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SOME APPLICATIONS OF NON-STANDARD ANALYSIS TO PROXIMITY SPACES

BY

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0. Introduction. Techniques of non-standard analysis have been used by Machover and Hirschfeld in [4] to prove several results in the theory of proximity and uniform spaces. Two such results which are of fundamental importance are the characterization of filters by means of nuclear sets and the reduction of a proximity relation δ on a completely regular space X to an equivalence relation $\approx$ on the corresponding set $\hat{X}$ in an enlargement.

In this paper a characterization of round filters in terms of their nuclei is obtained. The nuclei of the maximal round filters on a proximity space (X, δ) then serve as points of the Smirnov compactification δX and, by this means, the Smirnov compactification may be constructed directly from (X, δ). Standard and non-standard characterizations of proximity spaces admitting a unique compatible proximity relation are then obtained. We also characterize, by means of maximal round filters on (X, δ), these points x of δX for which every real-valued proximity function on (X, δ) has a real-valued p-continuous extension to x, i.e., we determine the points of the real-completion of (X, δ) (see [5]).

1. Characterization of maximal round filters. Non-standard treatments of topological spaces may be found in [4] and [7], and a non-standard development of proximity and uniform spaces occurs in [2], [4] and [6]. In general, we follow the notation of [4], unless otherwise indicated. For (standard) background concerning proximity spaces and round filters, see [8]. Denote by $\bar{\delta}$ the negation of a proximity relation δ.

A filter on a proximity space (X, δ) will be denoted by $\mathscr{F}$, and the nucleus of $\mathscr{F}$ (see Section 5 of [4]) by nuc $\mathscr{F}$. Let $[x]$ be the monad of a point x in X relative to the unique precompact uniformity in the proximity class $\pi(\delta)$ of δ. Then for subsets A and B of X, by Theorem 8.2.2 of [4], we have $A\delta B$ if and only if $a, b \in [x]$ for some *points $a \,{}^*\!\in A$ and $b \,{}^*\!\in B$.

LEMMA 1.1. *A filter* $\mathscr{F}$ *on* (X, δ) *is round if and only if* $x \in \text{nuc}\,\mathscr{F}$ *implies* $[x] \subseteq \text{nuc}\,\mathscr{F}$.

Proof. Necessity. Let $\mathscr{F}$ be a round filter with $x \in \text{nuc}\,\mathscr{F}$ and $y \notin \text{nuc}\,\mathscr{F}$. Then $y\,{}^*\!\notin F$ for some $F \in \mathscr{F}$. Since $\mathscr{F}$ is round, choose $G \in \mathscr{F}$ such that $G \Subset F$. Then $x\,{}^*\!\in G$ and $y\,{}^*\!\in X - F$. Now $G\bar{\delta}(X-F)$ implies $y \notin [x]$.

Sufficiency. Choose $F \in \mathscr{F}$, a filter satisfying the condition. If G is an infinitesimal *member of $\mathscr{F}$, then $x \in G$ implies $x \in \text{nuc}\,\mathscr{F}$. If $y\,{}^*\!\in X - F$, then $y \notin \text{nuc}\,\mathscr{F}$, so that $y \notin [x]$. Thus, $G\bar{\delta}(X-F)$. Now the statement $\exists G[G \in \mathscr{F}$ and $G \Subset F]$ is true in $^*\mathscr{U}$, hence in $\mathscr{U}$, so that $\mathscr{F}$ is round. This completes the proof.

We can now characterize maximal round filters by means of monads.

THEOREM 1.1. *Let* $\mathscr{F}$ *be a filter on* (X, δ). *Then the following are equivalent*:

(i) $\text{nuc}\,\mathscr{F} = [x]$ *for some *point* $x\,{}^*\!\in X$.

(ii) $\mathscr{F}$ *is a maximal round filter.*

Proof. (i) implies (ii). By Lemma 1.1, $\mathscr{F}$ is round. If $\mathscr{F} \subseteq \mathscr{F}_1$, where $\mathscr{F}_1$ is a round filter on (X, δ), then $\text{nuc}\,\mathscr{F}_1 \subseteq \text{nuc}\,\mathscr{F}$ by Theorem 5.1.3 of [4]. But $\mathscr{F}_1$ is round, so that if $y \in \text{nuc}\,\mathscr{F}_1$ and $x \approx y$, then $x \in \text{nuc}\,\mathscr{F}_1$. Since $\text{nuc}\,\mathscr{F}_1$ is non-empty, we have $[x] \subseteq \text{nuc}\,\mathscr{F}_1$, hence $\mathscr{F} = \mathscr{F}_1$. Thus $\mathscr{F}$ is maximal.

(ii) implies (i). If $\mathscr{F}$ is a maximal round filter on (X, δ), then $x \in \text{nuc}\,\mathscr{F}$ implies $[x] \subseteq \text{nuc}\,\mathscr{F}$ by Lemma 1.1. But, by 7.3.3 and 7.4.5 of [4], $[x]$ is nuclear, hence $\mathscr{F} = \text{fil}(\text{nuc}\,\mathscr{F}) \subseteq \text{fil}[x]$. Since $\text{fil}[x]$ is round and $\mathscr{F}$ is maximal, we have $\text{nuc}\,\mathscr{F} = [x]$, and the proof is complete.

Theorem 1.1 shows that the nuclear sets described in Theorem 7.3.3 of [4] are precisely the nuclei of the maximal round filters on (X, δ), where the uniformity in Theorem 7.3.3 is taken to be the unique precompact uniformity in the proximity class of δ.

A collection $\{A_1, \ldots, A_n\}$ of subsets of X is a p-cover of (X, δ) if there exist sets $B_1, \ldots, B_n$ satisfying $B_i \Subset A_i$ for $i = 1, 2, \ldots, n$ and $\bigcup_i B_i = X$ (see [1]).

COROLLARY 1.1. *For a round filter* $\mathscr{F}$ *on* (X, δ), *the following are equivalent*:

(i) $\text{nuc}\,\mathscr{F} = [x]$ *for some *point* $x\,{}^*\!\in X$.

(ii) $\mathscr{F}$ *is maximal.*

(iii) *If* $\{A_1, \ldots, A_n\}$ *is a p-cover of* (X, δ), *then* $A_i \in \mathscr{F}$ *for some* i.

Proof. The equivalence of (i) and (ii) is already established. Thus, let $\{A_1, \ldots, A_n\}$ be a p-cover of X, and let $\mathscr{F}$ be a maximal round filter. Since $\text{nuc}\,\mathscr{F} = [x]$ for some $x\,{}^*\!\in X$, we must have $x\,{}^*\!\in B_i$ for some i. Now $B_i \Subset A_i$ implies that $[x]\,{}^*\!\subseteq A_i$, so that $A_i \in \mathscr{F}$.

Conversely, if (iii) holds, take $x \in \text{nuc}\,\mathscr{F}$. Since $\mathscr{F}$ is round, $[x] \subseteq \text{nuc}\,\mathscr{F}$. If $y \in \text{nuc}\,\mathscr{F}$, but $y \notin [x]$, then there exist subsets A and B of X such that $x\,{}^*\!\in A$, $y\,{}^*\!\in B$ and $A\bar{\delta}B$. Now $\{X-A, X-B\}$ is a p-cover of (X, δ), and neither $X-A$ nor $X-B$ is a member of $\mathscr{F}$, which contradicts (iii). Hence $\text{nuc}\,\mathscr{F} = [x]$, and $\mathscr{F}$ is maximal. This completes the proof.

The equivalence of (ii) and (iii) was proved in [1] by standard methods.

Since $[x]$ is nuclear for each *point of X, Theorem 1.1 provides a one-one correspondence between the maximal round filters on (X, δ) and the monads of *points of X. Denote the maximal round filter $\text{fil}[x]$ by $\mathscr{F}_{[x]}$. It is natural, then, to construct the Smirnov compactification δX of (X, δ) by taking the monads $[x]$, $x\,{}^*\!\in X$, to serve as the points of δX.

THEOREM 1.2 (Smirnov). *A proximity space* (X, δ) *is p-isomorphic to a dense p-subspace of a compact Hausdorff space* δX.

Proof. Take $\delta X = \{[x]: x\,{}^*\!\in X\}$. Define basic closed sets in δX as follows. For each $A \subseteq X$, take $[x] \in \bar{A}$ if and only if $B \in \mathscr{F}_{[x]}$ whenever $A \Subset B$. Then $\bar{X} = \delta X$ and $\bar{\emptyset} = \emptyset$. Suppose $[x] \notin \bar{A}_1 \cup \bar{A}_2$. Then there exist subsets B_1 and B_2 of X such that $A_1 \Subset B_1$, $A_2 \Subset B_2$, $[x]\ {}^*\!\nsubseteq B_1$ and $[x]\ {}^*\!\nsubseteq B_2$. Choose C_1 and C_2 such that $A_1 \Subset C_1 \Subset B_1$ and $A_2 \Subset C_2 \Subset B_2$. Then

$$[x]\ {}^*\!\subseteq (X-C_1) \cap (X-C_2),$$

so that $C_1 \cup C_2 \notin \mathscr{F}_{[x]}$. But $A_1 \cup A_2 \Subset C_1 \cup C_2$, hence

$$[x] \notin \overline{A_1 \cup A_2} \quad \text{and} \quad \overline{A_1 \cup A_2} \subseteq \bar{A}_1 \cup \bar{A}_2.$$

If $A_1 \cup A_2 \Subset B$, then $A_1 \Subset B$ and $A_2 \Subset B$, and it follows that

$$\bar{A}_1 \cup \bar{A}_2 \subseteq \overline{A_1 \cup A_2}.$$

Thus the collection $\{\bar{A}: A \subseteq X\}$ is a base for the closed sets of a topology on δX. Define an injection φ of X into δX by setting $\varphi(x) = [x]$. We note that, for $A \subseteq X$, $\text{Cl}_{\delta X}\, \varphi[A] = \bar{A}$, so that $\text{Cl}_{\delta X}\, \varphi[X] = \bar{X} = \delta X$.

Next we show that δX is compact and Hausdorff. If $[x] \neq [y]$ in δX, then there are subsets A and B of X such that $x\,{}^*\!\in A$, $y\,{}^*\!\in B$ and $A\bar{\delta}B$. Choose subsets C, D, E, F of X such that E and F are disjoint, $A \Subset C \Subset E$ and $B \Subset D \Subset F$. Since $\{X-C, X-D\}$ is a p-cover of (X, δ), Corollary 1.1 implies that $(X-D) \in \mathscr{F}_{[x]}$, and $(X-C) \in \mathscr{F}_{[y]}$. Now $(X-C) \Subset (X-A)$ and $x\,{}^*\!\notin X-A$ imply that $[x] \notin \overline{X-C}$. Hence $[x] \in \delta X - (\overline{X-C})$. Similarly, $[y] \in \delta x - (\overline{X-D})$. Since $\delta X - (\overline{X-C})$ and $\delta X - (\overline{X-D})$ are disjoint, δX is Hausdorff.

Let $\{\bar{A}_a: a \in I\}$ be a collection of basic closed sets with the finite intersection property. If there is $A_{a_i} \Subset B_{a_i}$ for each $i = 1, \ldots, n$, then $[x] \in \bigcap \{\bar{A}_{a_i}: i = 1, \ldots, n\}$ implies that

$$\bigcap \{B_{a_i}: i = 1, \ldots, n\} \in \mathscr{F}_{[x]}.$$

Thus, the collection $\mathscr{B} = \{B \subseteq X: A_\alpha \Subset B$ for some $\alpha \in I\}$ generates a round filter on (X, δ) which can be embedded in some maximal round filter $\mathscr{F}_{[x]}$. Then we have $[x]\, {}^*\!\subseteq B$ for all $B \in \mathscr{B}$, so that $[x] \in \bar{A}_\alpha$ for all $\alpha \in I$, and $\bigcap \{\bar{A}_\alpha: \alpha \in I\} \neq \emptyset$. Hence δX is compact.

Now $A\delta B$ in X if and only if there are $x^* \in A$ and $y^* \in B$, where $y \in [x]$. This implies that $[x] \in \bar{A} \cap \bar{B}$ in δX, so that $\varphi[A]$ is close to $\varphi[B]$ in δX. Conversely, if $A\bar{\delta}B$, take disjoint subsets C and D of X for which $A \Subset C$ and $B \Subset D$. Then

$$\bigcup \{[x]: [x] \in \bar{A}\}\, {}^*\!\subseteq C \quad \text{and} \quad \bigcup \{[y]: [y] \in \bar{B}\}\, {}^*\!\subseteq D.$$

But C and D are disjoint, hence $\bar{A} \cap \bar{B} = \emptyset$, and $\varphi[A]$ and $\varphi[B]$ are remote. Thus φ is a p-isomorphism and the proof is complete.

Next, non-standard methods are used to establish the "universal mapping property" for δX, which is a characteristic of the Smirnov compactification. We first prove a preliminary lemma.

LEMMA 1.2. *Let $A \subseteq \delta X$ and suppose that E is open in δX with* $\mathrm{Cl} A \subseteq E$. *If $C = \varphi^{-1}(E)$, then $\bigcup \{[x]: [x] \in \mathrm{Cl} A\}\, {}^*\!\subseteq C$.*

Proof. By normality, choose G open in δX such that $\mathrm{Cl} A \subseteq G \subseteq \mathrm{Cl} G \subseteq E$. Take $[x] \in \mathrm{Cl} A$, and set $H = \varphi^{-1}(\delta X - G)$. Then $\{C, H\}$ is a p-cover of (X, δ). If $H \in \mathscr{F}_{[x]}$, then

$$[x] \in \bar{H} = \mathrm{Cl}\varphi[H] \subseteq (\delta X - G),$$

which contradicts $[x] \in G$. Thus, by Lemma 1.1, $[x]\, {}^*\!\subseteq C$, and the proof is complete.

THEOREM 1.3. *Let f be a p-mapping of (X, δ) onto a proximity space (X_1, δ_1). Then f has a continuous extension (necessarily a p-mapping) from δX onto $\delta_1 X_1$.*

Proof. Let $\bar{f}$ be the mapping of δX into $\delta_1 X_1$ defined by $\bar{f}([x]) = [f(x)]_1$. Since f is a p-mapping, $\bar{f}$ is well defined (see Theorem 9.1.3 of [4]).

Suppose that $\bar{f}[A]$ is remote from $\bar{f}[B]$ in $\delta_1 X_1$. By normality, choose open sets C_1 and D_1 in $\delta_1 X_1$ satisfying $\mathrm{Cl}\bar{f}[A] \subseteq C_1$ and $\mathrm{Cl}\bar{f}[B] \subseteq D_1$, where C_1 and D_1 are remote. Let $C = \varphi_1^{-1}(C_1)$ and $D = \varphi_1^{-1}(D_1)$, where φ_1 is the canonical injection of X_1 into $\delta_1 X_1$. Then, by Lemma 1.2,

$$\bigcup \{[x]: [x] \in \mathrm{Cl}\bar{f}[A]\}\, {}^*\!\subseteq C \quad \text{and} \quad \bigcup \{[x]: [x] \in \mathrm{Cl}\bar{f}[B]\}\, {}^*\!\subseteq D.$$

If $U = f^{-1}(C)$ and $V = f^{-1}(D)$, we have $U\bar{\delta}V$ in X. Now, for $[x] \in A$, suppose $[x] \cap (\hat{X} - \hat{U}) \neq \emptyset$. Then there is $x'\, {}^* \in X - U$, where $x' \in [x]$. It follows that $f(x') \in \hat{X}_1 - \hat{C}_1$, so that $[f(x)]_1 = \bar{f}([x])\, {}^*\!\nsubseteq C$, which is a contradiction. Thus $\bigcup \{[x]: [x] \in A\}\, {}^*\!\subseteq U$ and, similarly, $\bigcup \{[x]: [x] \in B\}\, {}^*\!\subseteq V$. Since U and V are remote and $A \subseteq \bar{U}$ and $B \subseteq \bar{V}$, we know that A and B are remote in δX. Thus $\bar{f}$ is a p-mapping, and the proof is complete.

2. Locally compact proximity spaces. The equivalence of conditions (i) through (iv) of the following theorem was established by Hirschfeld in Theorem 5.5.4 of [4]:

THEOREM 2.1. *For a completely regular non-compact space X, the following are equivalent:*

(i) *X is locally compact.*

(ii) *Every near-standard *point *belongs to some compact subset of X.*

(iii) *Every convergent ultrafilter has a compact member.*

(iv) *The collection of remote *points is the nucleus of a filter.*

(v) *There exists an admissible precompact uniformity $\mathscr{H}$ on X for which the collection of all remote *points is precisely one equivalence class of the nuclear equivalence relation for $\mathscr{H}$.*

(vi) *There is a compatible proximity relation δ on X for which $\delta X - X$ consists of a single point.*

Proof. We need only establish that (v) and (vi) are equivalent with (i) through (iv).

(i) implies (v). Let $\approx$ be the equivalence relation on $\hat{X}$ whose equivalence classes are the topological monads $[x]$ of standard points $x \in X$, and the set of remote *points is the remaining equivalence class. We show that $\approx$ is nuclear. Thus, if $x_0 \not\approx y_0$, we show that there is a set $H \subseteq X \times X$ such that $(x, y) {}^*\!\in H$ whenever $x \approx y$, but $(x_0, y_0) {}^*\!\notin H$.

If $x_0 \not\approx y_0$, we may assume that x_0 is near-standard. Thus, x_0 is a member of the topological monad of some standard point x_1. Since $y_0 {}^*\!\in X$, there is some open neighborhood V of x_1 such that $y_0 {}^*\!\notin V$. Let W be a compact neighborhood of x_1 contained in V. Then

$$(x_0, y_0) {}^*\!\notin H = V \times V \cup (X-W) \times (X-W).$$

Suppose next that $x \approx y$. If both x and y are near-standard, hence in the topological monad of some standard point z, then either V or $X-W$ is an open neighborhood of z. Thus $(x, y) {}^*\!\in H$. If x and y are remote, then $x {}^*\!\notin W$ and $y {}^*\!\notin W$, since W is compact. Again $(x, y) {}^*\!\in H$. This shows that $\approx$ is nuclear, and so determines the desired admissible uniformity $\mathscr{H}$ on X. It follows from (iv) and Theorem 7.4.5 of [4] that $\mathscr{H}$ is precompact.

(v) implies (vi). If δ is the proximity relation on X induced by $\mathscr{H}$, then the equivalence classes on X which determine δX are precisely the monads of $\mathscr{H}$, so that $\delta X - X$ consists of a single point, by the construction in Theorem 1.2.

(vi) implies (iv). If $\delta X - X$ consists of a single point, then the collection of remote *points of X is the nucleus of the unique free maximal round filter on (X, δ). This completes the proof.

The following result shows that if X is not locally compact, then X admits infinitely many distinct compatible proximity relations.

PROPOSITION 2.1. *If X is a non-locally compact, completely regular space, then there is an infinite ascending chain of compatible proximity relations for X.*

Proof. Since X is not locally compact, the cardinal of $\beta X - X$ is infinite, where βX is the Stone-Čech compactification of X. Now βX is also the Smirnov compactification of X relative to the smallest compatible proximity δ_0 on X. Let p and q be distinct points of $\beta X - X$, and let x_1 be any object not in βX. Let $\delta_1 X = (\beta X - \{p, q\}) \cup \{x_1\}$, and define a mapping τ_1 of βX onto $\delta_1 X$ by $\tau_1(x) = x$ if $x \neq p, q$, and by $\tau_1(p) = \tau_1(q) = x_1$. Provide $\delta_1 X$ with the quotient topology relative to τ_1. The restriction of τ_1 to X is the identity, and $\tau_1[\beta X] = \delta_1 X$ insures that $\delta_1 X$ is compact and Hausdorff. If δ_1 is the proximity relation on X induced by $\delta_1 X$, then τ_1 is the Smirnov extension of $\tau_1 | X$ to βX. Clearly, $\delta_0 \subsetneq \delta_1$. Since $\delta_1 X - X$ is infinite, this process may be repeated to obtain a compatible proximity δ_2 on X satisfying $\delta_0 \subsetneq \delta_1 \subsetneq \delta_2$. Thus, inductively, we can generate an infinite ascending chain of compatible proximities for X, and the proof is complete.

We next consider proximity spaces (X, δ) for which $\delta X - X$ is finite.

THEOREM 2.2. *If (X, δ) is a proximity space for which $\delta X - X$ contains exactly n points, then the proximity class $\pi(\delta)$ contains only the unique precompact uniformity $\mathscr{H}(\delta)$.*

Proof. Assume that $\pi(\delta)$ contains $\mathscr{H}_1$, where $\mathscr{H}_1 \neq \mathscr{H}(\delta)$. Since $\mathscr{H}_1$ is not precompact, we can choose a symmetric entourage $H \in \mathscr{H}_1$ and a countably infinite subset $A = \{x_i : i = 1, 2, \ldots\}$ of X such that

$$x_{i+1} \notin \bigcup \{H(x_k) \colon k = 1, 2, \ldots, i\} \quad \text{for each } i.$$

Select a collection of $n+1$ pairwise disjoint infinite subsets $A_1, \ldots, A_{n+1}$ of A. Since no A_i is compact, each A_i has a remote *member y_i. Thus, for some i and j, $i \neq j$, we have $[y_i] = [y_j]$ which implies that $A_i \delta A_j$. But $H[A_i] \cap A_j = \emptyset$, contradicting $\mathscr{H} \in \pi(\delta)$.

Hence $\pi(\delta)$ contains only the precompact uniformity $\mathscr{H}(\delta)$, and the proof is complete.

To summarize, we state the following corollary:

COROLLARY 2.1. (i) *If δ is the locally compact proximity relation (of Theorem 2.1) on X, then $\pi(\delta)$ contains only the precompact uniformity $\mathscr{H}(\delta)$.*

(ii) *If $\mathscr{H}$ is an admissible non-precompact uniformity for X, and if $\delta_{\mathscr{H}}$ is the proximity induced by $\mathscr{H}$, then $\delta X - X$ is infinite.*

(iii) *If X admits a non-precompact uniformity $\mathscr{H}$, then there is an infinite ascending chain of compatible proximities for X, for which $\delta_{\mathscr{H}}$ is the initial member of the chain.*

Proof of (iii). If $\mathscr{H}$ is not precompact, card $\delta_{\mathscr{H}} X - X$ is infinite. Now applying the argument of the proof of Proposition 2.1 completes the proof.

Proposition 2.1 insures that if X admits a unique compatible proximity, then X is locally compact. There are many conditions equivalent to the condition that a completely regular space X has a unique compactification (e.g., see 6.J and 15.R of [3]). The next result, which follows immediately from our previous results, provides a non-standard characterization of such "almost compact" spaces.

COROLLARY 2.2. *For a completely regular space X, the following are equivalent*:

(i) *X has a unique compatible proximity relation δ.*

(ii) *X admits a unique uniformity (necessarily precompact).*

(iii) *There is a unique nuclear equivalence relation on X.*

(iv) *X has a unique compactification (necessarily the one-point compactification).*

3. Extensions of real-valued p-functions. Let δ_R denote the proximity relation in the real numbers induced by the standard metric, and let $P(X)$ be the class of real-valued proximity mappings on (X, δ). By Theorem 1.3, each member f of $P(X)$ has an extension $\bar{f}$ mapping δX into $\delta_R R$. Thus $\bar{f}([x])$ is real if and only if $[f(x)]_R = [r]_R$ for some $r \in R$. Take $[x] \in \delta X$ and consider $\mathscr{F}_{[x]}$. If f is bounded on some member F of $\mathscr{F}_{[x]}$, then $\mathrm{Cl}_R f[F]$ is compact, so that every point of $f[F]$ is near-standard (see Theorem 5.5.2 of [4]). Thus $\bar{f}([x])$ is real. Thus, if $\bar{f}([x])$ is not real, then f is unbounded on every member of $\mathscr{F}_{[x]}$. The set $\varepsilon_\delta X$ of all $[x]$ in δX for which $\bar{f}([x])$ is real for every member f of $P(X)$ with proximity inherited from δX is the real-completion of (X, δ) (see [5]). If $[x] \in \varepsilon_\delta X$, we say that $\mathscr{F}_{[x]}$ is a real maximal round filter.

We next provide a characterization of the real maximal round filters. A subset S of (X, δ) is called *relatively p-pseudocompact* if every $f \in P(X)$ is bounded on S.

THEOREM 3.1. *$\mathscr{F}_{[x]}$ is real if and only if $\mathscr{F}_{[x]}$ contains a relatively p-pseudocompact member.*

Proof. If F is a relatively p-pseudocompact member of $\mathscr{F}_{[x]}$, then $f[F]$ is bounded for each $f \in P(X)$, and $\bar{f}([x])$ is real.

Conversely, suppose $\mathscr{F}_{[x]}$ is real, so that every member f of $P(X)$ has a real-extension $\bar{f}$ at $[x]$. Choose an infinitesimal *member F of $\mathscr{F}_{[x]}$. Now

$$f[F] \subseteq f[x] \subseteq [f(x)]_R = [r]_R \subseteq \hat{A},$$

where A is the interval $[r-1, r+1]$ in R. Thus the statement

$$\exists F \in \mathscr{F}_{[x]} \big[\forall f \in P(X) \, [\exists a, b \in R \colon f[F] \subseteq [a, b]] \big]$$

is true in $^*\mathscr{U}$. Reinterpreting this in $\mathscr{U}$, there exists an $F \in \mathscr{F}_{[x]}$ on which every $f \in P(X)$ is bounded, so F is relatively p-pseudocompact. This completes the proof.

In metric spaces, where δ is the metric proximity, every totally bounded set is relatively p-pseudocompact. But examples can be found to show that the converse need not be true. Thus, even in metric spaces relatively p-pseudocompact sets are not coincidental with the totally bounded sets. Finally, we observe that (X, δ) is compact if and only if $\varepsilon_\delta X = X$ and X is relatively p-pseudocompact.

REFERENCES

[1] E. M. Alfsen and J. E. Fenstad, *A note on completion and compactification*, Mathematica Scandinavica 8 (1960), p. 97-104.

[2] J. E. Fenstad and A. M. Nyberg, *Standard and non-standard methods in uniform topology*, in Logic Colloquium (1969), Amsterdam 1971.

[3] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton 1960.

[4] M. Machover and J. Hirschfeld, *Lectures on non-standard analysis*, Lecture Notes in Mathematics 94, New York 1969.

[5] O. Njastad, *On real-valued proximity mappings*, Mathematische Annalen 154 (1964), p. 413-419.

[6] K. Potthoff, *Untersuchungen über Nichtstandard-Modelle*, Ph. D. Dissertation, Technische-Hochschüle, Hanover 1967.

[7] A. Robinson, *Non-standard analysis*, Amsterdam 1966.

[8] W. Thron, *Topological structures*, New York 1966.

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