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ON GENERALIZED PROJECTIVE CLASSES

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In [3] a relation between similar structures was introduced: A^* is said to be *elementarily lifted over* A, $A < A^*$, iff every projective class $\mathscr{P}$ containing A also contains A^* (for projective classes, see [4], [1] and [3]; we use the notations as in [3]). In this paper we generalize this notion to a relation between subclasses $\mathfrak{A}$ of the similarity class $\mathfrak{A}_\varrho$ and structures A of type ϱ and we describe the closed classes by means of ultraproducts. The situation is quite similar to the well-known case where one has the elementary equivalence $\equiv$ as a relation between structures, an induced compact closure on $\mathfrak{A}_\varrho$ as a relation between the subclasses and the structures and, finally, a description of this closure by the ultraproduct theorem.

Let $\mathfrak{A}$ be an arbitrary subclass of $\mathfrak{A}_\varrho$. We say that A^* *belongs to the projective closure of* $\mathfrak{A}$, $A^* \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$, iff every projective class $\mathscr{P}$ which contains $\mathfrak{A}$ contains A^*. So we have $A^* \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$ iff $\mathscr{C}_{\mathscr{P}}(A^*) \subseteq \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$, where $\mathscr{C}_{\mathscr{P}}(A^*)$ is the class of all elementarily lifted extensions of A^* or, equivalently [3], the class of extensions of A^* by diagonal-like embeddings.

THEOREM 1. *$\mathscr{C}_{\mathscr{P}}$ defines a topological closure on the class $\mathfrak{A}_\varrho$ of all relational systems of type ϱ.*

Proof. Obviously, $\mathscr{C}_{\mathscr{P}}$ is a closure operator:

(α) $\mathscr{C}_{\mathscr{P}}(\mathfrak{A}) \supseteq \mathfrak{A}$,

(β) $\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \frown \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_1) \supseteq \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_2)$,

(γ) $\mathscr{C}_{\mathscr{P}}\mathscr{C}_{\mathscr{P}}\mathfrak{A} = \mathscr{C}_{\mathscr{P}}\mathfrak{A}$.

We show that $\mathscr{C}_{\mathscr{P}}$ is topological:

(δ) $\mathscr{C}_{\mathscr{P}}(\mathfrak{A}_1 \cup \mathfrak{A}_2) = \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_1) \cup \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_2)$.

Clearly, $\mathscr{C}_{\mathscr{P}}(\mathfrak{A}_1 \cup \mathfrak{A}_2) \supseteq \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_1) \cup \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_2)$ by (β); let $A' \notin \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_1) \cup \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_2)$. By definition, we then have projective classes $\mathscr{P}_1$ and $\mathscr{P}_2$ such that $\mathfrak{A}_1 \subseteq \mathscr{P}_1$ but $A' \notin \mathscr{P}_1$; $\mathfrak{A}_2 \subseteq \mathscr{P}_2$ but $A' \notin \mathscr{P}_2$. Let $\mathscr{P}_i = \mathfrak{A}\Gamma_i\mathfrak{B}_i$ where the axiomatic correspondence between $\mathfrak{A}$ and the axiomatic class $\mathfrak{B}_i$ of structures of type σ_i is given by a set K_i of sentences in $L_{\tau_i} \supseteq L_{[\varrho,\sigma_i]}$,

$$A \in \mathscr{P}_i \frown (A \dot{\cup} B_i)^{\wedge} \models K_i, \quad i = 1, 2,$$

for suitable $B_i \in \mathfrak{B}_i$ and for suitable interpretations of the connecting relational symbols occurring in Θ_i (see [3]).

Consider the set $K_{\mathfrak{A},\mathfrak{B}_1,\mathfrak{B}_2} \in L_{[\varrho,\sigma_1,\sigma_2]}$, where

$$C \in \mathfrak{L} = \mathrm{Md}(K_{\mathfrak{A},\mathfrak{B}_1,\mathfrak{B}_2}) \frown C = A \dot{\cup} B_1 \dot{\cup} B_2, \qquad A \in \mathfrak{A},\ B_i \in \mathfrak{B}_i.$$

We relativize all the relational symbols in K_1 to A and B_1 and those in K_2 to A and B_2. If Φ is a theorem in K_i, we denote by $\overline{\Phi}$ the transformed theorem. Then the set

$$\overline{K} = \{\overline{\Phi}_1 \vee \overline{\Phi}_2 \mid \Phi_i \in K_i\}$$

defines an axiomatic relation Γ between $\mathfrak{A}$ and $\mathfrak{B}$. Clearly, $\mathfrak{A}_1 \subseteq \mathscr{P}$ and $\mathfrak{A}_2 \subseteq \mathscr{P}$, $\mathscr{P} = \mathfrak{A}\Gamma\mathfrak{B}$.

Assume $A' \in \mathscr{P}$. We then would have

$$(A' \dot{\cup} B_1 \dot{\cup} B_2)^{\wedge} \models \overline{K}.$$

But $(A' \dot{\cup} B_i)^{\wedge} \models \overline{K}_i$ is false. Therefore, there are $\Phi_i \in K_i$ such that

$$(A \dot{\cup} B_i)^{\wedge} \models \neg \overline{\Phi}_i, \qquad i = 1, 2.$$

But then $(A' \dot{\cup} B_1 \dot{\cup} B_2)^{\wedge} \models \overline{K}$ cannot be.

We call a class $\mathfrak{A}$ *generalized projective* iff it is closed with respect to $\mathscr{C}_{\mathscr{P}}$: $\mathfrak{A}$ is generalized projective iff $\mathfrak{A}$ is the intersection of all projective classes containing $\mathfrak{A}$. This is clear because the projective classes are a basis for the closure $\mathscr{C}_{\mathscr{P}}$.

COROLLARY 1. *Generalized projective classes are closed under ultraproducts and under elementarily lifted extensions (for example, ultrapowers and ultralimits).*

COROLLARY 2. *Let E be a finite set of models, $E = \{A_1, \ldots, A_n\}$. Then $A' \in \mathscr{C}_{\mathscr{P}}(E)$ iff A' is an elementarily lifted extension of one of the A_i.*

A closure on $\mathfrak{A}_\varrho$ is called *compact* iff any set of closed classes with the finite intersection property has a non-empty intersection.

COROLLARY 3. *$\mathscr{C}_{\mathscr{P}}$ is compact.*

Proof. A projective class is closed under ultraproducts and it is well known that the classes closed under the formation of ultraproducts define a compact topology $\mathscr{C}_{\mathfrak{U}}$ on $\mathfrak{A}_\varrho$. But $\mathscr{C}_{\mathscr{P}}$ is finer than $\mathscr{C}_{\mathfrak{U}}$, so $\mathscr{C}_{\mathscr{P}}$ is also compact.

Note that the closure defined by the elementary classes is compact in a much stronger sense. Every class of axiomatic classes with the finite intersection property has a non-empty intersection. This no longer holds for $\mathscr{C}_{\mathscr{P}}$. Let $\mathscr{P}$ be any projective class and denote by $\mathscr{P}_a$ the projective class of all models in $\mathscr{P}$ of a cardinal greater or equal to a. Clearly, $\mathscr{P}_a$, a cardinal, has the finite intersection property but the intersection over all $\mathscr{P}_a$ is empty. Similarly, the underlying class for the Kolmogoroff contraction of $(\mathfrak{A}_\varrho, \mathscr{C}_\mathscr{P})$ is not a set. Define in $\mathfrak{A}_\varrho$ an equivalence by $A \sim_\mathscr{P} A'$ iff $\mathscr{C}_\mathscr{P}(A) = \mathscr{C}_\mathscr{P}(A')$. $\mathfrak{A}_\varrho | \sim_\mathscr{P}$ with the quotient topology is the T_0-space of the projective types in $\mathfrak{A}_\varrho$. Two models of the same projective type are elementarily equivalent and of the same cardinal.

PROPOSITION 1. *Let $\mathfrak{A}$ be a generalized projective class and assume that $\mathfrak{A}$ is categorical for all cardinals greater or equal to $\mathfrak{m}$. This implies that if $A \in \mathfrak{A}$ is of a cardinal greater or equal to $\mathfrak{m}$, then A' is projectively equivalent to A iff A' and A are isomorphic.*

For algebraically closed fields, one even infers that two projectively equivalent fields have the same transcendence degree over the common prime field, and so they are isomorphic.

PROPOSITION 2. *A generalized projective class $\mathfrak{A}$ is connected iff all models in $\mathfrak{A}$ are elementarily equivalent. The connected components of a generalized projective class $\mathfrak{A}$ are just the classes of elementarily equivalent models in $\mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be a class in $\mathfrak{A}_\varrho$ and assume $A_i \in \mathfrak{A}$, $i = 1, 2$, are not elementarily equivalent. Then there is a theorem $p \in L_\varrho$ where $A_1 \models p$ and $A_2 \models \neg p$. But then

$$\mathfrak{A} = (\mathfrak{A} \cap \mathrm{Md}(p)) \dot{\cup} (\mathfrak{A} \cap \mathrm{Md}(\neg p))$$

shows that $\mathfrak{A}$ is disconnected. Let $\mathfrak{A}$ be a generalized projective class of elementarily equivalent models and assume $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$ with generalized projective classes $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Then if $A_i \in \mathfrak{A}_i$, $i = 1, 2$, one has isomorphic ultralimits $A_{i\omega}$ of A_i and, therefore, $\mathfrak{A}_1 \cap \mathfrak{A}_2 \neq \emptyset$, $\mathfrak{A}$ is connected. A connected component $\mathfrak{A}_K$ of a generalized projective class $\mathfrak{A}$ is generalized projective, so

$$\mathfrak{A}_K \subseteq \mathrm{Md}(K) \cap \mathfrak{A} = \overline{\mathfrak{A}},$$

where K is the complete theory of a model in $\mathfrak{A}_K$. But $\overline{\mathfrak{A}}$ is connected, so $\mathfrak{A}_K = \overline{\mathfrak{A}}$.

The following theorem is similar to the ultraproduct theorem:

THEOREM 2. *Let $\mathfrak{A}$ be an arbitrary class of relational structures of type ϱ. Then $A' \in \mathfrak{A}_\varrho$ belongs to the generalized projective class $\mathscr{C}_\mathscr{P}(\mathfrak{A})$ iff there is a subset γ of $\mathfrak{A}$ and an ultrafilter $\mathfrak{u}$ on γ such that every projective class containing $\mathfrak{u}$-almost all $A \in \gamma$ contains A'.*

$\mathscr{P}$ contains $\mathfrak{u}$-almost all $A \in \gamma$ means $\{A \mid A \in \mathscr{P} \cap \gamma\} \in \mathfrak{u}$.

Proof. Assume first that, for $A' \in \mathfrak{A}_\varrho$, one has a subset γ of $\mathfrak{A}$ with the properties described in the theorem. Then, clearly, $A \in \mathscr{C}_\mathscr{P}(\gamma)$ because a projective class $\mathscr{P}$ containing γ contains $\mathfrak{u}$-almost all $A \in \gamma$ and, therefore, $A' \in \mathscr{P}$. Then, $\mathscr{C}_\mathscr{P}(\gamma) \subseteq \mathscr{C}_\mathscr{P}(\mathfrak{A})$ yields $A' \in \mathscr{C}_\mathscr{P}(\mathfrak{A})$.

Assume now, conversely, $A' \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$. Then we have $A' \in \mathscr{C}_{\mathscr{P}}(\gamma)$ for a subset γ of $\mathfrak{A}$. Let, namely, $\operatorname{card}(A) = \alpha$ and let

$$\mathfrak{A}_\alpha = \{A \mid A \in \mathfrak{A},\ \operatorname{card} A \geqslant \alpha\},$$
$$\mathfrak{A}_{\alpha^+} = \{A \mid A \in \mathfrak{A},\ \operatorname{card} A \geqslant \alpha^+\},$$

α^+ successor of α. The isomorphism types in $\mathfrak{A}_\alpha$ form a set represented by a subset γ of $\mathfrak{A}$ and it is easy to see that all models in $\mathscr{C}_{\mathscr{P}}(\mathfrak{A}_{\alpha^+})$ are of a cardinal greater or equal to α^+. So

$$\mathscr{C}_{\mathscr{P}}(\mathfrak{A}) = \mathscr{C}_{\mathscr{P}}(\gamma) \cup \mathscr{C}_{\mathscr{P}}(\mathfrak{A}_{\alpha^+})$$

and $A' \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$ yields $A' \in \mathscr{C}_{\mathscr{P}}(\gamma)$.

Consider the neighbourhood filter $\mathfrak{B}(A')$ of A'. A basis for $\mathfrak{B}(A')$ is given by the complements $\mathfrak{A}_\varrho - \mathscr{P} = \overline{\mathscr{P}}$ of all projective classes $\mathscr{P}$ not containing A'. If $A' \in \overline{\mathscr{P}}$, then $\overline{\mathscr{P}} \cap \gamma \neq \emptyset$, otherwise $\mathscr{P} \supseteq \gamma$ and, therefore, $A' \in \mathscr{P}$. Moreover,

$$F_{\mathscr{P}} = \{A \mid A \in \gamma,\ A \notin \mathscr{P}\}, \qquad A' \notin \mathscr{P},$$

is a filterbasis $\mathfrak{f}$ on γ. $\mathfrak{f}$ is finer than $\mathfrak{B}(A')$, so A' is a limit point for $\mathfrak{f}$. Now, there is an ultrafilter $\mathfrak{u}$ on γ finer than $\mathfrak{f}$ and also converging to A'. If $\mathscr{P}$ is any projective class containing $\mathfrak{u}$-almost all $A \in \gamma$, then $A' \in \mathscr{P}$, otherwise $F_{\mathscr{P}} \in \mathfrak{u}$ would yield a contradiction.

COROLLARY 1. *$A' \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$ iff $A' \in \mathscr{C}_{\mathscr{P}}(\gamma)$ for some subset γ of $\mathfrak{A}$.*

We call a projective class $\mathscr{P}$ *compatible with an ultraproduct*

$$A = \mathop{\mathsf{X}}_{i \in I} A_i \mid \mathfrak{u}$$

iff $\mathscr{P}$ contains $\mathfrak{u}$-almost all factors A_i. Clearly, $A \in \mathscr{P}$ if $\mathscr{P}$ is $\mathfrak{u}$-compatible with A.

COROLLARY 2. *$A' \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$ iff there is an ultraproduct*

$$A = \mathop{\mathsf{X}}_{i \in I} A_i \mid \mathfrak{u}$$

with factors $A_i \in \mathfrak{A}$ such that any projective class which is $\mathfrak{u}$-compatible with A contains A'.

Proof. Let $A' \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$ and choose γ and $\mathfrak{u}$ according to the theorem. Then, if $\mathscr{P}$ is any projective class, $\mathfrak{u}$-compatible with

$$\overline{A} = \mathop{\mathsf{X}}_{A \in \gamma} A \mid \mathfrak{u},$$

we have $A' \in \mathscr{P}$. Otherwise, we would have $F_{\mathscr{P}} \in \mathfrak{u}$, a contradiction because $\mathfrak{u}$ is proper. Now, assume that there is an ultraproduct

$$\overline{A} = \mathop{\mathsf{X}}_{i \in I} A_i \mid \mathfrak{u}$$

with the properties of the corollary. Consider $\gamma = \{A \mid A = A_i$ for some $i \in I\}$ and the ultrafilter on γ corresponding to $\mathfrak{u}$ call it also $\mathfrak{u}$. Then, if $\mathscr{P}$ is any projective class containing $\mathfrak{u}$-almost all $A \in \gamma$, we get that $\mathscr{P}$ is compatible with $\bar{A}$ and, therefore, $A' \in \mathscr{P}$. So $A' \in \mathscr{C}_{\mathscr{P}}(\mathfrak{A})$ by the theorem.

The corollary suggests a relation $\underset{\mathfrak{u}}{\prec}$, $A \underset{\mathfrak{u}}{\prec} A'$ iff A is an ultraproduct with an ultrafilter $\mathfrak{u}$ and any $\mathfrak{u}$-compatible projective class of A contains A'.

COROLLARY 3. *$\mathfrak{A}$ is generalized projective iff $\mathfrak{A}$ is closed under the relation* $\underset{\mathfrak{u}}{\prec}$.

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