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UNIFORM COMPLETENESS OF TOPOLOGICAL SPACES

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In this note there are proved some theorems which are generalizations of three known theorems, namely of a theorem of Alexandroff [1] and Hausdorff [3] that each G_δ-subset of a complete metric space is homeomorphic to a complete metric space, of a theorem of Nagata [7] and Kelley [5] that each paracompact space is topologically complete, and of a theorem of Čech [2] that each metric G_δ-subspace of a compact space is homeomorphic to a complete metric space.

Our theorems extend those known theorems in two directions. First, we consider intersections of arbitrary families of open subsets instead of G_δ's and a property induced by the uniformity with a base having prescribed cardinality (not necessarily $\aleph_0$) instead of metrizability. Second, we consider generalized uniformities, namely f-uniformities from our paper [5], instead of uniformities. As a consequence, results are concerned with general topological spaces without assuming any separation axioms.

In [5] there was introduced a notion of an f-uniformity on a set X, being a generalization of a uniformity in the sense of Tukey.

A family $\mathcal{U} \subset 2^{2^X}$ is called an *f-uniformity* on the set X if the following conditions are satisfied:

F1. $X = \bigcup\{\bigcup P\colon P \in \mathcal{U}\}$.

F2. $Q \in \mathcal{U}$ iff for each $x \in \bigcup Q$ there exists $P(x) \in \mathcal{U}$ such that $x \in \bigcup P(x)$ and $P(x) \succ Q$.

F3. If $P_1, P_2 \in \mathcal{U}$, then for each $x \in \bigcup P_1 \cap \bigcup P_2$ there exists $P(x) \in \mathcal{U}$ such that $P(x) \underset{*}{\succ} P_1 \wedge P_2$.

F4. If P, $Q \in \mathcal{U}$ and $x \in \bigcup Q$, then $\operatorname{st}(x, Q) \cap \bigcup P \neq \emptyset$.

The symbols $P \succ Q$ and $P \underset{*}{\succ} Q$ mean that P is a refinement and a star-refinement, respectively, of Q.

If we assume that the elements of $\mathcal{U}$ are coverings of X, then we obtain axioms of uniformity without the axiom of separation.

A *weight* of an f-uniformity $\mathscr{U}$ is the minimum of card$\mathscr{B}$, where $\mathscr{B}$ is a base for $\mathscr{U}$.

Each f-uniformity $\mathscr{U}$ on a set X induces a topology $T_{\mathscr{U}}$ on X: $G \in T_{\mathscr{U}}$ iff for each $x \in G$ there exists $P \in \mathscr{U}$ such that $x \in \bigcup P$ and $\operatorname{st}(x, P) \subset G$. Conversely, each topology on X is induced by some f-uniformity on X (see [5]).

A filter $\xi \subset 2^X$ is called a *Cauchy filter* if, for each $P \in \mathscr{U}$, $P \cap \xi$ is non-empty. For each Cauchy filter ξ there exists a minimal Cauchy filter $\xi_0 \subset \xi$ such that if $\eta \subset \xi$ is a Cauchy filter, then $\xi_0 \subset \eta$. The filter ξ_0 has a base of the form $\{\operatorname{st}(A, P)\colon A \in \xi,\ P \in \mathscr{U}\}$. Let $\xi(x)$ be the filter of neighbourhoods of the point x. The filter $\xi(x)$ is induced by the base

$$\{\operatorname{st}(x, P)\colon x \in \bigcup P,\ P \in \mathscr{U}\}.$$

If ξ is a Cauchy filter, then the conditions

(a) $x \in \lim \xi$,

(b) $x \in \lim \xi_0$, where ξ_0 is the minimal Cauchy filter of ξ,

(c) $x \in \bigcap \{\operatorname{cl} A\colon A \in \xi\}$,

where $x \in \lim \xi$ means that $\xi(x) \subset \xi$, are equivalent.

An f-uniformity $\mathscr{U}$ on X is *complete* if each Cauchy filter converges.

We say that *f-completeness* (*completeness*) of a space X is not greater than $\mathfrak{m}$, and write $\operatorname{fcp} X \leqslant \mathfrak{m}$ ($\operatorname{cp} X \leqslant \mathfrak{m}$), if there exists a complete f-uniformity (uniformity) $\mathscr{U}$ of the weight not greater than $\mathfrak{m}$ compatible with the topology on X.

We say that a space X is *well embedded* in a space Y if $Y = \operatorname{cl}_Y X$ and $\bigcap \{\operatorname{cl}_Y V\colon V \in \xi(x)\} \subset X$ for each $x \in X$.

A space X is said to be *f-compact* if for each open covering P there exists a finite subfamily $Q \subset P$ such that $X = \operatorname{cl} \bigcup Q$. If a space X is Hausdorff, then it is usually called H-closed. Note that there is an equivalence between the f-compactness and the convergence of each open filter.

THEOREM 1. *Each space X can be well embedded in an f-compact space Y. If the topology on X is induced by a uniformity, then X can be well embedded in a compact space Y.*

Proof. In [5] it was proved that each space X has a totally bounded f-uniformity compatible with the topology on X. There was constructed an f-completion $(\tilde{X}, \tilde{\mathscr{U}})$ of the f-uniform space $(X, \mathscr{U})$, where $\tilde{X} = X \cup X_0$, X_0 is the set of ξ_0 such that ξ_0 is a minimal Cauchy filter in $\mathscr{U}$ having an empty limit, and the f-uniformity $\tilde{\mathscr{U}}$ is induced by the base $\{\tilde{P}\colon P \in \mathscr{U}\}$ with

$$\tilde{P} = \{\tilde{U}\colon U \in P\} \quad \text{and} \quad \tilde{U} = U \cup \{\xi_0 \in X_0\colon U \in \xi_0\}.$$

It was proved also there that X is densely embedded in X and that the topology $T_{\tilde{\mathscr{U}}}$ is f-compact. To see that X is well embedded in $Y = \tilde{X}$

notice that if $x \in X$ and $\xi_0 \in X_0$, then there exist $P \in \mathscr{U}$ and $U \in \xi_0 \cap P$ such that $x \notin \mathrm{cl}_X U$. Hence and from the axiom F3 we infer that there exist open (in Y) neighbourhoods V_x and V_{ξ_0} of the points x and ξ_0, respectively, such that $V_x \cap V_{\xi_0} = \emptyset$.

If the space X is uniformizable, then the topology on X is also induced by a totally bounded uniformity $\mathscr{U}$. Since the above-described completion $\tilde{\mathscr{U}}$ is also a uniformity, the topology $T_{\tilde{\mathscr{U}}}$ is compact, which completes the proof.

A set X is a $G_{\mathfrak{m}}$*-subset* of a space Y if X is an intersection of no more than $\mathfrak{m}$ open subsets of Y.

THEOREM 2. *If* $\mathrm{fcp}\, X \leqslant \mathfrak{m}$ *and a space X is well embedded in Y, then X is a* $G_{\mathfrak{m}}$*-subset of Y.*

Proof. Let $\mathscr{B} \subset \mathscr{U}$ with $\mathrm{card}\, \mathscr{B} \leqslant \mathfrak{m}$ be a base of a complete f-uniformity $\mathscr{U}$ compatible with the topology on X. For each $P \in \mathscr{B}$ and $x \in X$ let $U_P(x)$ be an open subset of Y such that $U_P(x) \cap X \subset V \in P$. Put

$$R_P = \bigcup \{U_P(x) \colon x \in X\}.$$

It suffices to prove that $X = \bigcap \{R_P \colon P \in \mathscr{B}\}$. Clearly, $X \subset \bigcap \{R_P \colon P \in \mathscr{B}\}$. Now, suppose that there exists $y \in \bigcap \{R_P \colon P \in \mathscr{B}\}$ such that $y \in Y \setminus X$. For each $P \in \mathscr{B}$ choose $x_P \in X$ such that $y \in U_P(x_P)$. The family $\{U_P(x_P) \cap X \colon P \in \mathscr{B}\}$ is a base for a Cauchy filter ξ. Since $\mathscr{U}$ is complete, there exists

$$x \in \bigcap \{\mathrm{cl}_X U_P(x_P) \cap X \colon P \in \mathscr{B}\}.$$

Now ξ is a Cauchy filter, and so, for each $V \in \xi(x)$, $V \subset Y$, there exists $P \in \mathscr{U}$ such that $U_P(x_P) \cap X \subset V$. Since $\mathrm{cl}_Y X = Y$, we have

$$\mathrm{cl}_Y\big(U_P(x_P) \cap X\big) = \mathrm{cl}_Y U_P(x_P) \subset \mathrm{cl}_Y V.$$

Thus $y \in \mathrm{cl}_Y V$ for each neighbourhood $V \subset Y$ of x. This implies

$$y \in \bigcap \{\mathrm{cl}_Y V \colon V \in \xi(x)\} \subset X,$$

which contradicts $y \in Y \setminus X$.

THEOREM 3. *Each topological space has a complete f-uniformity compatible with the topology on X.*

Proof. Let Y be an f-compact extension of X such that $Y = \tilde{X}$, where $\tilde{X}$ is a space with the topology induced by the completion of the greatest totally bounded f-uniformity $\mathscr{U}$ on X. For each $y \in Y \setminus X$ let $P(y)$ be the set of V such that V is an open set in X with $y \notin \mathrm{cl}_Y V$ (cf. the remark in the proof of Theorem 1: for each $x \in X$ and $y \in Y \setminus X$ there exist open neighbourhoods V_x and V_y with $V_x \cap V_y = \emptyset$). The covering $P(y)$ of X belongs to the greatest f-uniformity $\mathscr{U}^*$ on the space X (see [5]). Now, it is easy to see that $\mathscr{U}^*$ is complete.

Let $\xi_0 \subset 2^X$ be a minimal Cauchy filter. For each $A \in \xi_0$ there is $\mathrm{int}_X A \neq \emptyset$. We have

$$\bigcap\{\mathrm{cl}_X A\colon A \in \xi_0\} \supset \bigcap\{\mathrm{cl}_X \mathrm{int}_X A\colon A \in \xi_0\} = \bigcap\{\mathrm{cl}_Y \mathrm{int}_X A\colon A \in \xi_0\} \cap X,$$

but, in view of the f-compactness of Y,

$$\emptyset \neq \bigcap\{\mathrm{cl}_Y \mathrm{int}_X A\colon A \in \xi_0\} \subset \bigcap\{\mathrm{cl}_Y \mathrm{int}_X A\colon A \in \xi_0 \cap P(y),\, y \in Y \setminus X\} \subset X.$$

Hence $\bigcap\{\mathrm{cl}_X A\colon A \in \xi_0\} \neq \emptyset$.

Let X be a uniformizable space and let $X \subset Y$. An open set $G \subset Y$ is said to be a *uniform neighbourhood* of X if there exists an open covering P of X such that $\bigcup\{\mathrm{cl}_Y U\colon U \in P\} \subset G$ and P belongs to the greatest uniformity compatible with the topology on X.

Note that if X is paracompact, then each open set G containing X is a uniform neighbourhood of X, since each open covering of X belongs to the greatest uniformity on X.

THEOREM 4. *Let $X \subset Y$, $\mathrm{cl}_Y X = Y$, be an intersection of no more than $\mathfrak{m}$ uniform neighbourhoods of X. If the topology on Y is induced by a complete uniformity of the weight not greater than $\mathfrak{m}$, then the topology on X is induced by a complete uniformity of the weight not greater than $\mathfrak{m}\cdot\mathfrak{n}$, i.e., $\mathrm{cp}\, Y \leqslant \mathfrak{n}$ implies $\mathrm{cp}\, X \leqslant \mathfrak{m}\cdot\mathfrak{n}$.*

Proof. Let $\mathcal{G}$ be a family of open neighbourhoods of X with $\mathrm{card}\,\mathcal{G} \leqslant \mathfrak{m}$ and such that $X = \bigcap\{G\colon G \in \mathcal{G}\}$. For each $G \in \mathcal{G}$ choose a P_G belonging to the greatest uniformity $\mathcal{U}^*$ on the space X such that

$$\bigcup\{\mathrm{cl}_Y U\colon U \in P_G\} \subset G.$$

Let $\mathcal{V}$ be a complete uniformity on Y with weight $\mathcal{V} \leqslant \mathfrak{n}$. By a countable operation (see [6], p. 246) we can find a uniformity

$$\mathcal{U}_0 \supset \mathcal{V} \cap X \cup \{P_G\colon G \in \mathcal{G}\}$$

compatible with the topology on X and such that

$$\mathrm{weight}\,\mathcal{U}_0 = \mathrm{weight}\,\mathcal{V}\cdot \mathrm{card}\,\mathcal{G}.$$

To see that $\mathcal{U}_0$ is complete notice that each minimal Cauchy filter $\xi \subset 2^X$ in the sense of $\mathcal{U}_0$ is a Cauchy filter in the sense of $\mathcal{V}$ and

$$\bigcap\{\mathrm{cl}_X A\colon A \in \xi\} \supset \bigcap\{\mathrm{cl}_X \mathrm{int}_X A\colon A \in \xi\} = \bigcap\{\mathrm{cl}_Y \mathrm{int}_X A\colon A \in \xi\} \cap X \neq \emptyset,$$

since $\mathcal{V}$ is complete and

$$\bigcap\{\mathrm{cl}_Y \mathrm{int}_X A\colon A \in \xi\} \subset \bigcap\{\mathrm{cl}_Y \mathrm{int}_X A\colon A \in \xi \cap P_G,\, G \in \mathcal{G}\} \subset X.$$

Theorem 4 is a generalization of the Alexandroff [1] and Hausdorff [3] Theorem that each G_δ-subset of a complete metric space is metrizable in a complete manner.

Put $\mathrm{uw}\, X = \min\{\mathrm{weight}\,\mathcal{U}\colon \mathcal{U}$ is a uniformity compatible with the topology on $X\}$.

THEOREM 5. *Suppose that* $\mathrm{uw}X \leqslant \mathfrak{n}$, X *is a dense subspace of an f-compact space* Y, *and* X *is an intersection of no more than* $\mathfrak{m}$ *uniform neighbourhoods of* X. *Then* $\mathrm{cp}X \leqslant \mathfrak{n}\cdot\mathfrak{m}$.

Proof. The idea of the proof is the same as that of Theorem 4. Let $\mathcal{G}$ be a family of uniform neighbourhoods of X with $\mathrm{card}\,\mathcal{G} \leqslant \mathfrak{m}$. For each $G \in \mathcal{G}$ choose an open covering $P_G \in \mathcal{U}^*$ such that

$$\bigcup \{\mathrm{cl}_Y U:\ U \in P_G\} \subset G.$$

Let $\mathcal{U}$ be a uniformity on X with weight not greater than $\mathfrak{n}$. There exists a uniformity $\mathcal{U}_0 \subset \mathcal{U}^*$ such that

$$\mathcal{U}_0 \supset \mathcal{U} \cup \{P_G:\ G \in \mathcal{G}\} \quad \text{and} \quad \mathrm{weight}\,\mathcal{U}_0 \leqslant \mathfrak{m}\cdot\mathfrak{n}.$$

Take a minimal Cauchy filter $\xi \subset 2^X$ in the sense of $\mathcal{U}_0$. We have

$$\bigcap \{\mathrm{cl}_X A:\ A \in \xi\} \supset \bigcap \{\mathrm{cl}_X \mathrm{int}_X A:\ A \in \xi\} \cap X = \bigcap \{\mathrm{cl}_Y \mathrm{int}_X A:\ A \in \xi\} \neq \emptyset,$$

since Y is f-compact and

$$\bigcap \{\mathrm{cl}_Y \mathrm{int}_X A:\ A \in \xi\} \subset \bigcap \{\mathrm{cl}_Y \mathrm{int}_X A:\ A \in \xi \cap P_G,\ G \in \mathcal{G}\} \subset X.$$

Theorem 5 is a generalization of the fact that each paracompact space X is topologically complete (Nagata [7], Kelley [4]) and it is also a generalization ($\mathfrak{m} = \mathfrak{n} = \aleph_0$) of the theorem that each metrizable G_δ of a compact space is completely metrizable (Čech [2]).

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