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ON WEAKLY ZERO-DIMENSIONAL MAPPINGS

BY

K. KUPERBERG AND W. KUPERBERG (WARSZAWA)

A continuous mapping $f\colon X \to Y$ is said to be *zero-dimensional* if each counter image $f^{-1}(y)$, where $y \in f(X)$, has dimension zero; it is said to be *weakly zero-dimensional* if each counter image $f^{-1}(y)$, where $y \in f(X)$, contains a point x at which it is zero-dimensional, i.e. $\dim_x f^{-1}(y) = 0$.

A finite-dimensional connected compact metric space X is called a *Cantorian manifold* if X remains connected after removing any subset of dimension less than $\dim X - 1$.

In [3], Lelek posed the following question (P 614, p. 316): does there exist a Cantorian manifold X and a weakly zero-dimensional mapping of X onto Y such that $\dim Y < \dim X$?

The answer is affirmative. Indeed, let X, Y and $f\colon X \to Y$ be defined as follows: X is the subset of the euclidean plane

$$X = \left\{(x, y)\colon\ 0 < x \leqslant 1,\ \left(\sin\frac{1}{x}\right) - x \leqslant y \leqslant \left(\sin\frac{1}{x}\right) + x,\ -1 \leqslant y \leqslant 1\right\} \cup$$

$$\cup \{(x, y)\colon\ x = 0,\ -1 \leqslant y \leqslant 1\},$$

Y is the segment $\langle -1, 1\rangle$, and $f\colon X \to Y$ is the projection defined by $f(x, y) = y$. It is easy to see that X, Y and $f\colon X \to Y$ satisfy all the requirements of conjecture.

The space X in the above example is not locally connected. But it is possible to construct a weakly zero-dimensional mapping of the two-dimensional disk Q^2 onto a segment; moreover, the following theorem is true:

THEOREM. *If X is a compact space which contains topologically a Cantor set as a G_δ-subset and Y is a non-degenerate locally connected continuum, then there exists a weakly zero-dimensional mapping of X onto Y.*

The aim of this note is to prove this theorem.

First we prove some lemmas.

LEMMA 1. *Let f be a continuous mapping of a compact space X into a space Y and let $x \in X$. If $\dim_{f(x)} Y = \dim_x f^{-1}(f(x)) = 0$, then $\dim_x X = 0$.*

Proof. We shall prove that the quasicomponent C of x (i.e., the intersection of all closed-open subsets containing x) consists of one point x only. Indeed, by the compactness of X, C is connected, hence $C \subset f^{-1}(f(x))$, because $\dim_{f(x)} Y = 0$. Since $\dim_x f^{-1}(f(x)) = 0$ and $x \in C$, we have $C = \{x\}$. Thus the point x has arbitrarily small closed-open neighbourhoods in X, because X is compact, and the proof is completed.

COROLLARY 1. *If X and Y are compact, $f: X \to Y$, $g: Y \to Z$ are continuous mappings, $y = f(x)$, $z = g(y)$ and $\dim_y g^{-1}(z) = \dim_x f^{-1}(y) = 0$, then $\dim_x (gf)^{-1}(z) = 0$.*

In particular, if f and g are weakly zero-dimensional mappings, then also the composition $g \circ f$ is weakly zero-dimensional.

Denote by I the unit segment of the real line, by D the standard Cantor set and by $J_1, J_2, \ldots$ the segments which are the closures of components of the complement of D in I. Let K_i be a plane circular disk the diameter of which is J_i $(i = 1, 2, \ldots)$, and let K be the union of all K_i. Let us define four subsets of the euclidean plane as follows:

$$Q = \{(x, y): 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\},$$

$$Q' = \overline{Q - K},$$

$$R = \{(r, \varphi): 1 \leqslant r \leqslant 2,\ 0 \leqslant \varphi \leqslant 2\pi\},$$

$$S = \{(r, \varphi): r = 1,\ 0 \leqslant \varphi \leqslant 2\pi\},$$

where the pairs (x, y) and (r, φ) are cartesian and polar coordinates, respectively (see Fig. 1).

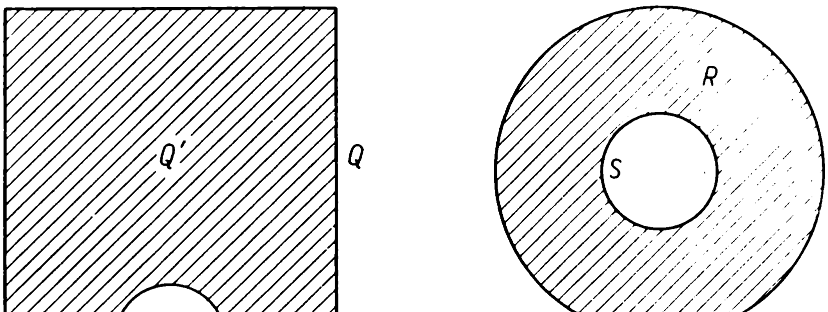

Fig. 1

LEMMA 2. *There exists a mapping f of Q' onto S such that $f(D) = S$ and, for each $y \in S$, if $x \in D \cap f^{-1}(y)$, then $\dim_x f^{-1}(y) = 0$.*

Proof. Let $\bar{f}_1$ be the well-known mapping of D onto I (see, e.g., [1], p. 186). Denote by $\bar{\bar{f}}_1: I \to I$ the extension of $\bar{f}_1$ which maps the segment J_i into the point being the image of its ends. There is an extension

$f_1\colon Q \overset{\text{onto}}{\to} Q$ of $\bar{\bar{f}}_1$ which is one-to-one on $Q-I$. To prove this, let us take (see Fig. 2)

$$f_1(x, y) = \big((1-y)\bar{\bar{f}}_1(x)+xy,\, y\big) \quad \text{for all } (x, y)\in Q.$$

Let $f_2\colon Q \to R$ be defined by $f_2(x, y) = \big(r(y), \varphi(x)\big)$, where $r(y) = y+1$, $\varphi(x) = 2\pi x$ (see Fig. 3).

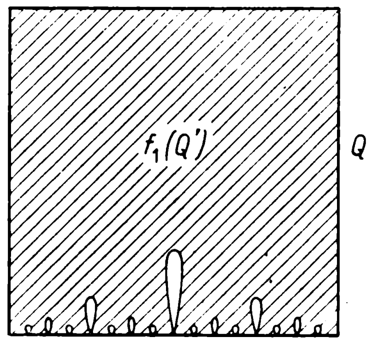

Fig. 2

The set $R-f_2f_1(Q')$ has infinitely many components, say $P_1, P_2, \ldots$; let f_3 be a homeomorphism of R onto R, which does not move the points of S and such that each ray whose endpoint is the origin intersects infinitely many of the sets $f_3(P_i)$ (see Fig. 4). Finally, we define $f_4\colon R\to S$ by $f_4(r, \varphi) = (1, \varphi)$. We shall show that $f = (f_4\circ f_3\circ f_2\circ f_1)|_{Q'}$ is a required

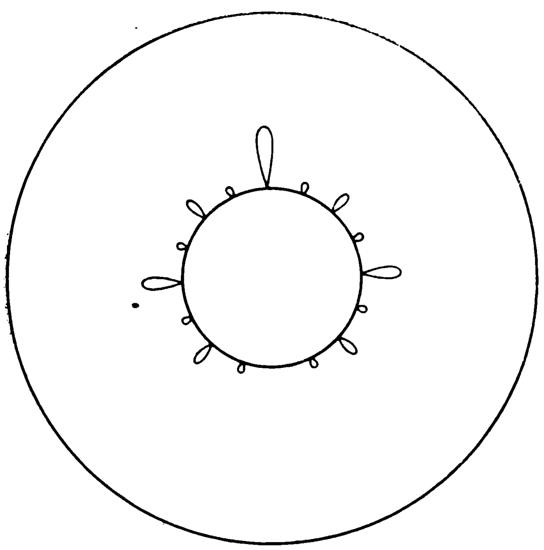

Fig. 3

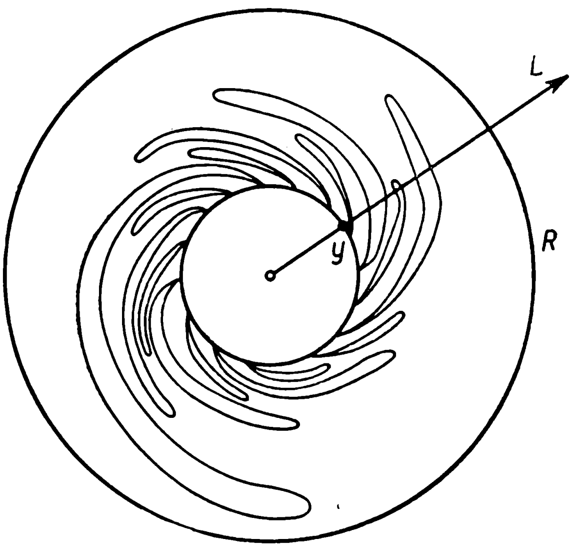

Fig. 4

mapping. It is clear that $f(D) = S$. Let $y\in S$ and $x\in D\cap f^{-1}(y)$. The set $(f_3\circ f_2\circ f_1)^{-1}(y)\cap Q'$ is finite, hence, by Corollary 1, in order to prove $\dim_x f^{-1}(y) = 0$ it is sufficient to show that $\dim_y f_4^{-1}(y) = 0$. But $f_4^{-1}(y)$ lies on a segment L which intersects infinitely many of sets $f_3(P_i)$, the sequence of diameters of $f_3(P_i)$ converges to zero, thus $L - f_4^{-1}(y)$ contains a sequence of points converging to y. This implies that $\dim_y f_4^{-1}(y) = 0$ which completes the proof.

LEMMA 3. *If Y is a non-degenerate locally connected continuum, then there exists a zero-dimensional mapping ψ of the segment I onto Y (i.e., there exists a ψ such that, for each $y \in Y$, $\dim \psi^{-1}(y) = 0$).*

Proof. Let φ be a continuous mapping of I onto Y, and let $\{A_i\}_{i=1}^{\infty}$ be the collection of all maximal subsegments of I, which φ maps into a point. We write $B = I - \bigcup_{i=1}^{\infty} \operatorname{Int} A_i$. Clearly, $\varphi(B) = Y$. Since Y is non-degenerate, there exist $\bar{\varphi}\colon B \xrightarrow{\text{onto}} I$ and $\psi\colon I \to Y$ such that $\psi\bar{\varphi} = \varphi|_B$. For each $y \in Y$ the set $\psi^{-1}(y)$ contains no segment. Thus ψ is a zero-dimensional mapping.

Proof of the theorem. By the assumption of the theorem, the compact space X contains a G_δ-subset D' homeomorphic with the Cantor set D. Then there exists a continuous mapping $\bar{g}\colon X \to [0, \frac{1}{2}]$, such that $\bar{g}^{-1}(0) = D'$ (see [2], p. 134). Let $h\colon D' \to D$ be a homeomorphism of D' onto D. Then h can be extended to a continuous mapping $\bar{\bar{g}}\colon X \to Q'$, because Q' is an absolute retract. We can assume, for $(x, y) \in \bar{\bar{g}}(X)$, that $y \leqslant \frac{1}{2}$. Now, let $g\colon X \to Q'$ be defined by $g(z) = \bar{\bar{g}}(z) + (0, \bar{g}(z))$. By the definition, $g^{-1}(D) = D'$ and, by Corollary 1, the composition $f \circ g$ (where f is the mapping from Lemma 2) is a weakly zero-dimensional mapping of X onto S. It is easy to see that there exists a weakly zero-dimensional mapping h of the circle S onto the segment I, and, by Lemma 3, there exists a weakly zero-dimensional mapping of I onto Y. The composition $\psi \circ h \circ f \circ g$ is a weakly zero-dimensional mapping of X onto Y. This completes the proof.

COROLLARY 2. *If there exists a continuous mapping of a compact metric space X onto the segment, then, for each non-degenerate locally connected continuum Y, there exists a weakly zero-dimensional mapping of X onto Y.*

REFERENCES

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[2] — *Topology*, Vol. 1, Warszawa 1966.

[3] A. Lelek, *On mappings on Cantorian manifolds*, Colloquium Mathematicum 17 (1967), p. 315–318.

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