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SYMMETRIC TWOFOLD CR SUBMANIFOLDS IN A EUCLIDEAN SPACE R^{4m}

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0. Introduction. Bejancu initiated to study CR submanifolds of a Kaehler manifold ([2], [3]). Since then many papers on CR submanifolds of Kaehler manifolds and Sasakian manifolds have been published ([5]–[7], [9], [12], [13]).

Barros et al. studied quaternion CR submanifolds of a Kaehler quaternion manifold.

The purpose of this paper is to study, what we call, symmetric twofold CR submanifolds in a Euclidean space R^{4m} with global Kaehler quaternion structure.

Section 1 is mainly devoted to the definition of symmetric twofold CR submanifolds in R^{4m}.

In Section 2, we give a characterization of symmetric twofold CR submanifolds in terms with the naturally induced structures on submanifolds.

In the last Section 3, we introduce the notion of the symmetric twofold CR products and determine them.

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1. Preliminaries. Let R^{4m} be a $4m$-dimensional Euclidean space with standard Kaehler quaternion metric and standard Kaehler quaternion structure ϕ_λ ($\lambda = 1, 2, 3$). Then we have

$$\phi_\lambda^2 = -I, \quad \phi_\lambda \phi_\mu = \phi_\varkappa, \tag{1.1}$$

$$\tilde{\nabla}\phi_\lambda = 0 \tag{1.2}$$

for any cyclic permutation (λ, μ, $\varkappa$) of (1, 2, 3) and where $\tilde{\nabla}$ is the Riemannian connection determined by $\langle\ ,\ \rangle$.

Let M be an isometrically immersed submanifold in R^{4m} with induced connection ∇. We denote by $\langle\ ,\ \rangle$ the induced metric on M. Then the Gauss and Weingarten formulas for M are given respectively by

$$\tilde{\nabla}_U V = \nabla_U V + B(U, V), \tag{1.3}$$

$$\tilde{\nabla}_U N = -A_N U + D_U N \tag{1.4}$$

for any vector fields U and V tangent to M and any vector field N normal to M, where B denotes the second fundamental form of M, and D the linear connection, called the *normal connection induced in the normal bundle* $T^{\perp}(M)$. Then the second fundamental tensor (or the Weingarten map) A_N is related to B by

$$\langle A_N U, V\rangle = \langle B(U, V), N\rangle. \tag{1.5}$$

The mean curvature vector field H is defined by

$$H = \frac{1}{n}\operatorname{trace} B, \tag{1.6}$$

where $n = \dim M$. If $H = 0$ identically, then M is said to be *minimal* and if $B = 0$ identically, then M is said to be *totally geodesic*.

Definition. A submanifold M in R^{4m} is said to be a *CR submanifold with respect to* ϕ_λ if there exists a differentiable distribution $\mathscr{D}$ on M such that

$$\phi_\lambda \mathscr{D} = \mathscr{D}, \quad \phi_\lambda \mathscr{D}^{\perp} \subset T^{\perp}(M), \tag{1.7}$$

where $\mathscr{D}^{\perp}$ is the orthogonal complementary distribution of $\mathscr{D}$.

Definition. A submanifold M in R^{4m} is called a *symmetric twofold CR submanifold of type* (ϕ_λ, ϕ_μ) if there exists a differentiable distribution $\mathscr{D}$ on M such that

$$\begin{aligned} \phi_\lambda \mathscr{D} &= \mathscr{D}, & \phi_\lambda \mathscr{D}^{\perp} &\subset T^{\perp}(M), \\ \phi_\mu \mathscr{D}^{\perp} &= \mathscr{D}^{\perp}, & \phi_\mu \mathscr{D} &\subset T^{\perp}(M), \end{aligned} \tag{1.8}$$

where $\mathscr{D}^{\perp}$ is the orthogonal complementary distribution of $\mathscr{D}$. We say that M is *proper* if neither $\mathscr{D}$ nor $\mathscr{D}^{\perp}$ is null.

Remark. Let $\mathscr{V}$ be a 12-dimensional vector space over reals with almost quaternion structure ϕ_λ $(\lambda = 1, 2, 3)$ and e_i $(i = 1, 2, \dots, 12)$ an orthonormal base of $\mathscr{V}$. Let $\mathscr{W}$ be a 6-dimensional subspace generated by $e_1, \dots, e_6$. We put

$$\mathscr{D} = \langle e_1, \dots, e_4\rangle \quad \text{and} \quad \mathscr{D}^{\perp} = \langle e_5, e_6\rangle.$$

Now, we write

$$\phi_1 e_1 = -e_2, \quad \phi_1 e_3 = e_4, \quad \phi_1 e_5 = e_7, \quad \phi_1 e_6 = e_8,$$
$$\phi_1 e_9 = e_{10}, \quad \phi_1 e_{11} = e_{12};$$
$$\phi_2 e_5 = e_6, \quad \phi_2 e_1 = e_9, \quad \phi_2 e_2 = e_{10}, \quad \phi_2 e_3 = e_{11},$$
$$\phi_2 e_4 = -e_{12}, \quad \phi_2 e_7 = -e_8.$$

Then we see that

$$\phi_3 e_1 = e_{10}, \quad \phi_3 e_{10} = -e_1, \quad \phi_3 e_2 = -e_9, \quad \phi_3 e_9 = e_2,$$

$$\phi_3 e_3 = e_{12}, \quad \phi_3 e_{12} = -e_3, \quad \phi_3 e_4 = e_{11}, \quad \phi_3 e_{11} = -e_4,$$

$$\phi_3 e_5 = e_8, \quad \phi_3 e_8 = -e_5, \quad \phi_3 e_6 = -e_7, \quad \phi_3 e_7 = e_6,$$

$$\phi_1 \mathscr{D} = \mathscr{D}, \quad \phi_2 \mathscr{D}^\perp = \mathscr{D}^\perp, \quad \phi_1 \mathscr{D}^\perp \subset \mathscr{W}^\perp \quad \text{and} \quad \phi_2 \mathscr{D} \subset \mathscr{W}^\perp,$$

which guarantees the existence of our symmetric twofold CR submanifold of type (ϕ_1, ϕ_2).

2. Characterization of symmetric twofold CR submanifolds. Let M be a submanifold in R^{4m}. For a vector field U tangent to M and a vector field N normal to M, we put

$$\phi_\lambda U = T_\lambda U + F_\lambda U, \tag{2.1}$$

$$\phi_\lambda N = t_\lambda N + f_\lambda N \quad (\lambda = 1, 2, 3), \tag{2.2}$$

where $T_\lambda U$ and $t_\lambda U$ are the tangent parts of $\phi_\lambda U$ and $\phi_\lambda N$, respectively, and $F_\lambda U$ and $f_\lambda N$ are the normal parts of $\phi_\lambda U$ and $\phi_\lambda N$, respectively. Now, using (1.1), we have the following identities:

$$\langle T_\lambda U, V\rangle = -\langle U, T_\lambda V\rangle, \tag{2.3}$$

$$\langle f_\lambda N_1, N_2\rangle = -\langle N_1, f_\lambda N_2\rangle, \tag{2.4}$$

$$\langle F_\lambda U, N\rangle = -\langle U, t_\lambda N\rangle, \tag{2.5}$$

$$T_\lambda^2 U = -U - t_\lambda F_\lambda U, \tag{2.6}$$

$$F_\lambda T_\lambda U + f_\lambda F_\lambda U = 0, \tag{2.7}$$

$$T_\lambda t_\lambda N + t_\lambda f_\lambda N = 0, \tag{2.8}$$

$$f_\lambda^2 N + F_\lambda t_\lambda N = -N, \tag{2.9}$$

$$T_\lambda T_\mu U + t_\lambda F_\mu U = -T_\mu T_\lambda U - t_\mu F_\lambda U = T_\varkappa U, \tag{2.10}$$

$$F_\lambda T_\mu U + f_\lambda F_\mu U = -F_\mu T_\lambda U - f_\mu F_\lambda U = F_\varkappa U, \tag{2.11}$$

$$T_\lambda t_\mu N + t_\lambda F_\mu N = -T_\mu t_\lambda N - t_\mu f_\lambda N = t_\varkappa N, \tag{2.12}$$

$$F_\lambda t_\mu N + f_\lambda f_\mu N = -F_\mu t_\lambda N - f_\mu f_\lambda N = f_\varkappa N. \tag{2.13}$$

We begin with

LEMMA 2.1. *Let M be a symmetric twofold CR submanifold of type (ϕ_λ, ϕ_μ) in R^{4m}. Then*

$$\phi_\varkappa(T(M)) \subset T^\perp(M). \tag{2.14}$$

Proof. For $X \in \mathscr{D}$ we have

$$\phi_\varkappa X = -\phi_\mu \phi_\lambda X \in \phi_\mu \mathscr{D} \subset T^\perp(M),$$

and for $W \in \mathscr{D}^{\perp}$ we have

$$\phi_{\varkappa} W = \phi_{\lambda} \phi_{\mu} W \in \phi_{\lambda} \mathscr{D}^{\perp} \subset T^{\perp}(M),$$

which proves (2.14).

LEMMA 2.2. *Let M be a symmetric twofold CR submanifold of type $(\phi_{\lambda}, \phi_{\mu})$ in R^{4m}. Then $\phi_{\lambda} \mathscr{D}^{\perp}$ and $\phi_{\mu} \mathscr{D}$ are mutually orthogonal subbundles in $T^{\perp}(M)$.*

Proof. This follows from

$$\langle \phi_{\lambda} \mathscr{D}^{\perp}, \phi_{\mu} \mathscr{D} \rangle = -\langle \mathscr{D}^{\perp}, \phi_{\lambda} \phi_{\mu} \mathscr{D} \rangle = -\langle \mathscr{D}^{\perp}, \phi_{\varkappa} \mathscr{D} \rangle = 0$$

(by Lemma 2.1).

By Lemmas 2.1 and 2.2, we may decompose the normal bundle $T^{\perp}(M)$ as

$$T^{\perp}(M) = \phi_{\lambda} \mathscr{D}^{\perp} \oplus \phi_{\mu} \mathscr{D} \oplus \nu, \tag{2.15}$$

where ν is the orthogonal subbundle of $\phi_{\lambda} \mathscr{D}^{\perp} \oplus \phi_{\mu} \mathscr{D}$ in $T^{\perp}(M)$ and it is easily seen that ν is invariant under the action of ϕ_{λ} and ϕ_{μ}.

LEMMA 2.3. *Let M be a symmetric twofold CR submanifold of type $(\phi_{\lambda}, \phi_{\mu})$ in R^{4m}. Then*

$$\text{(a) } t_{\lambda}(T^{\perp}(M)) = \mathscr{D}^{\perp}, \quad \text{(b) } t_{\mu}(T^{\perp}(M)) = \mathscr{D}, \tag{2.16}$$

$$\text{(a) } F_{\mu} t_{\lambda} = 0, \quad \text{(b) } F_{\lambda} t_{\mu} = 0. \tag{2.17}$$

Proof. Using (2.15), we have

$$\phi_{\lambda}(T^{\perp}(M)) = (-\mathscr{D}^{\perp}) + \phi_{\lambda} \phi_{\mu} \mathscr{D} + \phi_{\lambda} \nu = (-\mathscr{D}^{\perp}) + \phi_{\varkappa} \mathscr{D} + \phi_{\lambda} \nu.$$

Since $\phi_{\varkappa} \mathscr{D} \oplus \phi_{\lambda} \nu$ is the normal subbundle, we have

$$t_{\lambda}(T^{\perp}(M)) = \mathscr{D}^{\perp}.$$

Similarly, we have

$$t_{\mu}(T^{\perp}(M)) = \mathscr{D}.$$

Then (2.17) follows immediately from (2.16).

We are now in a position to seek the conditions that a submanifold M in R^{4m} to be a symmetric twofold CR submanifold of type $(\phi_{\lambda}, \phi_{\mu})$. We first assume that M is a symmetric twofold CR submanifold of type $(\phi_{\lambda}, \phi_{\mu})$. Let $\boldsymbol{l}$ and $\boldsymbol{m}$ be the projection operators corresponding to $\mathscr{D}$ and $\mathscr{D}^{\perp}$, respectively. Then we have

$$\boldsymbol{l} + \boldsymbol{m} = I, \quad \boldsymbol{lm} = \boldsymbol{ml} = 0, \quad \boldsymbol{l}^2 = \boldsymbol{l}, \quad \boldsymbol{m}^2 = \boldsymbol{m}. \tag{2.18}$$

From (2.1) we obtain

$$\phi_{\lambda} \boldsymbol{l} U = T_{\lambda} \boldsymbol{l} U + F_{\lambda} \boldsymbol{l} U, \quad \phi_{\lambda} \boldsymbol{m} U = T_{\lambda} \boldsymbol{m} U + F_{\lambda} \boldsymbol{m} U,$$

$$\phi_\mu lU = T_\mu lU + F_\mu lU, \quad \phi_\mu mU = T_\mu mU + F_\mu mU,$$

$$\phi_\varkappa lU = T_\varkappa lU + F_\varkappa lU, \quad \phi_\varkappa mU = T_\varkappa mU + F_\varkappa mU,$$

whence

$$(2.19)\qquad \begin{cases} \phi_\lambda l = T_\lambda l, & F_\lambda l = 0, \\ \phi_\mu l = F_\mu l, & T_\mu l = 0, \quad \phi_\varkappa l = F_\varkappa l, \end{cases}$$

$$(2.20)\qquad \begin{cases} \phi_\lambda m = F_\lambda m, & T_\lambda m = 0, \\ \phi_\mu m = T_\mu m, & F_\mu m = 0, \quad \phi_\varkappa m = F_\varkappa m, \quad T_\varkappa m = 0. \end{cases}$$

Then, by (2.18)–(2.20), we have

$$(2.21)\qquad T_\lambda l = T_\lambda(I-m) = T_\lambda - T_\lambda m = T_\lambda,$$

$$(2.22)\qquad T_\mu m = T_\mu(I-l) = T_\mu - T_\mu l = T_\mu,$$

$$(2.23)\qquad T_\varkappa = 0, \quad F_\varkappa = \phi_\varkappa.$$

Next, from (2.7) we obtain

$$(2.24)\qquad \text{(a) } F_\lambda T_\lambda lU + f_\lambda F_\lambda lU = 0, \quad \text{(b) } F_\lambda T_\lambda mU + f_\lambda F_\lambda mU = 0,$$

$$(2.25)\qquad \text{(a) } F_\mu T_\mu lU + f_\mu F_\mu lU = 0, \quad \text{(b) } F_\mu T_\mu mU + f_\mu F_\mu mU = 0.$$

Since $T_\lambda l = T_\lambda$, $F_\lambda l = 0$, $T_\mu m = T_\mu$ and $F_\mu m = 0$, we have, by (a) of (2.24) and (b) of (2.25),

$$(2.26)\qquad F_\lambda T_\lambda = 0, \quad F_\mu T_\mu = 0,$$

which is equivalent to

$$(2.27)\qquad f_\lambda F_\lambda = 0, \quad f_\mu F_\mu = 0$$

by virtue of (2.7). Since

$$\langle F_\lambda U, V\rangle = -\langle U, t_\lambda V\rangle \quad \text{and} \quad \langle F_\mu U, V\rangle = -\langle U, t_\mu V\rangle,$$

formula (2.26) is also equivalent to

$$(2.28)\qquad T_\lambda t_\lambda = 0, \quad T_\mu t_\mu = 0.$$

Therefore, (2.27) is also equivalent to

$$(2.29)\qquad t_\lambda f_\lambda = 0, \quad t_\mu f_\mu = 0$$

by virtue of (2.8). Also, since $F_\mu m = 0$ and $T_\varkappa = 0$, from (2.10) we infer that $T_\lambda T_\mu m = 0$. Then, from (2.22), we have

$$(2.30)\qquad T_\lambda T_\mu = 0.$$

Similarly, we have

$$(2.31)\qquad T_\mu T_\lambda = 0.$$

The conditions (2.30) and (2.31) are equivalent to

$$t_\lambda F_\mu = 0, \quad t_\mu F_\lambda = 0 \tag{2.32}$$

by virtue of (2.10).

Conversely, we assume that

$$F_\mu t_\lambda = 0, \quad F_\lambda T_\lambda = 0, \quad F_\mu T_\mu = 0, \quad T_\lambda T_\mu = 0 \quad \text{and} \quad T_\mu T_\lambda = 0.$$

We put

$$\mathscr{D} = \{U \in T(M);\ F_\lambda U = 0,\ T_\mu U = 0\}, \tag{2.33}$$

$$\mathscr{D}' = \{U \in T(M);\ F_\mu U = 0,\ T_\lambda U = 0\}. \tag{2.34}$$

We first show that $\mathscr{D}'$ is the orthogonal complementary distribution $\mathscr{D}^\perp$ of $\mathscr{D}$ in $T(M)$. For $X \in \mathscr{D}$ and $Z \in \mathscr{D}'$, we have

$$\langle X, Z\rangle = \langle \phi_\lambda X, \phi_\lambda Z\rangle = \langle T_\lambda X, F_\lambda Z\rangle = 0. \tag{2.35}$$

And, from (2.6), we have

$$U = -T_\lambda^2 - t_\lambda F_\lambda U.$$

Here we have

$$F_\lambda(-T_\lambda^2 U) = -(F_\lambda T_\lambda) T_\lambda U = 0,$$

$$F_\mu(-t_\lambda F_\lambda U) = -(F_\mu t_\lambda) F_\lambda U = 0,$$

$$T_\mu(-T_\lambda^2 U) = -(T_\mu T_\lambda) T_\lambda U = 0,$$

$$T_\lambda(-t_\lambda F_\lambda U) = -(T_\lambda t_\lambda) F_\lambda U = 0,$$

which, together with (2.35), implies $\mathscr{D}' = \mathscr{D}^\perp$. Thus we have

$$F_\lambda \phi_\lambda X = F_\lambda T_\lambda X = 0, \qquad T_\mu \phi_\lambda X = T_\mu T_\lambda X = 0,$$

which shows that $\phi_\lambda \mathscr{D} = \mathscr{D}$. We next show that $\phi_\lambda \mathscr{D}^\perp \subset T^\perp(M)$. To see this we first show that $t_\lambda(T^\perp(M)) = \mathscr{D}^\perp$. In fact, for $X \in \mathscr{D}$ and $N \in T^\perp(M)$ we have

$$\langle t_\lambda N, X\rangle = -\langle N, F_\lambda X\rangle = 0,$$

which implies that $\mathscr{D}^\perp \supset t_\lambda(T^\perp(M))$. And, for any $W \in \mathscr{D}^\perp$ we have

$$t_\lambda(-F_\lambda W) = -t_\lambda F_\lambda W = -(\phi_\lambda - f_\lambda) F_\lambda W = -\phi_\lambda F_\lambda W$$

$$= -\phi_\lambda(\phi_\lambda W - T_\lambda W) = -\phi_\lambda^2 W = W,$$

which implies that $\mathscr{D}^\perp \subset t_\lambda(T^\perp(M))$ and, consequently, we have

$$t_\lambda(T^\perp(M)) = \mathscr{D}^\perp.$$

Then, for a tangent vector field U to M, we have, by putting $W = t_\lambda N$

$(N \in T^{\perp}(M))$,

$$\langle \phi_\lambda W, U \rangle = \langle \phi_\lambda t_\lambda N, U \rangle = -\langle t_\lambda N, \phi_\lambda U \rangle$$
$$= -\langle t_\lambda N, T_\lambda U \rangle = \langle N, F_\lambda T_\lambda U \rangle = 0,$$

which shows that $\phi_\lambda \mathscr{D}^{\perp} \subset T^{\perp}(M)$. Similarly, we see that $\phi_\mu \mathscr{D}^{\perp} = \mathscr{D}^{\perp}$ and $\phi_\mu \mathscr{D} \subset T^{\perp}(M)$.

Summing up, we have

THEOREM 2.1. *Let M be a submanifold in R^{4m}. Then M is a symmetric twofold CR submanifold of type (ϕ_λ, ϕ_μ) if and only if one of the following conditions is satisfied*:

(a) $F_\mu t_\lambda = 0$, $F_\lambda T_\lambda = 0$, $F_\mu T_\mu = 0$ *and* $T_\lambda T_\mu = T_\mu T_\lambda = 0$ *or* $t_\lambda F_\mu = t_\mu F_\lambda = 0$.

(b) $F_\mu t_\lambda = 0$, $f_\lambda F_\lambda = 0$, $f_\mu F_\mu = 0$ *and* $T_\lambda T_\mu = T_\mu T_\lambda = 0$ *or* $t_\lambda F_\mu = t_\mu F_\lambda = 0$.

(c) $F_\mu t_\lambda = 0$, $T_\lambda t_\lambda = 0$, $T_\mu t_\mu = 0$ *and* $T_\lambda T_\lambda = T_\mu T_\lambda = 0$ *or* $t_\lambda F_\mu = t_\mu F_\lambda = 0$.

(d) $F_\mu t_\lambda = 0$, $t_\lambda f_\lambda = 0$; $t_\mu f_\mu = 0$ *and* $T_\lambda T_\mu = T_\mu T_\lambda = 0$ *or* $t_\lambda F_\mu = t_\mu F_\lambda = 0$.

3. Symmetric twofold *CR* products in R^{4m}. Let M be a symmetric twofold *CR* submanifold of type (ϕ_1, ϕ_2) in R^{4m}. The following results are known:

LEMMA A ([2], p. 138). *Let L be a CR submanifold of a Kaehler manifold with CR structure $(\mathscr{D}, \mathscr{D}^{\perp}, J)$. Then the holomorphic distribution $\mathscr{D}$ is integrable if and only if*

$$B(X, JY) = B(Y, JX) \quad \textit{for all } X, Y \in \mathscr{D}.$$

LEMMA B ([5], p. 308). *The notation being as above, the distribution $\mathscr{D}^{\perp}$ is always integrable.*

Now, we consider the *CR* structure $(\mathscr{D}, \mathscr{D}^{\perp}, \phi_1)$ of a symmetric twofold *CR* submanifold M of type (ϕ_1, ϕ_2) in R^{4m}. Then, by Lemma B, we see that $\mathscr{D}^{\perp}$ is integrable. Next we consider the *CR* structure $(\mathscr{D}^{\perp}, \mathscr{D}, \phi_2)$. Then, again by Lemma B, we see that $\mathscr{D}$ is integrable. Thus we have

THEOREM 3.1. *Let M be a symmetric twofold CR submanifold in R^{4m}. Then both distributions $\mathscr{D}$ and $\mathscr{D}^{\perp}$ are integrable.*

Also, from Lemma A and Theorem 3.1 we have easily

THEOREM 3.2. *Let M be a symmetric twofold CR submanifold in R^{4m}. Then M is a minimal submanifold.*

COROLLARY. *There does not exist a compact symmetric twofold CR submanifold in R^{4m}.*

We now give the following definitions:

Definition. A submanifold M' of M which is also a submanifold in R^{4m} is said to be a *ϕ_λ-invariant submanifold* ($\lambda = 1$ or 2) if the tangent space of M' is invariant under the action of ϕ_λ at each point $p' \in M'$.

Definition. A symmetric twofold CR submanifold M of type (ϕ_1, ϕ_2) is called a *symmetric twofold CR product* if M is a Riemannian product of a ϕ_1-invariant submanifold and a ϕ_2-invariant submanifold in R^{4m} locally.

LEMMA 3.1. *We have*

$$\langle \phi_1 B(X, Z), N \rangle = \langle B(\phi_1 X, Z), N \rangle, \tag{3.1}$$

$$\langle \phi_2 B(X, Z), N \rangle = \langle B(X, \phi_2 Z), N \rangle, \tag{3.2}$$

$$\langle B(\mathscr{D}, \mathscr{D}^\perp), \nu \rangle = \{0\} \tag{3.3}$$

for $X \in \mathscr{D}$, $Z \in \mathscr{D}^\perp$ *and* $N \in \nu$.

Proof. We have

$$\begin{aligned} \langle B(X, Z), \phi_1 N \rangle &= \langle \tilde{\nabla}_Z X, \phi_1 N \rangle = -\langle X, \tilde{\nabla}_Z \phi_1 N \rangle \\ &= -\langle X, -\phi_1 A_N Z + \phi_1 D_Z N \rangle = -\langle \phi_1 X, A_N Z \rangle \\ &= \langle B(\phi_1 X, Z), N \rangle, \end{aligned}$$

which proves (3.1). Similarly we have (3.2). Then, for (3.3), we have

$$\begin{aligned} \langle B(\phi_1 X, \phi_2 Z), N \rangle &= \langle \phi_1 B(X, \phi_2 Z), N \rangle = -\langle B(X, \phi_2 Z), \phi_1 N \rangle \\ &= -\langle \phi_2 B(X, Z), \phi_1 N \rangle = \langle B(X, Z), \phi_2 \phi_1 N \rangle. \end{aligned}$$

This, together with $\phi_1 \phi_2 = -\phi_2 \phi_1$, implies (3.3).

Now, calculating $\tilde{\nabla}_U \phi_1 V$ and $\tilde{\nabla}_U \phi_2 V$ in two ways and taking the tangent parts, we have

$$(\bar{\nabla}_U T_1) V = A_{F_1 V} U + t_1 B(U, V), \tag{3.4}$$

$$(\bar{\nabla}_U T_2) V = A_{F_2 V} U + t_2 B(U, V), \tag{3.5}$$

where $(\bar{\nabla}_U T_1) = \nabla_U T_1 V - T_1 \nabla_U V$.

LEMMA 3.2. *If T_1 (respectively, T_2) is parallel, then*

$$t_1 B(X, U) = 0 \quad (\textit{respectively, } t_2 B(Z, U) = 0) \tag{3.6}$$

for $X \in \mathscr{D}$ *and* $Z \in \mathscr{D}^\perp$. *If both T_1 and T_2 are parallel, then*

$$B(\mathscr{D}, \mathscr{D}^\perp) = \{0\}. \tag{3.7}$$

Proof. Putting $V = X$ in (3.4) or $V = Z$ in (3.5), we have (3.6) immediately because of $F_1 X = T_2 X = 0$ and $F_2 Z = T_1 Z = 0$. Then, for $W \in \mathscr{D}^\perp$ and $Y \in \mathscr{D}$, we have

$$\langle B(X, Z), \phi_1 W \rangle = -\langle t_1 B(X, Z), W \rangle = 0,$$

$$\langle B(X, Z), \phi_2 Y \rangle = -\langle t_2 B(X, Z), Y \rangle = 0.$$

From these equalities and (3.3) we have (3.7).

LEMMA 3.3. *If M is a symmetric twofold CR product, then both T_1 and T_2 are parallel.*

Proof. Since M is a symmetric twofold CR product, for $Y \in \mathscr{D}$ and $Z \in \mathscr{D}^\perp$ and a tangent vector field U to M we have $\nabla_U Y \in \mathscr{D}$ and $\nabla_U Z \in \mathscr{D}^\perp$. Now, we obtain

$$\tilde{\nabla}_U \phi_1 Y = \nabla_U T_1 Y + B(U, T_1 Y).$$

On the other hand, we have

$$\tilde{\nabla}_U \phi_1 Y = \phi_1 \tilde{\nabla}_U Y = \phi_1 \nabla_U Y + \phi_1 B(U, Y).$$

Since $t_1 B(U, Y) \in \mathscr{D}^\perp$ by (a) of (2.16), taking the tangent part, we have $\nabla_U T_1 Y = T_1 \nabla_U Y$, which implies $(\bar{\nabla}_U T_1) Y = 0$. This, together with $(\bar{\nabla}_U T_1) Z = 0$ $(Z \in \mathscr{D}^\perp)$, implies $\bar{\nabla} T_1 = 0$. Similarly we have $\bar{\nabla} T_2 = 0$.

THEOREM 3.3. *Let M be a proper symmetric twofold CR product of type (ϕ_1, ϕ_2) in R^{4m}. Then M is a Riemannian product of a ϕ_1-invariant submanifold M^T in some complex linear space C^d and a ϕ_2-invariant submanifold $M^\perp$ in C^{2m-d} locally, i.e.,*

$$M = M^\mathrm{T} \times M^\perp \subset C^d \times C^{2m-d} = R^{4m}$$

and M is minimal in R^{4m}.

Proof. The minimality is already done (Theorem 3.2). Since M is a symmetric twofold CR product, we see that T_1 and T_2 are parallel (Lemma 3.3). Then we see that $B(\mathscr{D}, \mathscr{D}^\perp) = \{0\}$ (Lemma 3.1). Since the ambient manifold is a Euclidean space R^{4m}, our assertion follows by applying a lemma of Moore [11].

THEOREM 3.4. *Let M be a $2n$-dimensional complete twofold CR product in R^{4m}. If $T_i A_N = A_N T_i$ $(i = 1, 2)$, then $M = R^{2n}$.*

Proof. Since $B(\mathscr{D}, \mathscr{D}^\perp) = \{0\}$, it suffices to show that

$$B(\mathscr{D}, \mathscr{D}) = \{0\} \quad \text{and} \quad B(\mathscr{D}^\perp, \mathscr{D}^\perp) = \{0\}.$$

Since $T_1 A_N U = A_N T_1 U$ for a tangent vector field U, putting $U = X \in \mathscr{D}$ and then making an inner product of this with $Y \in \mathscr{D}$, we have

$$\langle T_1 A_N X, Y \rangle = \langle A_N T_1 X, Y \rangle,$$

whence

$$\langle B(X, T_1 Y) + B(Y, T_1 X), N \rangle = 0.$$

Since N is an arbitrary normal vector field to M, we have

$$B(X, T_1 Y) + B(Y, T_1 X) = 0.$$

This, together with $B(X, T_1 Y) - B(Y, T_1 X) = 0$, implies

$$B(X, T_1 Y) = 0.$$

Replacing Y by $T_1 Y$ we have $B(X, Y) = 0$, i.e.,

$$B(\mathscr{D}, \mathscr{D}) = \{0\}.$$

Similarly, using $T_2 A_N = A_N T_2$, we have

$$B(\mathscr{D}^{\perp}, \mathscr{D}^{\perp}) = \{0\},$$

which completes the proof.

REFERENCES

[1] M. Barros, B. Y. Chen and F. Urabana, *Quaternion CR submanifolds of quaternion manifolds*, Kōdai Math. J. 4 (1981), pp. 399–417.

[2] A. Bejancu, *CR submanifolds of a Kaehlerian manifolds. I*, Proc. Amer. Math. Soc. 69 (1978), pp. 135–142.

[3] – *CR submanifolds of a Kaehlerian manifold. II*, Trans. Amer. Math. Soc. 250 (1979), pp. 333–345.

[4] – *Umbilical semi-invariant submanifolds of a Sasakian manifold*, Tensor N. S. 37 (1982), pp. 203–212.

[5] B. Y. Chen, *CR submanifolds of a Kaehler manifold. I*, J. Differential Geom. 16 (1981), pp. 305–322.

[6] – *CR submanifolds of a Kaehler manifold. II*, ibidem 16 (1981), pp. 493–509.

[7] – *Geometry of Submanifolds and its Applications*, Sci. Univ. Tokyo, 1981.

[8] S. Ishihara, *Quaternion Kaehler manifolds*, J. Differential Geom. 9 (1974), pp. 483–500.

[9] M. Kobayashi, *CR submanifolds of a Sasakian space form with flat normal connection*, Tensor N. S. 36 (1982), pp. 207–214.

[10] – *3-contact CR submanifolds of manifolds with Sasakian 3-structure*, ibidem 40 (1983), pp. 57–69.

[11] J. D. Moore, *Isometric immersions of Riemannian products*, J. Differential Geom. 5 (1971), pp. 159–168.

[12] K. Yano and M. Kon, *Differential geometry of CR submanifolds*, Geom. Dedicata 10 (1981), pp. 369–391.

[13] – *Contact CR submanifolds*, Kōdai Math. J. 5 (1982), pp. 238–252.

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