COLLOQUIUM MATHEMATICUM

VOL. XXVIII 1973 FASC. 2

LENGTH-PRESERVING DEFORMATIONS OF CLOSED PLANE CURVES

BY

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1. Introduction. Let $C = \{s \bmod L : s \in E\}$, where E denotes the real numbers, be a simple closed curve. Let

$$\varphi: C \to E^2 \tag{1.1}$$

be an immersion of class C^r $(r \geqslant 2)$ of C into the Euclidean plane E^2 such that

$$\int_0^L k(s)\,ds = 2\pi n, \quad \text{where } 1 \leqslant n < \infty \text{ and } k(s) > 0 \text{ for } n \geqslant 2, \tag{1.2}$$

$k(s)$ denoting the curvature as a function of arc-length parameter s of $\varphi(C)$. In the case $n = 1$ the mapping (1.1) is supposed to be an imbedding of class $C^r (r \geqslant 2)$. In the sequel (1.1) is called a *closed plane curve*.

The purpose of this paper is to give a construction of a homotopy of class C^r if φ is an immersion (resp. an isotopy of the same class if φ is an imbedding) with the following properties: if

$$\varphi(s, \tau) = \big(x(s, \tau), y(s, \tau)\big), \quad 0 \leqslant \tau \leqslant \infty, \tag{1.3}$$

denotes this homotopy (resp. isotopy), then

$$\varphi(s, 0) = \varphi(s) \tag{1.4}$$

is the immersion (resp. imbedding) (1.1) and

$$\varphi(s, \infty) = \left(a + \frac{L}{2\pi n}\cos\frac{2\pi n}{L}s,\ b + \frac{L}{2\pi n}\sin\frac{2\pi n}{L}s\right) \tag{1:5}$$

and

$$|\dot{\varphi}(s, \tau)| = 1 \tag{1.6}$$

for every τ, $0 \leqslant \tau \leqslant \infty$, where $\dot{\varphi}(s, \tau) = \partial\varphi/\partial s$.

A homotopy (1.3) which satisfies (1.6) is called *length-preserving*.

By

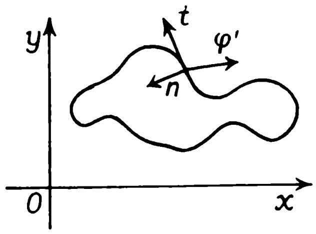

Fig. 1

$$t(s,\tau) = \dot{\varphi}(s,\tau), \qquad n(s,\tau) = (-\dot{y}, \dot{x})$$

we denote the unit tangent and unit normal vector field of immersion (resp. imbedding) $\varphi_\tau(s) = \varphi(s,\tau)$. For s fixed we denote by $\alpha(s,\tau)$, $\beta(s,\tau)$ coordinates of the vector $\varphi_s'(\tau) = \varphi'(s,\tau)$ tangent to the curve $\varphi_s(\tau) = \varphi(s,\tau)$ with respect to the moving frame t, n (Fig. 1). Hence we have

$$\varphi'(s,\tau) = \alpha(s,\tau)t(s,\tau) + \beta(s,\tau)n(s,\tau). \tag{1.7}$$

2. Length-preserving homotopies. In coordinates, system (1.7) has the form

$$\begin{aligned} x' &= \alpha\dot{x} - \beta\dot{y}, \\ y' &= \beta\dot{x} + \alpha\dot{y}. \end{aligned} \tag{2.1}$$

The homotopy (1.3) is called *regular* if

$$x'^2 + y'^2 = \alpha^2 + \beta^2 > 0. \tag{2.2}$$

From (2.1) we have

$$\begin{aligned} \alpha &= \dot{x}x' + \dot{y}y', \\ \beta &= \dot{x}y' - \dot{y}x'. \end{aligned} \tag{2.3}$$

Using (2.3) and the Frenet formulas $\dot{x} = -k\dot{y}$, $\dot{y} = k\dot{x}$, we get for a length-preserving homotopy

$$\dot{\alpha} = k\beta, \tag{2.4}$$

$$k' = (\dot{\beta} + \alpha k)^{\cdot}. \tag{2.5}$$

A vector field (α, β) which satisfies (2.4) and (2.5) is called *length-preserving*.

In section 3 we prove that for every length-preserving vector field (α, β) there exists a solution $\varphi(s,\tau)$ of (1.6) for which $k(s,\tau)$ is the curvature function and, therefore, $\varphi(s,\tau)$ is length-preserving. Thus (2.4) and (2.5) are necessary and sufficient conditions for a deformation to be length-preserving.

If $k(s,\tau) \neq 0$, then it follows from (2.4) and (2.5) that

$$\ddot{\alpha}k - \dot{\alpha}\dot{k} + \alpha k^3 - k^2\int_0^s k'\,d\sigma = 0, \tag{2.6}$$

$$\ddot{\beta}k - \dot{\beta}\dot{k} + \beta k^3 - kk' + \dot{k}\int_0^s k'\,d\sigma = 0. \tag{2.7}$$

If $k(s, \tau)$ is a periodic function of s with the period L and

$$\int_0^L k' d\sigma = 0, \tag{2.8}$$

then, since α and β appear explicitly in (2.6) and (2.7), their solutions are also periodic functions of s with the period L. General solutions of (2.6) and (2.7) can be written in an explicit form. Namely, homogeneous equations assigned to (2.6) and (2.7) have partial solutions

$$\alpha_1 = -\beta_2 = \cos\int_0^s k(\sigma, \tau)d\sigma, \qquad \alpha_2 = \beta_1 = \sin\int_0^s k(\sigma, \tau)d\sigma.$$

Since

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \dot{\alpha}_1 & \dot{\alpha}_2 \end{vmatrix} = \begin{vmatrix} \beta_1 & \beta_2 \\ \dot{\beta}_1 & \dot{\beta}_2 \end{vmatrix} = k,$$

the special solution of equation (2.6) can be written in the form (after dividing (2.6) by k)

$$\alpha_0 = \alpha_2\int_0^s \Big(\alpha_1\int_0^\sigma k' d\eta\Big)d\sigma - \alpha_1\int_0^s\Big(\alpha_2\int_0^\sigma k' d\eta\Big)d\sigma. \tag{2.9}$$

We have

$$\dot{\alpha}_0 = k\Big(\alpha_1\int_0^s\Big(\alpha_1\int_0^\sigma k' d\eta\Big)d\sigma + \alpha_2\int_0^s\Big(\alpha_2\int_0^\sigma k' d\eta\Big)d\sigma\Big).$$

We write

$$\beta_0 = \alpha_1\int_0^s\Big(\alpha_1\int_0^\sigma k' d\eta\Big)d\sigma + \alpha_2\int_0^s\Big(\alpha_2\int_0^\sigma k' d\eta\Big)d\sigma. \tag{2.10}$$

Immediate verification shows that (2.9) and (2.10) is a solution of the system (2.4) and (2.5). Instead of (2.6) we can start with (2.7) to get a solution of the system (2.4), (2.5). Formulas (2.9) and (2.10) are defined also if $k(s, \tau) = 0$ for some s and, therefore, define a solution of (2.4) and (2.5) also in this case. Thus the assumption $k(s, \tau) \neq 0$ can be neglected.

It follows that if $k(s, \tau)$ is a periodic function with respect to s with the period L, then there exist periodic solutions with respect to s of (2.4) and (2.5). Periodicity of β_0 follows from that of α_0, $k(s, \tau)$ and (2.4).

Let us suppose that $k(s, \tau)$ changes with respect to τ according to the law

$$k'(s, \tau) = \frac{2\pi}{L}n - k(s, \tau), \tag{2.11}$$

where n denotes the integer defined by (1.2). We consider the special solution of (2.11)

$$k(s, \tau) = \frac{2\pi}{L} n + \left(k(s) - \frac{2\pi}{L} n\right) e^{-\tau}, \quad 0 \leqslant \tau \leqslant \infty. \tag{2.12}$$

The solution (2.12) satisfies (1.2) and (2.8). We also have

$$k(s, 0) = k(s), \quad k(s, \infty) = \frac{2\pi}{L} n.$$

Thus, applying the existence theorem of section 3, we have the following theorems:

THEOREM 1 (the homotopy theorem). *If φ is an immersion for which* (1.2) *is satisfied, then there exists a regular length-preserving homotopy $\varphi(s, \tau)$ of class C^r with the curvature function* (2.12) *such that* (1.4) *and* (1.5) *are satisfied.*

THEOREM 2 (the isotopy theorem). *If φ is an imbedding of class C^r ($r \geqslant 2$), then there exists a regular length-preserving isotopy $\varphi(s, \tau)$ of class C^r for which* (2.12) *is the curvature function and $\varphi(s, 0) = \varphi(s)$, $\varphi(s, \infty)$ is a circle of length L.*

As an immediate consequence of Theorem 1 and (2.12) we have

THEOREM 3. *If $k(s)$ is the curvature function of a closed curve with the property* (1.2), *then for every τ, $0 \leqslant \tau \leqslant 1$,*

$$k(s, \tau) = (1-\tau) k(s) + \tau \frac{2\pi}{L} n$$

is a curvature function of a closed (plane) curve.

Theorem 2 will follow from Theorem 1 if we prove

LEMMA 1. *For $n = 1$,* (2.12) *is the curvature function of an imbedded closed plane curve provided $k(s)$ is the curvature function of such a curve.*

Proof. Let us write

$$k_0 = \max_{0 \leqslant s \leqslant L} |k(s)|, \quad \varepsilon_1 = \frac{1}{k_0}.$$

Then the mapping

$$\bar{\varphi}(s, \varepsilon) = \varphi(s) + n\varepsilon, \quad |\varepsilon| < \varepsilon_1, \tag{2.13}$$

where n denotes the unit normal vector to $\varphi(C)$ at $\varphi(s)$, is an immersion as follows from the equality

$$|\dot{\bar{\varphi}}(s, \varepsilon)|^2 = (1 - \varepsilon k)^2.$$

Let $0 < \varepsilon_0 \leqslant \varepsilon_1$ denote a number such that, for $|\varepsilon| < \varepsilon_0$, (2.13) is an imbedding. Since C is compact, ε_0 exists.

We write

$$D = \{x \in E^2 : x = \bar{\varphi}(s', \pm\varepsilon_0) = \bar{\varphi}(s'', \pm\varepsilon_0),\ s', s'' \in C\}.$$

To every point $x \in D$ there correspond exactly two points $s' = s'(x)$ and $s'' = s''(x)$ of C. The tangent lines of $\varphi(C)$ at $\varphi(s')$ and $\varphi(s'')$ are parallel. The points $s', s'' \in C$ divide C into two parts C_1, C_2 with the property

$$C_1 \cup C_2 = C, \quad C_1 \cap C_2 = \{s', s''\}.$$

It follows that

$$\text{length}(C_1),\ \text{length}(C_2) \geqslant \pi\varepsilon_0. \tag{2.14}$$

Indeed, we have

$$\int_{s'}^{s''} k(\sigma)\,d\sigma = \pi.$$

If we suppose to the contrary that $|s''-s'| < \pi\varepsilon_0$, then, for some $\bar{\sigma} \in (s', s'')$, $k(\bar{\sigma})(s''-s') = \pi$ and, therefore, $k(\bar{\sigma}) > 1/\varepsilon_0$, contrary to the definition of ε_0.

Let us observe that if we change φ on C_2, keeping φ unchanged on C_1, in a way that the resulting mapping φ^* is again an imbedding of C for which

$$L_2 = \int_{s'}^{s''} |\dot{\varphi}(\sigma)|\,d\sigma = \int_{s'}^{s''} |\dot{\varphi}^*(\sigma)|\,d\sigma = \text{length of } C_2, \tag{2.15}$$

then the curvature functions $k(s)$ and $k^*(s)$ of φ and φ^*, respectively, coincide on C_1 and, therefore, the curves $\varphi_\tau(C)$ and $\varphi_\tau^*(C)$ with the curvature functions

$$\frac{2\pi}{L} + \left(k(s) - \frac{2\pi}{L}\right)e^{-\tau}, \quad \frac{2\pi}{L} + \left(k^*(s) - \frac{2\pi}{L}\right)e^{-\tau}, \quad 0 \leqslant \tau \leqslant \infty$$

coincide on C_2 provided $\varphi_\tau(s') = \varphi_\tau^*(s')$ and $\dot{\varphi}_\tau(s') = \dot{\varphi}_\tau^*(s')$ (Fig. 2). For instance, the number

$$|\varphi_\tau(s'') - \varphi_\tau(s')| \tag{2.16}$$

is independent of the choice of φ on C_2 if φ remains unchanged on C_1. Thus for the determination of (2.16) it suffices to change φ to a form convenient for calculations.

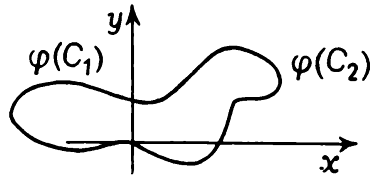

Fig. 2

Let us suppose that $s' = 0$ and $s'' = L_2$, where L_2 is defined by (2.15). We choose a coordinate system in E^2 such that

$$\varphi(0) = (0, 0), \quad \dot{\varphi}(0) = (1, 0).$$

We define a subsidiary imbedding $\tilde{\varphi}\colon C\to E^2$ of class C^1 by the following formulas:

$$\tilde{\varphi}(s)=\begin{cases}(s,0) & \text{for } 0\leqslant s\leqslant \dfrac{L_2-\pi\varepsilon_0}{2},\\ \left(\dfrac{L_2-\pi\varepsilon_0}{2}+\varepsilon_0\cos\dfrac{2s-L_2}{2\varepsilon_0},\ \varepsilon_0+\varepsilon_0\sin\dfrac{2s-L_2}{2\varepsilon_0}\right) & \\ & \text{for } \dfrac{L_2-\pi\varepsilon_0}{2}\leqslant s\leqslant \dfrac{L_2+\pi\varepsilon_0}{2},\\ (s,2\varepsilon_0) & \text{for } \dfrac{L_2+\pi\varepsilon_0}{2}\leqslant s\leqslant L_2,\\ \varphi(s) & \text{for } L_2\leqslant s\leqslant L.\end{cases}$$

It follows from (2.14) that $\tilde{\varphi}$ can be always defined. The curvature function of $\tilde{\varphi}$ is defined by the formulas

$$\tilde{k}(s)=\begin{cases}0 & \text{for } 0\leqslant s\leqslant \dfrac{L_2-\pi\varepsilon_0}{2},\ \dfrac{L_2+\pi\varepsilon_0}{2}\leqslant s\leqslant L_2,\\ \dfrac{1}{\varepsilon_0} & \text{for } \dfrac{L_2-\pi\varepsilon_0}{2}< s< \dfrac{L_2+\pi\varepsilon_0}{2},\\ k(s) & \text{for } L_2<s<L.\end{cases}$$

Using the formula

$$\tilde{\varphi}_\tau(s)=\left(\int_0^s\left[\cos\int_0^\sigma k(\eta,\tau)\,d\eta\right]d\sigma,\ \int_0^s\left[\sin\int_0^\sigma k(\eta,\tau)\,d\eta\right]d\sigma\right),$$

where

$$\tilde{k}(s,\tau)=\frac{2\pi}{L}+\left(\tilde{k}(s)-\frac{2\pi}{L}\right)e^{-\tau},\qquad 0\leqslant s\leqslant L_2 \text{ and } 0\leqslant\tau\leqslant\infty,$$

we get

$$|\tilde{\varphi}_\tau(L_2)-\tilde{\varphi}_\tau(0)|=\frac{2\sin\left[\dfrac{\pi}{L}(1-e^{-\tau})+\dfrac{1}{2}e^{-\tau}\dfrac{1}{\varepsilon_0}\right]\pi\varepsilon_0}{\dfrac{2\pi}{L}(1-e^{-\tau})+e^{-\tau}\dfrac{1}{\varepsilon_0}},$$

(2.16′)
$$|\tilde{\varphi}(L_2)-\tilde{\varphi}(0)|=2\varepsilon_0=|\varphi(L_2)-\varphi(0)|.$$

For every $a>0$, the imbedding $\tilde{\varphi}$ can be approximated by an imbedding $\varphi^*\colon C\to E^2$ of class C^r $(r\geqslant 2)$ such that

$$|\varphi^*(s)-\tilde{\varphi}(s)|<a\quad \text{for } L-a<s\leqslant L \text{ and } 0\leqslant s<L_2+a,$$

$$\varphi^*(s)=\varphi(s)\quad \text{for } L_2+a\leqslant s\leqslant L-a.$$

This can be achieved by smoothing $\tilde{\varphi}$ in sufficiently small neighbourhoods of the points $0, L_2 - \pi\varepsilon_0/2, L_2 + \pi\varepsilon_0/2, L_2$. It follows that the absolute value of the difference of numbers (2.16) and $|\varphi_\tau^*(L_2) - \varphi_\tau^*(0)|$ can be made arbitrarily small and therefore

$$|\tilde{\varphi}_\tau(L_2) - \tilde{\varphi}_\tau(0)| = |\varphi_\tau(L_2) - \varphi_\tau(0)|.$$

Hence, by (2.16), we have for every τ, $0 \leqslant \tau \leqslant \infty$,

$$|\varphi_\tau(L_2) - \varphi_\tau(0)| \geqslant 2\varepsilon_0,$$

i.e., the number $\varepsilon_0(\varphi_\tau(C))$ evaluated for $\varphi_\tau(C)$ is a non-decreasing function of τ. This proves our lemma.

From (2.16) it follows that if $\varepsilon(\tau) = \varepsilon_0(\varphi_\tau(C))$ is defined, then

$$\varepsilon(\tau + \Delta\tau) = \frac{\sin\left[\frac{\pi}{L}(1 - e^{-\Delta\tau}) + \frac{1}{2}e^{-\Delta\tau}\frac{1}{\varepsilon(\tau)}\right]\pi\varepsilon(\tau)}{\frac{2\pi}{L}(1 - e^{-\Delta\tau}) + e^{-\Delta\tau}\frac{1}{\varepsilon(\tau)}}. \tag{2.17}$$

From (2.17) it follows that

$$\frac{d\varepsilon}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\varepsilon(\tau + \Delta\tau) - \varepsilon(\tau)}{\Delta\tau} = -\frac{2\pi}{L}\varepsilon^2(\tau) + \varepsilon(\tau).$$

The solution of the differential equation

$$\frac{d\varepsilon}{d\tau} + \frac{2\pi}{L}\varepsilon^2 - \varepsilon = 0, \qquad \varepsilon(0) = \varepsilon_0,$$

has the form

$$\varepsilon(\tau) = \frac{1}{\frac{2\pi}{L} + \left(\frac{1}{\varepsilon_0} - \frac{2\pi}{L}\right)e^{-\tau}}, \qquad 0 \leqslant \tau \leqslant \infty. \tag{2.18}$$

Formula (2.18) has the following geometrical meaning:

THEOREM 4. *If a circle of radius ε_0 can be freely rolled in a simple closed plane curve $\varphi(C)$ with the curvature function $k(s)$, then a circle of radius $\varepsilon(\tau)$, defined by* (2.18), *can be freely rolled in the simple closed curve $\varphi_\tau(C)$, whose curvature function is defined by* (2.12), *where $n = 1$.*

Using theorem 3 we also have

THEOREM 5 (the four-vertex theorem). *Every imbedded closed plane curve $\varphi(C)$ admits at least four vertices, i.e., points s_i such that $\dot{k}(s_i) = 0$, $i = 1, 2, 3, 4$.*

Proof. If $k(s)$ denotes the curvature function of $\varphi(C)$, then

$$k(s, \tau) = (1 - \tau)k(s) + \tau\frac{2\pi}{L}, \qquad 0 \leqslant \tau \leqslant 1,$$

is positive for τ sufficiently close to 1 and hence it is a curvature function of a convex closed curve. Since $k(s, \tau)$ and $k(s)$ have the same stationary points, the theorem follows from the vertex theorem for convex closed curves.

3. THEOREM 6 (the existence theorem for the system (1.7)). *Let (α, β) be a length-preserving vector field with the curvature function $k(s, \tau)$, $0 \leqslant \tau \leqslant \infty$. Then there exists a function $\varphi(s, \tau)$ defined on $C \times [0, \infty]$, which satisfies* (1.7) *and whose curvature function is $k(s, \tau)$. The solution with the initial condition* (1.4) *is unique.*

First we prove this theorem under the assumption that $k(s, \tau)$ is of class C^∞, hence also $\alpha(s, \tau)$ and $\beta(s, \tau)$ are of class C^∞, and that all derivatives of $k(s, \tau)$, $\alpha(s, \tau)$, $\beta(s, \tau)$ are uniformly bounded by a constant A. The proof is carried out in the following parts. We do not assume that $k(s, \tau)$ is periodic with respect to s.

a. The closed interval $[0, \infty]$ can be endowed with the topology of the interval $[0, 1]$. Let P_n denote a division of $[0, 1]$ by the points k/n, $k = 0, 1, \ldots, n$. We write

$$\varphi_0(s, 0) = \varphi(s), \quad t_0(s, 0) = t(s), \quad n_0(s, 0) = n(s), \tag{3.1}$$

$$\varphi_1(s, h) = \varphi_0(s, 0) + \big(\alpha(s, 0)t_0 + \beta(s, 0)n_0\big)h, \tag{3.2}$$

where $0 \leqslant h \leqslant 1/n$ throughout this section.

If

$$\varphi_{k-1}\left(s, \frac{k-2}{n} + h\right)$$

is a smooth function with respect to s and

$$t_{k-1}\left(s, \frac{k-1}{n}\right) = \dot{\varphi}_{k-1}\left(s, \frac{k-1}{n}\right), \tag{3.3}$$

$$n_{k-1}\left(s, \frac{k-1}{n}\right) = (-\dot{y}_{k-1}, \dot{x}_{k-1}), \tag{3.4}$$

where $\varphi_{k-1} = (x_{k-1}, y_{k-1})$, then

$$\begin{aligned}\varphi_k\left(s, \frac{k-1}{n} + h\right) = \varphi_{k-1}\left(s, \frac{k-1}{n}\right) + \\ + \left[\alpha\left(s, \frac{k-1}{n}\right)t_{k-1} + \beta\left(s, \frac{k-1}{n}\right)n_{k-1}\right]h.\end{aligned} \tag{3.5}$$

We write

$$\varphi^n(s, \tau) = \varphi_k\left(s, \frac{k-1}{n} + h\right), \quad n = 1, 2, \ldots, \tag{3.6}$$

for $\tau = (k-1)/n + h$.

b. In this part we evaluate

$$\dot{\varphi}_k\left(s, \frac{k-1}{n}+h\right).$$

We write

$$\dot{\varphi}_i = \dot{\varphi}_i\left(s, \frac{i}{n}\right), \quad \alpha_i = \alpha\left(s, \frac{i}{n}\right), \quad \beta_i = \beta\left(s, \frac{i}{n}\right),$$

$$k_i = k\left(s, \frac{i}{n}\right), \quad t_i = t_i\left(s, \frac{i}{n}\right), \quad n_i = n_i\left(s, \frac{i}{n}\right), \quad 1 \leqslant i \leqslant n.$$

We have

$$\dot{\varphi}_1 = t_1 = t_0 + (\dot{\beta}_0 + \alpha_0 k_0) n_0 \frac{1}{n}, \tag{3.7}$$

$$n_1 = n_0 - (\dot{\beta}_0 + \alpha_0 k_0) t_0 \frac{1}{n}.$$

Using (2.5) and definition (3.5), we get

$$\dot{\varphi}_2 = t_2 = t_0 + [(\dot{\beta}_0 + \alpha_0 k_0) + (\dot{\beta}_1 + \alpha_1 k_1)] n_0 \frac{1}{n} + a_2 \frac{1}{n^2} + b_2 \frac{1}{n^2} = c_2 \frac{1}{n^2}, \tag{3.8}$$

$$c_2 = a_2 + b_2 \frac{1}{n},$$

where

$$a_2 = [-(\dot{\beta}_1 + \alpha_1 k_0) t_0 + (\dot{\alpha}_1 - \beta_1 k_0) n_0](\dot{\beta}_0 + \alpha_0 k_0),$$

$$b_2 = \frac{1}{2} k''\left(s, \theta\frac{1}{n}\right)(\beta_1 t_0 - \alpha_1 n_0), \quad 0 < \theta < 1,$$

$$n_2 = n_0 - [(\dot{\beta}_0 + \alpha_0 k_0) + (\dot{\beta}_1 + \alpha_1 k_1)] t_0 \frac{1}{n} + \bar{c}_2 \frac{1}{n^2}, \tag{3.9}$$

and where $\bar{c}_2$ is a vector such that $c_2 \bar{c}_2 = 0$, $|c_2| = |\bar{c}_2|$, and $(c_2, \bar{c}_2)$ defines the positive orientation of E^2.

Inductively we have

$$\dot{\varphi}_k = t_k = \dot{\varphi}_{k-1} + \left[\dot{\alpha}_{k-1} - \left(k_0 + \sum_{i=0}^{k-2} k_i' \frac{1}{n}\right)\beta_{k-1}\right] t_0 \frac{1}{n} + \tag{3.10}$$

$$+(\dot{\alpha}_{k-1} - \beta_{k-1} k_0)\left[\sum_{i=0}^{k-2} (\dot{\beta}_i + \alpha_i k_i)\right] n_0 \frac{1}{n^2} +$$

$$+\left[\dot{\beta}_{k-1} + \left(k_0 + \sum_{i=0}^{k-2} k_i' \frac{1}{n}\right)\alpha_{k-1}\right] n_0 \frac{1}{n} -$$

$$-(\dot{\beta}_{k-1}+a_{k-1}k_0)\Big[\sum_{i=0}^{k-2}(\dot{\beta}_i+a_ik_i)\Big]t_0\frac{1}{n^2}+$$

$$+(\dot{a}_{k-1}c_{k-1}+a_{k-1}\dot{c}_{k-1}+\dot{\beta}_{k-1}\bar{c}_{k-1}+\beta_{k-1}\dot{\bar{c}}_{k-1})\frac{1}{n^3},$$

where c_2 and $\bar{c}_2$ are defined in (3.8) and (3.9), and if c_{k-1} is defined, then

$$c_k = a_k+b_k\frac{1}{n}+(\dot{a}_{k-1}c_{k-1}+a_{k-1}\dot{c}_{k-1}+\dot{\beta}_{k-1}\bar{c}_{k-1}+\beta_{k-1}\dot{\bar{c}}_{k-1})\frac{1}{n}, \tag{3.11}$$

where

$$a_k = [-(\dot{\beta}_{k-1}+a_{k-1}k_0)t_0+(\dot{a}_{k-1}-\beta_{k-1}k_0)n_0]\sum_{i=0}^{k-2}(\dot{\beta}_i+a_ik_i),$$

$$b_k = \frac{\beta_{k-1}}{2}\sum_{i=1}^{k-1}k''\Big(s,\frac{i-1}{n}+\theta_i\frac{1}{n}\Big)-\frac{a_{k-1}}{2}\sum_{i=1}^{k-1}k''\Big(s,\frac{i-1}{n}+\theta_i\frac{1}{n}\Big),$$

$$0<\theta_i<1,\quad i=1,2,\ldots,k-1.$$

Formula (3.10) can be written in the form

$$\dot{\varphi}_k = t_k = t_0+\sum_{i=0}^{k-1}(\dot{\beta}_i+a_ik_i)n_0\frac{1}{n}+\frac{1}{n^2}\sum_{i=2}^{k}c_i. \tag{3.12}$$

For the complementary vector to t_k, we have

$$n_k = n_0-\sum_{i=0}^{k-1}(\dot{\beta}_i+a_ik_i)t_0\frac{1}{n}+\frac{1}{n^2}\sum_{i=2}^{k}\bar{c}_i, \tag{3.13}$$

where $|c_i| = |\bar{c}_i|$, $c_i\bar{c}_i = 0$, and $(c_i, \bar{c}_i)$ defines the positive orientation of E^2.

c. *A combinatorial formula.* For the estimation of t_k the following combinatorial facts are useful. Symbol L is called a *linear scheme on the indeterminate* x if the symbols

$$L(L^ix+L^jx) \quad\text{and}\quad L^{i+1}x+L^{j+1}x \tag{3.14}$$

are identified, where

$$L^ix = \underbrace{LL\ldots Lx}_{i\text{ times}},\quad L^0x = x.$$

LEMMA 2. *Let L be a linear scheme on the symbol x. If $P_1(x) = x$ and*

$$P_n(x) = P_{n-1}(x)+L\big(P_{n-1}(x)\big),\quad n = 2,3,\ldots,$$

then

$$P_n(x) = (1+L)^nx.$$

The proof of the lemma is a simple verification.

Remark. If L is as in Lemma 2, $P_1(x) = x$ and

$$P_n(x) = P_{n-1}(x) + nx + L(P_{n-1}(x)), \quad n = 2, 3, \ldots,$$

then

$$P_n(x) = (1+L)^{n-1}x + 2(1+L)^{n-2}x + \ldots + nx. \tag{3.15}$$

Formula (3.15) follows immediately by Lemma 2.

d. *The estimation of* $n^{-2}\sum_{i=2}^{k} c_i$. We have

$$|c_2|\frac{1}{n^2} \leqslant KA^3\frac{1}{n^2}, \qquad |\dot{c}_2|\frac{1}{n^2} \leqslant 3KA^3\frac{1}{n^2},$$

where K denotes a constant.

In the estimation of $\dot{c}_2$ we use the fact that if the product of m functions is estimated by A^m, then the derivative of this product is estimated by mA^m, provided all these functions and their derivatives are uniformly bounded by A.

For c_3 we have

$$|c_3|\frac{1}{n^2} \leqslant 2KA^3\frac{1}{n^2} + 2AKA^3\frac{1}{n^3} + 2A\cdot 3KA^3\frac{1}{n^3} \leqslant 2KA^3\frac{1}{n^2} + 4A\cdot 3KA^3\frac{1}{n^3}.$$

If $\sum_{i=0}^{k} a_i A^i$ is a polynomial in A, then we define the linear operation

$$L_A\Big(\sum_{i=0}^{k} a_i A^i\Big) = \sum_{i=0}^{k} i a_i A^{i+1}. \tag{3.16}$$

Hence

$$\begin{aligned} |c_2 + c_3|\frac{1}{n^2} &\leqslant KA^3\frac{1}{n^2} + 2KA^3\frac{1}{n^2} + 4\cdot 3KA^4\frac{1}{n^3} \\ &= KA^3\frac{1}{n^2} + 2KA^3\frac{1}{n^2} + \frac{4}{n}L_A\left(KA^3\frac{1}{n^2}\right). \end{aligned}$$

By induction, if $n^{-2}\sum_{i=2}^{k+1} c_i$ is estimated by a polynomial $P_k(A, 1/n)$ of degree $k+2$ with respect to A and $k+1$ with respect to $1/n$, then

$$\Big|\sum_{i=2}^{k+2} c_i\Big|\frac{1}{n^2} \leqslant P_k\left(A, \frac{1}{n}\right) + (k+1)KA^3\frac{1}{n^2} + 4L_A\left(P_k\left(A, \frac{1}{n}\right)\right)\frac{1}{n}, \tag{3.17}$$

where

$$P_1\left(A, \frac{1}{n}\right) = KA^3\frac{1}{n^2}, \quad P_0\left(A, \frac{1}{n}\right) = 0.$$

Using formula (3.15), we can represent the right-hand member of (3.17) in the form

$$P_{k+1}\left(A,\frac{1}{n}\right)=P_k\left(A,\frac{1}{n}\right)+(k+1)KA^3\frac{1}{n^2}+4L_A\left(P_k\left(A,\frac{1}{n}\right)\right)\frac{1}{n}$$

$$=\frac{K}{n^2}\sum_{i=1}^{k+1}i\left(1+\frac{4}{n}L_A\right)^{k-i+1}A^3,\quad \text{where } L=\frac{4}{n}L_A.$$

For $k=n$ we get

$$P_n\left(A,\frac{1}{n}\right)=\frac{K}{n^2}\sum_{i=1}^{n}i\left(1+\frac{4}{n}L_A\right)^{n-i}A^3.$$

Let us take

$$n\geqslant 2kl, \tag{3.18}$$

where l is an integer such that

$$\frac{4}{l}A<1. \tag{3.19}$$

Then we have

$$P_{k+1}\left(A,\frac{1}{2kl}\right)=\frac{K}{(2kl)^2}\sum_{i=1}^{k+1}i\left(1+\frac{4}{2kl}L_A\right)^{k-i+1}A^3. \tag{3.20}$$

For a fixed summand of (3.20) we have

$$\frac{K}{(2kl)^2}i\left(1+\frac{4}{2kl}L_A\right)^{k-i+1}A^3$$

$$=\frac{K}{(2kl)^2}i\sum_{j=0}^{k-i+1}\frac{1}{2j!}\,\frac{4^jA^j(j+2)!}{(2kl)^j}(k-i+1)\dots(k-i-j+2)$$

$$\leqslant\frac{KA^3}{2(2kl)^2}i\sum_{j=0}^{k-i+1}\frac{(j+2)^2}{2^j}\left(\frac{4A}{l}\right)^j.$$

Since $(j+2)^2/2^j$ $(j=0,1,2,\dots)$ is bounded by a constant M we get, using (3.19),

$$\frac{K}{(2kl)^2}i\left(1+\frac{4}{2kl}L_A\right)^{k-i+1}A^3\leqslant\frac{KA^3M}{2(2kl)^2}i\sum_{j=0}^{k-i+1}\left(\frac{4A}{l}\right)^j\leqslant i\frac{KA^3M}{2(2kl)^2}\,\frac{1}{1-\dfrac{4A}{l}}.$$

Thus for (3.20) we have the estimation

$$P_{k+1}\left(A,\frac{1}{2kl}\right)\leqslant\frac{KA^3M}{4\left(1-\dfrac{4A}{l}\right)}\,\frac{k(k+1)}{(2kl)^2}.$$

If $k\to\infty$, then from $k(k+1)/k^2\leqslant 2$ we get

$$P_{k+1}\left(A,\frac{1}{2kl}\right)\leqslant\frac{KA^3M}{8l(l-4A)},\quad\text{where } n\geqslant 2kl. \tag{3.21}$$

e. Let $[0,1/2l]\subset[0,1]$. Then from (3.12) and (3.21) we have

$$|\dot{\varphi}_k| = |t_k|\leqslant 1+AL+\frac{KMA^3}{8l(l-4A)}\quad\text{for every } \tau\in\left[0,\frac{1}{2l}\right]. \tag{3.22}$$

Thus the sequence $(\dot{\varphi}_k)$, $1\leqslant k<\infty$, is uniformly bounded on the segment $[0,1/2l]$.

Formula (3.5) can be written in the form

$$\varphi_k\left(s,\frac{k-1}{n}+h\right) = \varphi_0(s,0)+\frac{1}{n}\sum_{i=0}^{k-2}(a_it_i+\beta_in_i)+(a_{k-1}t_{k-1}+\beta_{k-1}n_{k-1})h. \tag{3.23}$$

Since $|t_i| = |n_i|$, we get by the use of (3.22) the estimation

$$\left|\varphi_k\left(s,\frac{k-1}{n}+h\right)\right|\leqslant A+2AG\quad\text{for } n\geqslant 2kl,$$

where G denotes the right-hand member of inequality (3.21). Hence the sequence (3.6) is uniformly bounded on the interval $[0,1/2l]$.

Functions (3.6) are equicontinuous with respect to $\tau\in[0,1/2l]$. If we evaluate (3.23) for

$$\tau'' = \frac{k''-1}{n}+h'',\quad \tau' = \frac{k'-1}{n}+h',\quad k''>k',$$

and take the difference of these values, we get

$$|\varphi^n(s,\tau'')-\varphi^n(s,\tau')|\leqslant\frac{1}{n}\left|\sum_{i=k'-1}^{k''-2}(a_it_i+\beta_in_i)\right|+$$
$$+|(a_{k''-1}t_{k''-1}+\beta_{k''-1}n_{k''-1})h''-(a_{k'-1}t_{k'-1}+\beta_{k'-1}n_{k'-1})h'|$$
$$\leqslant 2AG\left|\frac{k''-k'-1}{n}+h''+h'\right|\leqslant 4AG|\tau''-\tau'|.$$

By the theorem of Ascoli, there exists a subsequence

$$(3.24)\qquad \left(\varphi^{n_r}(s,\tau)\right),\quad r=1,2,\ldots$$

of the sequence (3.6), uniformly convergent with respect to $\tau\in[0,1/2l]$. We write

$$(3.25)\qquad \varphi(s,\tau)=\lim_{r\to\infty}\varphi^{n_r}(s,\tau).$$

The sequence (3.24) can be supposed to converge uniformly also with respect to $s\in C$. Indeed, we have

$$|\varphi^n(s'',\tau)-\varphi^n(s',\tau)|\leqslant\int_{s'}^{s''}|\dot\varphi^n(s,\tau)|\,ds\leqslant G|s''-s'|.$$

Moreover, since $\dot\varphi^n(s,\tau)$ is a smooth function with respect to s, there exists a constant H such that

$$|\dot\varphi^n(s'',\tau)-\dot\varphi^n(s',\tau)|\leqslant H|s''-s'|.$$

Hence we may assume that $\left(\dot\varphi^{n_r}(s,\tau)\right)$, $r=1,2,\ldots$, converges uniformly with respect to s. By induction, it follows that the function (3.24) is smooth with respect to s.

Let us write

$$(3.26)\qquad t(s,\tau)=\lim_{r\to\infty}\dot\varphi^{n_r}(s,\tau)=\lim_{r\to\infty}t^{n_r}(s,\tau),\qquad \tau\in\left[0,\frac{1}{2l}\right],$$

and

$$(3.27)\qquad n(s,\tau)=\lim_{r\to\infty}n^{n_r}(s,\tau),$$

where n^{n_r} is the complementary vector to t^{n_r}, of the same length as t^{n_r}.

We prove that $\varphi(s,\tau)$, defined by (3.25), is differentiable with respect to τ and that $\varphi(s,\tau)$ satisfies system (1.7).

We have

$$\begin{aligned}\varphi(s,\tau'')-\varphi(s,\tau')&=\lim_{r\to\infty}\left(\varphi^{n_r}(s,\tau'')-\varphi^{n_r}(s,\tau')\right)\\&=\lim_{r\to\infty}\frac{1}{n_r}\left(\sum_{i=k_r'-1}^{k_r''-2}a_i t\left(s,\frac{i}{n_r}\right)+\beta_i n\left(s,\frac{i}{n_r}\right)\right)+\\&\qquad+\lim_{r\to\infty}\frac{1}{n_r}\left[\sum_{i=k_r'-1}^{k_r''-2}a_i\left(t_i-t\left(s,\frac{i}{n_r}\right)\right)+\beta_i\left(n_i-n\left(s,\frac{i}{n_r}\right)\right)\right]\\&=\int_{\tau'}^{\tau''}(at+\beta n)\,d\tau.\end{aligned}$$

The last equality follows from (3.26), (3.27) and the definition of the integral, where

$$\tau'' = \frac{k_r''-1}{n_r} + h_r'', \quad \tau' = \frac{k_r'-1}{n_r} + h_r'$$

and

$$\lim_{r\to\infty} \frac{k_r''}{n_r} = \tau'', \quad \lim_{r\to\infty} \frac{k_r'}{n_r} = \tau', \quad \lim_{r\to\infty} h_r'' = \lim_{r\to\infty} h_r' = 0.$$

Hence we have

$$\frac{\varphi(s,\tau'')-\varphi(s,\tau')}{\tau''-\tau'} = at+\beta n,$$

where the right-hand member is evaluated at $\big(s, \tau'+\theta(\tau''-\tau')\big)$, $0<\theta<1$, and therefore

$$\varphi'(s,\tau) = at+\beta n, \quad \tau\in\left[0, \frac{1}{2l}\right].$$

It follows that $\varphi(s,\tau)$, defined by (3.25), is a solution of (1.7). Since a, β is a length-preserving vector field, the curvature function of $\varphi(s,\tau)$ is $k(s,\tau)$. Therefore s is the arc-length parameter of $\varphi(s,\tau)$ for a fixed τ, $0\leqslant\tau\leqslant 1/2l$, and t, n are unit vectors. Since $k(s,\tau)$ is the curvature function of any solution $\varphi(s,\tau)$ of (1.7), it follows that this solution is uniquely determined by the initial condition (1.4).

We repeat the construction, which led us to the solution $\varphi(s,\tau)$ on $C\times[0, 1/2l]$, for the segment $[1/2l, 1+1/2l]$ with the initial condition

$$\varphi_{1/2l}\left(s, \frac{1}{2l}\right) = \varphi\left(s, \frac{1}{2l}\right).$$

Functions a, β are defined on $C\times[0,1]$. However, they can be extended to the segment $[0,2]$ by setting, for instance,

$$a(s, 1+\tau) = a(s, 1-\tau), \quad \beta(s,1+\tau) = \beta(s, 1-\tau), \quad 0\leqslant\tau\leqslant 1.$$

Then, since the estimations of φ, a, β, k and its derivatives do not change, we get a solution of (1.7) on $C\times[0, 1/l]$. Thus, after a finite number of steps, we get a solution on the whole set $C\times[0,1]$.

f. In the general case, if φ is an immersion of class C^r, $r\geqslant 2$, then from the Weierstrass approximation theorem it follows that there exist polynomials $\tilde\beta, \tilde k$ such that

$$|\beta-\tilde\beta| < \frac{1}{\nu}, \quad |k-\tilde k| \leqslant \frac{1}{\nu}, \quad \nu = 1, 2, \ldots$$

We write

$$\tilde{a} = \int k\tilde{\beta}\,ds \quad \text{and} \quad k' = (\dot{\tilde{\beta}} + \tilde{a}\tilde{k})^{\cdot}.$$

Then $(\tilde{a}, \tilde{\beta})$ is a length-preserving vector field of class C^∞ such that $\tilde{a}, \tilde{\beta}, \tilde{k}$ and their derivatives are uniformly bounded. Therefore there exists a solution $\varphi_\nu(s, \tau)$ of (1.7), where a, β are replaced by $\tilde{a}, \tilde{\beta}$, respectively. If $\nu \to \infty$, then $\varphi_\nu(s, \tau)$ tends uniformly to a solution of (1.7). This completes the proof.

Reçu par la Rédaction le 3. 1. 1972