$$[f, g]_1 \colon X \multimap Y \sqcap Z$$

in the following way:

$$\operatorname{dom}[f, g]_1 = \operatorname{dom} f \cup \operatorname{dom} g,$$

$$[f, g]_1(x) = \begin{cases} (f(x), g(x)) & \text{if } x \in \operatorname{dom} f \cap \operatorname{dom} g, \\ f(x) & \text{if } x \in \operatorname{dom} f \setminus \operatorname{dom} g, \\ g(x) & \text{if } x \in \operatorname{dom} g \setminus \operatorname{dom} f. \end{cases}$$

It is easy to verify that the functions

(10) $$?_1 \sqcap ?_2, \quad \tau^1_{?_1,?_2}, \quad \tau^2_{?_1,?_2}, \quad [?, ?']_1$$

form an α_{23}-product (i.e. the usual product) in **Pfn**.

Using universes one may describe the category **Pfn** in an alternative way. Let U be a fixed universe and let V be a universe larger than U (i.e. such that $U \in V$). The symbol $\mathbf{Pfn}_U$ will denote the category whose objects are elements of U and whose arrows are partial maps between elements of U. By Set_V we denote the category of all sets belonging to V. For any fixed element $*$ of V such that $* \notin U$ let $\mathrm{Set}_U(*)$ be the subcategory of Set_V defined as follows: the objects are sets of the form $X^{(*)} = X \cup \{*\}$, where $X \in U$; the arrows from $X^{(*)}$ to $Y^{(*)}$ are maps $g \colon X^{(*)} \to Y^{(*)}$ such that $g(*) = *$. Let

$$?^{(*)} \colon \mathbf{Pfn}_U \to \mathrm{Set}_U(*)$$

be the functor with the object function $X \mapsto X^{(*)}$ and with the arrow function which assigns to each arrow $f \colon X \multimap Y$ of **Pfn** the arrow $f^{(*)} \colon X^{(*)} \to Y^{(*)}$ in $\mathrm{Set}_U(*)$ defined as

$$f^{(*)}(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f, \\ * & \text{otherwise.} \end{cases}$$

It is well known that the functor $?^{(*)}$ is an isomorphism of categories. The α_{13}-product (9) in **Pfn** yields an α_{13}-product in $\mathrm{Set}_U(*)$, and the product (10) in **Pfn** yields a product in $\mathrm{Set}_U(*)$.

4. Algebraic λ-pre-adjunctions.

4.1. Let $\mathfrak{X}$ be a category. By a *λ-pre-adjunction* of a functor $F \colon \mathfrak{X} \to \mathfrak{X}$ we mean any pre-adjunction $(G_0, \varphi; \psi)$ of F such that $G_0 = \mathrm{id}_{\mathrm{Ob}\,\mathfrak{X}}$. By a *$\lambda\alpha_i$-pre-adjunction* (*$\lambda\alpha_{ij}$-pre-adjunction*) of F we mean a λ-pre-adjunction of F satisfying the condition (α_i) (the conditions (α_i) and (α_j)) from Section 2.2. It is obvious that the composition (in the sense of Proposition 2.4) of two λ-pre-adjunctions is a λ-pre-adjunction.

4.2. We consider some λ-pre-adjunctions of functors defined in an algebraic theory in the sense of Lawvere [5].

By an *algebraic theory* we mean a triple $\mathscr{T} = (\mathfrak{T}, I, P)$ such that

(i) $\mathfrak{T}$ is a category, I is a bijection from N onto $\mathrm{Ob}\,\mathfrak{T}$, and $P = (p_n^i : n \in N^+, i \in n)$ is a family of arrows of $\mathfrak{T}$;

(ii) $p_n^i \in \mathfrak{T}(I(n), I(1))$ for all $n \in N^+$, $i \in n$;

(iii) the object $I(n)$ is a product of $I(1)$ with itself n times for all $n \in N$, and $(p_n^i : i \in n)$ is a family of product projections for all $n \in N^+$.

If $(\mathfrak{T}, I, P)$ is an algebraic theory, then the function I may be extended to the canonical functor $I\colon \mathbf{N}^{\mathrm{op}} \to \mathfrak{T}$. The arrow function of this functor assigns to each arrow $f\colon n \to m$ of $\mathbf{N}^{\mathrm{op}}$ (i.e. to each function $f\colon m \to n$) an arrow

$$\langle p_n^{f(0)}, \ldots, p_n^{f(m-1)} \rangle\colon I(n) \to I(m)$$

of $\mathfrak{T}$ (cf. Section 1.2).

4.3. Let $(\mathfrak{T}, I, P)$ be an algebraic theory. By an *algebraic λ-pre-adjunction* of a functor $F\colon \mathfrak{T} \to \mathfrak{T}$ we mean any λ-pre-adjunction $(\mathrm{id}_{\mathrm{Ob}\,\mathfrak{T}}, \varphi, \psi)$ of F satisfying the following additional condition:

$(\varkappa_1)$ the functions

$$\varphi_{I(n),I(m)}\colon \mathfrak{T}\big(F(I(n)), I(m)\big) \to \mathfrak{T}(I(n), I(m)),$$
$$\psi_{I(n),I(m)}\colon \mathfrak{T}(I(n), I(m)) \to \mathfrak{T}\big(F(I(n)), I(m)\big)$$

are *natural* in m, i.e. for each $n \in N$ the components $\varphi_{I(n),I(m)}$ and $\psi_{I(n),I(m)}$ for all m define natural transformations of functors from $\mathbf{N}^{\mathrm{op}}$ into $\mathbf{Set}$:

$$\varphi_{I(n),I(?)}\colon \mathfrak{T}\big(F(I(n)), I(?)\big) \dot{\to} \mathfrak{T}(I(n), I(?)),$$
$$\psi_{I(n),I(?)}\colon \mathfrak{T}(I(n), I(?)) \dot{\to} \mathfrak{T}\big(F(I(n)), I(?)\big).$$

It is easy to verify that the condition $(\varkappa_1)$ is equivalent to the following one:

$(\varkappa_2)$ the functions $\varphi_{I(n),I(m)}$ and the arrows $\varepsilon_{I(m)}\colon F(I(m)) \to I(m)$, where $\varepsilon_{I(m)}$ is defined by (2), are natural in m.

The naturality of $\varepsilon_{I(m)}$ in m means that the components $\varepsilon_{I(m)}$ define a natural transformation of functors $\varepsilon_{I(?)}\colon F \circ I \dot{\to} I$.

4.4. Let $(\mathfrak{T}, I, P)$ be an algebraic theory and let $(\mathrm{id}_{\mathrm{Ob}\,\mathfrak{T}}, \varphi, \psi)$ be a λ-pre-adjunction of a functor $F\colon \mathfrak{T} \to \mathfrak{T}$. Consider the following conditions:

(σ_1) for all $m, n \in N$ and all $i \in m$ the diagram

$$\begin{array}{ccc}
\mathfrak{T}\big(F(I(n)), I(m)\big) & \xrightarrow{\varphi_{I(n),I(m)}} & \mathfrak{T}(I(n), I(m)) \\
\Big\downarrow{\scriptstyle \mathfrak{T}(F(I(n)), p_m^i)} & & \Big\downarrow{\scriptstyle \mathfrak{T}(I(n), p_m^i)} \\
\mathfrak{T}\big(F(I(n)), I(1)\big) & \xrightarrow[\varphi_{I(n),I(1)}]{} & \mathfrak{T}(I(n), I(1))
\end{array}$$

is commutative;

(σ_2) for all m, $n \in N$ and all $i \in m$ the diagram

$$\begin{array}{ccc}
\mathfrak{T}(I(n), I(m)) & \xrightarrow{\psi_{I(n),I(m)}} & \mathfrak{T}(F(I(n)), I(m)) \\
\downarrow{\scriptstyle \mathfrak{T}(I(n), p^i_m)} & & \downarrow{\scriptstyle \mathfrak{T}(F(I(n)), p^i_m)} \\
\mathfrak{T}(I(n), I(1)) & \xrightarrow[\psi_{I(n),I(1)}]{} & \mathfrak{T}(F(I(n)), I(1))
\end{array}$$

is commutative;

(σ_2') for all $m \in N$ and all $i \in m$ the diagram

$$\begin{array}{ccc}
F(I(m)) & \xrightarrow{\varepsilon_{I(m)}} & I(m) \\
\downarrow{\scriptstyle F(p^i_m)} & & \downarrow{\scriptstyle p^i_m} \\
F(I(1)) & \xrightarrow[\varepsilon_{I(1)}]{} & I(1)
\end{array}$$

is commutative (where ε is defined by (2)).

It is easy to prove that for any λ-pre-adjunction $(\mathrm{id}_{\mathrm{Ob}\mathfrak{T}}, \varphi, \psi)$ of $F: \mathfrak{T} \to \mathfrak{T}$ the following conditions are equivalent:

(a) $(\mathrm{id}_{\mathrm{Ob}\mathfrak{T}}, \varphi, \psi)$ is algebraic;

(b) φ satisfies (σ_1) and ψ satisfies (σ_2);

(c) φ satisfies (σ_1) and ε defined by (2) satisfies (σ_2').

4.5. We shall now recall the following definitions due to Obtułowicz (cf. [7], [9]).

An *algebraic theory with application and abstraction* is an ordered triple $(\mathcal{T}, \varepsilon, (?)^*)$, where $\mathcal{T}$ is an algebraic theory, $\varepsilon: I(2) \to I(1)$ is a distinguished arrow of $\mathfrak{T}$, and $(?)^*$ is a function assigning to each arrow $f: I(n+1) \to I(1)$ $(n \in N)$ of $\mathfrak{T}$ an arrow $h: I(n) \to I(1)$ of $\mathfrak{T}$.

A *lambda-algebraic theory* is an algebraic theory with application and abstraction satisfying the following condition:

(ϱ) $(f)^* \circ g = (f \circ (g \times \mathrm{id}_{I(1)}))^*$ for all arrows $f \in \mathfrak{T}(I(n+1), I(1))$, and $g \in \mathfrak{T}(I(m), I(n))$, $m \in N$, $n \in N$.

A *Church algebraic theory* is a lambda-algebraic theory satisfying the following condition:

(β) $\varepsilon \circ ((f)^* \times \mathrm{id}_{I(1)}) = f$ for all $f \in \mathfrak{T}(I(n), I(1))$, $n \in N^+$.

An *algebraic theory of type $\lambda - \beta\eta$* is an algebraic theory with application and abstraction satisfying the condition (β) and the following condition:

(η) $(\varepsilon \circ (h \times \mathrm{id}_{I(1)}))^* = h$ for all $h \in \mathfrak{T}(I(n), I(1))$, $n \in N$.

These definitions originated from analysis of the syntax of the type free lambda calculus. The basic example of a lambda-algebraic theory is provided by the category $\mathfrak{T}[C]$ whose objects are natural numbers and whose arrows are labelled λ-terms; the composition of arrows is defined by the substitution of labelled λ-terms. Forming suitable quotient categories of $\mathfrak{T}[C]$ one obtains an example of a Church algebraic theory and an example of an algebraic theory of type $\lambda-\beta\eta$ (for details see [9]).

In any algebraic theory $(\mathfrak{T}, I, P)$ there are uniquely defined functors $?\times I(n)\colon \mathfrak{T}\to\mathfrak{T}$ $(n\in N)$. The following theorem describes relations between algebraic λ-pre-adjunctions of these functors and the Obtułowicz theories:

4.6. THEOREM. *For any algebraic theory $(\mathfrak{T}, I, P)$ the following conditions are equivalent in the sense that the data in one of these conditions determine uniquely the others:*

(i) *$(\mathfrak{T}, I, P)$ is equipped with a structure of an algebraic theory with application and abstraction (a structure of a lambda-algebraic theory, a structure of a Church theory, a structure of an algebraic theory of type $\lambda-\beta\eta$, respectively);*

(ii) *there is a specified algebraic λ-pre-adjunction ($\lambda\alpha_1$-pre-adjunction, $\lambda\alpha_{12}$-pre-adjunction, $\lambda\alpha_{23}$-pre-adjunction, respectively) of the functor $?\times I(1)\colon$ $\mathfrak{T}\to\mathfrak{T}$;*

(iii) *for each $n\in N$ there is a specified algebraic λ-pre-adjunction ($\lambda\alpha_1$-pre-adjunction, $\lambda\alpha_{12}$-pre-adjunction, $\lambda\alpha_{23}$-pre-adjunction, respectively) of the functor $?\times I(n)\colon \mathfrak{T}\to\mathfrak{T}$.*

The proof is straightforward by application of the results of Section 4.4 and Proposition 2.4.

In particular, it follows from Theorem 4.6 and Proposition 2.3 that an algebraic theory of type $\lambda-\beta\eta$ is a Cartesian closed category with exponentiation satisfying $I(m)^{I(n)} = I(m)$ for all $m, n\in N$ (cf. [7]).

REFERENCES

[1] H. P. Barendregt, *The type free lambda calculus*, p. 1091-1132 in: *Handbook of mathematical logic*, North-Holland, 1977.

[2] R. Börger and W. Tholen, *Abschwächungen des Adjunktionbegriffs*, Manuscripta Mathematica 19 (1976), p. 19-45.

[3] H. Herrlich and G. E. Strecker, *Category theory*, Boston 1973.

[4] J. Lambek, *Deductive systems and categories, III. Cartesian closed categories, intuitionist propositional calculus, and combinatory logic*, p. 57-82 in: *Toposes, algebraic geometry and logic*, Lecture Notes in Mathematics, 274, Berlin 1972.

[5] F. W. Lawvere, *Functorial semantics of algebraic theories*, Proceedings of the National Academy of Sciences of the U.S.A. 50 (1963), p. 869-872.

[6] S. Mac Lane, *Categories for the working mathematician*, Berlin 1971.

[7] A. Obtułowicz, *Functorial semantics of the type free $\lambda-\beta\eta$ calculus*, p. 302-307 in: *Fundamentals of computation theory*, Lecture Notes in Computer Science 56, Berlin 1977.
[8] — *The logic of categories of partial functions and its applications*, Dissertationes Mathematicae (to appear).
[9] — and A. Wiweger, *Categorical, functorial and algebraic aspects of the type free lambda calculus*, Banach Center Publications, Vol. 9 (1982), p. 399-422.

INSTITUTF OF MATHFMATICS
POLISH ACADFMY OF SCIENCES
WARSZAWA

Reçu par la Rédaction le 6. 12. 1979