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ON HOMOMORPHISMS OF PROJECTIVE LATTICES IN COMPLEX HILBERT SPACES

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1. Let H and $\mathscr{H}$ be separable complex Hilbert spaces. We assume that $\dim H \geqslant 3$. Let S_H and $S_{\mathscr{H}}$ denote the structures of all projective operators acting in H and $\mathscr{H}$, respectively.

A *homomorphism* of the structures S_H and $S_{\mathscr{H}}$ or a *spectral measure* (cf. [1]) on the structure S_H is any mapping $\pi\colon S_H \to S_{\mathscr{H}}$ such that, for any sequence of mutually orthogonal projections $P_1, P_2, \ldots$ from S_H, the projections $\pi(P_1), \pi(P_2), \ldots$ are mutually orthogonal and

$$\pi\Big(\sum_i P_i\Big) = \sum_i \pi(P_i).$$

A spectral measure π is said to be *normed* if $\pi(I_H) = I_{\mathscr{H}}$, where I_H and $I_{\mathscr{H}}$ stand for the unit operators acting in H and $\mathscr{H}$, respectively. A normed one-to-one spectral measure π mapping S_H onto the entire structure $S_{\mathscr{H}}$ is referred to as an isomorphism of the structures.

In the present paper we give a general form of a spectral measure $\pi\colon S_H \to S_{\mathscr{H}}$ (Theorem 2)(1). This result is a generalization of the known Wigner theorem stating that every isomorphism $a\colon S_H \to S_{\mathscr{H}}$ is of the form

$$a(P) = UPU^{-1}, \quad P \in S_H,$$

where U is a unitary or anti-unitary operator (cf. [3]).

We need the following consequence of Gleason's theorem (cf. [1]). A mapping $\xi\colon S_H \to \mathscr{H}$ is said to be an *orthogonal Gleason measure* if for any sequence of mutually orthogonal projectors $P_1, P_2, \ldots$ from S_H the vectors $\xi(P_1), \xi(P_2), \ldots$ are mutually orthogonal and

$$\xi\Big(\sum_i P_i\Big) = \sum_i \xi(P_i).$$

(1) For the case of a real space H an analogous result has been given in [1].

Then we have

THEOREM 1. *For every orthogonal Gleason measure* $\xi: S_H \to \mathscr{H}$ *there exist S-operators* M *and* M' *such that*

$$\big(\xi(P), \xi(Q)\big) = \operatorname{tr} MPQ + \operatorname{tr} M'QP$$

for any projections $P, Q \in S_H$.

By an *S-operator* we mean here a self-adjoint, non-negative operator with a finite trace.

Let H' be a subspace of H, its dimension being of no importance.

Definition. A (non-linear) operator $V: H' \to \mathscr{H}$ is said to be *subunitary* (or *subunitary with constant* k) if there exists a real number k, $-1 \leqslant k \leqslant 1$, such that

(0) $\quad (Va, Vb) = \operatorname{Re}(a, b) + ik\operatorname{Im}(a, b)$ for any vectors $a, b \in H'$.

Unitary, anti-unitary and subunitary operators being considered, we shall always assume that the image of the operator under investigation is a proper subspace of the space $\mathscr{H}$.

PROPOSITION 1. *If* $V: H' \to \mathscr{H}$ *is a subunitary operator with constant* k $(-1 < k < 1)$, *then there exists exactly one representation*

$$V = aV^+ + \beta V^-, \tag{1}$$

where V^+ *is unitary,* V^- *anti-unitary, and* $a, \beta \in R^+$; *namely,*

$$a = \sqrt{\frac{1+k}{2}}, \quad \beta = \sqrt{\frac{1-k}{2}}, \tag{2}$$

$$V^+a = \frac{1}{2a}\big(Va - iV(ia)\big), \quad V^-a = \frac{1}{2\beta}\big(Va + iV(ia)\big), \tag{3}$$

and $V^+(H') \perp V^-(H')$.

Proof. Let V satisfy condition (0). If a, β and V^+, V^- are defined by (2) and (3), then V^+ is a unitary operator, V^- is an anti-unitary operator, and condition (1) is clearly satisfied. Assume now that V has representation (1). Neither a nor β can vanish (since then $k = -1$ or $k = 1$) and (3) follows directly from (1). Making use of (0) we then obtain $(V^+a, V^-b) = 0$ for any vectors $a, b \in H'$. Thus, by (1),

$$(Va, Vb) = a^2(a, b) + \beta^2(b, a) = (a^2+\beta^2)\operatorname{Re}(a, b) + i(a^2-\beta^2)\operatorname{Im}(a, b),$$

i.e. (2) holds.

2. We introduce now some notation and give certain properties of spectral measures.

For an arbitrary set $M \subset H$ let $[M]$ denote a subspace spanned by vectors from M. When no misunderstanding can arise, we identify a projective operator with the subspace on which it projects. Let $\pi^i: S_H \to S_{\mathscr{H}}$

$(i = 1, 2)$ be spectral measures. The measures π^1 and π^2 are said to be *orthogonal* $(\pi^1 \perp \pi^2)$ if $\pi^1(P) \perp \pi^2(Q)$ for any projectors $P, Q \in S_H$. The measure π^1 is said to be *contained* in π^2 $(\pi^1 \leqslant \pi^2)$ if $\pi^1(P) \subset \pi^2(P)$ for any operator $P \in S_H$. One can easily verify the following propositions:

PROPOSITION 2. *If $\pi^1 \perp \pi^2$, then*

$$(\pi^1 + \pi^2)(P) = \pi^1(P) + \pi^2(P), \quad P \in S_H,$$

is a spectral measure.

PROPOSITION 3. *If $\pi^1 \leqslant \pi^2$, then*

$$(\pi^2 - \pi^1)(P) = \pi^2(P) - \pi^1(P), \quad P \in S_H,$$

is a spectral measure.

If V is a unitary or anti-unitary operator mapping H into $\mathscr{H}$, then the function $[e] \to [Ve]$, defined on one-dimensional projections in H, generates a spectral measure on S_H (denoted by $[V]$). If $\mathscr{H} = V(H)$, then $[V]$ is clearly a Wigner isomorphism $[V]P = VPV^{-1}$ $(P \in S_H)$. If V' is some other unitary or anti-unitary operator, then the conditions $[V] \perp [V']$ and $[V] \leqslant \pi$ are equivalent to $Va \perp Vb$ and $Va \in \pi([a])$, respectively, where a and b are arbitrary vectors from H.

PROPOSITION 4. *If $\pi: S_H \to S_{\mathscr{H}}$ is a spectral measure and the vector $f \in \mathscr{H}$, $\|f\| = 1$, belongs to $\pi([e])$ for some $e \in H$, $\|e\| = 1$, then one of the following three cases takes place:*

(a) *there is a unitary operator V^+ such that $f = V^+e$ and $[V^+] \leqslant \pi$;*

(b) *there is an anti-unitary operator V^- such that $f = V^-e$ and $[V^-] \leqslant \pi$;*

(c) *there are a unitary operator V^+ and an anti-unitary operator V^- such that $f = aV^+e + \beta V^-e$ $(a, \beta \in R^+)$, $[V^+] \perp [V^-]$ and $[V^+] + [V^-] \leqslant \pi$.*

Moreover, in each of these cases, the operators V^+ and V^- are uniquely determined.

The last proposition may be reformulated as follows:

PROPOSITION 4'. *If $\pi: S_H \to S_{\mathscr{H}}$ is a spectral measure and the vector $f \in \mathscr{H}$, $\|f\| = 1$, belongs to $\pi([e])$ for some $e \in H$, $\|e\| = 1$, then there is exactly one subunitary operator $V: H \to \mathscr{H}$ satisfying*

$$Ve = f \quad \textit{and} \quad Va \in \pi([a]),\ a \in H.$$

Indeed, it suffices to put $V = V^+$, $V = V^-$ or $V = aV^+ + \beta V^-$, respectively.

Proof of Proposition 4. Let $\pi: S_H \to S_{\mathscr{H}}$ be a spectral measure. For an arbitrary vector $x \in \mathscr{H}$ a mapping $\xi: S_H \to \mathscr{H}$ of the form $\xi(P) = \pi(P)x$, $p \in S_H$, is, as it can easily be verified, an orthogonal Gleason measure. Thus, by Theorem 1, there are S-operators M'_x and M''_x such that

$$(4) \qquad \big(\pi(P)x, \pi(Q)x\big) = \operatorname{tr} M'PQ + \operatorname{tr} M''QP$$

for any operators $P, Q \in S_H$. Assume that $x \in \pi([e])$ for some vector $e \in H$, $\|e\| = 1$. Then

$$\mathrm{tr}(M'_x + M''_x)[e] = \big(\pi([e])x, \pi([e])x\big) = \|x\|^2$$

and

$$\mathrm{tr}(M'_x + M''_x) I_H = \|x\|^2.$$

Moreover, $M'_x + M''_x$ is an S-operator. Making use of its spectral distribution one can easily verify that

$$M'_x + M''_x = \|x\|^2 [e].$$

Since the operators M'_x and M''_x are positive, we have

$$M'_x = \beta^2 [e] \quad \text{and} \quad M''_x = a^2 [e],$$

where $a^2 + \beta^2 = \|x\|^2$ $(a, \beta \in R^+)$. Then relation (4) yields

$$(5) \qquad \big(\pi([a])x, \pi([b])x\big) = \beta^2 \mathrm{tr}[e][a][b] + a^2 \mathrm{tr}[e][b][a]$$

$$= \frac{(e, a)(e, b)}{\|a\|^2 \|b\|^2} \big(a^2(a, b) + \beta^2(b, a)\big)$$

$$= \frac{(e, a)(e, b)}{\|a\|^2 \|b\|^2} \|x\|^2 \big(\mathrm{Re}(a, b) + ik_x \mathrm{Im}(a, b)\big)$$

(where $k_x = (a^2 - \beta^2)/\|x\|^2$) for any non-zero vectors $a, b \in \Sigma_e$,

$$\Sigma_e = \{a \in H : (a, e) \in R, \ (a, e) \neq 0\} \qquad (e \in H, \ \|e\| = 1).$$

For $x \in \pi([e])$ a function $U^e_x : \Sigma_e \to \mathscr{H}$ defined by

$$(6) \qquad U^e_x a = \frac{\|a\|^2}{(e, a)} \pi([a])x, \qquad a \in \Sigma_e,$$

satisfies the condition

$$(U^e_x a, U^e_x b) = \frac{\|a\|^2 \|b\|^2}{(e, a)(e, b)} \big(\pi([a])x, \pi([b])x\big)$$

$$= \|x\|^2 \big(\mathrm{Re}(a, b) + ik_x \mathrm{Im}(a, b)\big), \qquad a, b \in \Sigma_e.$$

If $f \in \pi([e])$ and $\|f\| = 1$ $(e \in H, \|e\| = 1)$, we have

(i) $(U^e_f a, U^e_f b) = \mathrm{Re}(a, b) + ik_f \mathrm{Im}(a, b)$ $(-1 \leqslant k_f \leqslant 1)$ for any vectors $a, b \in \Sigma_e$;

(ii) the vector $x = U^e_f a$ is uniquely determined by the conditions

$$(7) \qquad x \in \pi([a]), \qquad (x, f) = (a, e), \qquad \|x\| = \|a\|.$$

Indeed, if $x = U^e_f a$, then (7) follows from (6) and (i) as $f = U^e_f e$. If (7) holds, then

$$\pi([a])f = [x]f + y,$$

where $y = (\pi([a])-[x])f \perp [x]f$. Moreover,

$$[x]f = \frac{(f, x)}{\|x\|^2}x = \frac{(e, a)}{\|a\|^2}x.$$

Thus (cf. (5))

$$\|[x]f\| = \frac{(e, a)}{\|a\|} = \|\pi([a])f\| \quad \text{and} \quad y = 0.$$

By (6),

$$U_f^e a = \frac{\|a\|^2}{(e, a)}[x]f = x.$$

Let $\tilde{H}$ be a subspace of H containing the vector e and let $\dim\tilde{H} \geqslant 2$. Let $\tilde{V}^+$ and $\tilde{V}^-$ stand for unitary and anti-unitary operators, respectively, mapping $\tilde{H}$ into $\mathscr{H}$, and let V^+ and V^- denote unitary and anti-unitary operators, respectively, acting from H to $\mathscr{H}$.

(A) *If* $f \in \pi([e])$ ($\|f\| = \|e\| = 1$), *then for an arbitrary vector* $e' \in H$, $e' \perp e$, $\|e'\| = 1$, *there is a subunitary operator* $\tilde{V}: [e']^\perp \to \mathscr{H}$ *such that*

$$\tilde{V}e = f \quad \textit{and} \quad \tilde{V}a \in \pi([a]),\ a \in [e']^\perp$$

($[e']^\perp = I_H - [e']$).

(B) *Let* $\dim\tilde{H} \geqslant 2$. *If* $\tilde{V}^+a \in \pi([a])$ ($a \in \tilde{H}$), *then there is an operator* V^+, *identical with* $\tilde{V}^+$ *on* $\tilde{H}$, *for which*

$$[V^+] \leqslant \pi. \tag{8}$$

Similarly, if $\tilde{V}^-a \in \pi([a])$ ($a \in \tilde{H}$), *then there is an operator* V^- *identical with* $\tilde{V}^-$ *on* $\tilde{H}$ *and satisfying*

$$[V^-] \leqslant \pi. \tag{9}$$

(C) *Any subunitary operation* $V: H \to \mathscr{H}$ *satisfying the condition*

$$Va \in \pi([a]), \quad a \in H, \tag{10}$$

is uniquely determined by the vector $f = Ve$ ($e \in H$, $\|e\| = 1$), *i.e. the operators* V^+ *and* V^- *satisfying* (8) *and* (9) *are uniquely determined by the vectors* $f^+ = V^+e$ *and* $f^- = V^-e$, *respectively. Moreover, if* $f^+ \perp f^-$, *then*

$$[V^+] \perp [V^-]. \tag{11}$$

Proof of (A). Making use of the mapping $U_{f'}^{e'}$, where $\|f'\| = 1$ and $f' \in \pi([e'])$ for some vector $e' \in H$, $\|e'\| = 1$, we shall define a subunitary operation $V_{f'}^{e'}$ on the subspace $[e']^\perp = I_H - [e']$. Observe that for $a \in [e']^\perp$ the vectors

$$a_n = a + \frac{1}{n}e', \quad n = 1, 2, \ldots,$$

belong to $\Sigma_{e'}$ and, by (i), we have using (6)

$$\|U_{f'}^{e'}a_n - U_{f'}^{e'}a_m\|^2 = (a_n - a_m, a_n - a_m) \to 0 \quad \text{as } n, m \to \infty.$$

Thus there exists

$$V_{f'}^{e'}a = \lim_{n\to\infty} U_{f'}^{e'}a_n. \tag{12}$$

Condition (i) yields

$$(V_{f'}^{e'}a, V_{f'}^{e'}b) = \mathrm{Re}(a, b) + ik_{f'}\mathrm{Im}(a, b), \quad a, b \in [e']^{\perp},$$

i.e. the operation $V_{f'}^{e'}$ is subunitary.

We claim that

$$V_{f'}^{e'}a \in \pi([a]), \quad a \in [e']^{\perp}.$$

Indeed, if $f_n = U_{f'}^{e'}a_n$ and $e_n = a_n/\|a_n\|$ $(n = 1, 2, \ldots)$, then (cf. (6))

$$\|f_n - \pi([a])f_n\|^2 = \left\| U_{f_n}^{e_n}e_n - \frac{(e_n, a)}{\|a\|^2} U_{f_n}^{e_n}a \right\|^2$$

$$= \|f\|^2\left(e_n - \frac{(e_n, a)}{\|a\|^2}a, e_n - \frac{(e_n, a)}{\|a\|^2}a\right) \to 0 \quad \text{as } n \to \infty.$$

Thus

$$V_{f'}^{e'}a = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \pi([a])f_n \in \pi([a]). \tag{13}$$

Let now $e' \perp e$. Put

$$a_n = e + \frac{1}{n}e' \quad \text{and} \quad b_n = e' + \frac{1}{n}e \quad (n = 1, 2, \ldots)$$

and let

$$f' = \lim_{n\to\infty} U_f^e b_n \quad (f \in \pi([e]), \|f\| = 1).$$

Thus we have $f' = V^e e'$, i.e. $f' \in \pi([e'])$ and $\|f'\| = 1$. We show that

$$V_{f'}^{e'}e = f.$$

Since $U_f^e a_n \in \pi([a_n])$, $\|U_f^e a_n\| = \|a_n\|$ and

$$(f', U_f^e a_n) = \lim_{m\to\infty}(U_f^e b_m, U_f^e a_n) = \lim_{m\to\infty}(b_m, a_n) = (e', a_n),$$

by (ii) we have $U_{f'}^{e'}a_n = U_f^e a_n$. Hence

$$V_{f'}^{e'}e = \lim_{n\to\infty} U_{f'}^{e'}a_n = \lim_{n\to\infty} U_f^e a_n = f,$$

since

$$\|U_f^e a_n - f\| = \|U_f^e a_n - U_f^e e\| = \|a_n - e\| = \frac{1}{n}.$$

To complete the proof of (A) it suffices to take $\tilde{V} = V_{f'}^{e'}$.

Proof of (B). Making use of property (ii) of the function U_f^e we obtain, for $f^+ = \tilde{V}^+ e$ and $f^- = \tilde{V}^- e$,

$$U_{f^+}^e a = \tilde{V}^+ a \quad \text{and} \quad U_{f^-}^e a = \tilde{V}^- a$$

if $a \in \tilde{H} \cap \Sigma_e$. Since $\dim H \geqslant 2$, there are vectors $a_0, b_0 \in \tilde{H} \cap \Sigma_e$ such that $\operatorname{Im}(a_0, b_0) \neq 0$ (e.g., $a_0 = e + e'$, $b_0 = e + ie'$, $0 \neq e' \in \tilde{H}$, $e' \perp e$).

Evidently,

$$(U_{f^+}^e a_0, U_{f^+}^e b_0) = (V^+ a_0, V^+ b_0) = (a_0, b_0),$$

$$(U_{f^-}^e a_0, U_{f^-}^e b_0) = (V^- a_0, V^- b_0) = (b_0, a_0),$$

and hence (cf. property (i)) $k_{f^+} = 1$ and $k_{f^-} = -1$. Thus the functions $U_{f^+}^e$ and $U_{f^-}^e$ may be extended to unitary and anti-unitary operations V^+ and V^-, respectively, on H. It should be shown now that (8) and (9) hold, i.e. that

$$V^+ a,\ V^- a \in \pi([a]) \qquad (a \in H).$$

If $(a, e) \neq 0$, then $a' = (a, e)^{-1} a \in \Sigma_e$ and

$$V^+ a = (a, e) V^+ a' = (a, e)\, U_{f^+}^e a' \in \pi([a]),$$

$$V^- a = \overline{(a, e)}\, U_{f^-}^e a' \in \pi([a]).$$

Observe that $V^+ = V_{f^+}^e$ and $V^- = V_{f^-}^e$ on $[e]^\perp$ (cf. (12)). Thus, if $(a, e) = 0$, then, by (13),

$$V^+ a = V_{f^+}^e a \in \pi([a]), \qquad V^- a = V_{f^-}^e a \in \pi([a]).$$

Proof of (C). Let V be a subunitary operation satisfying (10). Again, making use of property (ii) one can show that

$$Va = U_f^e a \tag{14}$$

for $a \in \Sigma_e$ ($f = Ve$). Moreover, for $a \perp e$ we obtain

$$Va = \lim_{n \to \infty} V\left(a + \frac{1}{n} e\right) = \lim_{n \to \infty} U_f^e \left(a + \frac{1}{n} e\right) = V_f^e a,$$

i.e. the operator V is uniquely determined by the vector f on the subspace $[e]^\perp$. For any vector $a \notin [e]$ we have

$$a = \alpha e + \beta e', \qquad \alpha, \beta \in R,$$

where $e' \perp e$, $\|e'\| = 1$; thus $a \in \Sigma_{e'}$. Hence $Va = U_{f'}^{e'} a$, where $f' = Ve' = V_f^e e'$. Thus the operator V is uniquely determined by f on the whole space H.

Assume now that the operators V^+ and V^- satisfy (8) and (9). Let $e_1, e_2, \ldots$ be an orthonormal basis in H and let $e_i \in \Sigma_e$ $(i = 1, 2, \ldots)$. To show (10) it suffices to verify that $V^+e_i \perp V^-e_i$, i.e. (cf. (14)) that

$$U^e_{f^+}e_i \perp U^e_{f^-}e_j, \quad i, j = 1, 2, \ldots$$

For $i \neq j$ the orthogonality follows immediately from (8) and (9). Observe also that, by (6) and (i), the function $x \to U^e_x e_i$ is a linear operation on $\pi([e])$ which preserves norm, i.e. it is a unitary operation. Hence, if $f^+ \perp f^-$, then

$$U^e_{f^+}e_i \perp U^e_{f^-}e_i, \quad i = 1, 2, \ldots$$

Now we can achieve the proof of Proposition 4. Let f be a unit vector in $\pi([e])$ ($\|e\| = 1$). Consider a vector $e' \in H$, $e' \perp e$, $\|e'\| = 1$. The operation $\tilde{V}\colon [e']^\perp \to \mathscr{H}$ given in (A) may be either unitary or anti-unitary or subunitary with constant k, $-1 < k < 1$. Obviously, $\dim [e']^\perp \geqslant 2$. If the operation $\tilde{V} = \tilde{V}^+$ is unitary, then there exists, by (B), its unitary extension V^+ satisfying (8), and thus case (a) holds.

Similarly, if $\tilde{V}\colon [e']^\perp \to \mathscr{H}$ is an anti-unitary operation, then case (b) holds. By (C), the operator V^+ in case (a) and the operator V^- in case (b) are uniquely determined by the vector f.

If the operation $\tilde{V}$ defined in (A) is subunitary with constant k, $-1 < k < 1$, then applying Proposition 1 we can write

$$\tilde{V} = a\tilde{V}^+ + \beta\tilde{V}^-,$$

where $\tilde{V}^+\colon [e']^\perp \to \mathscr{H}$ and $\tilde{V}^-\colon [e']^\perp \to \mathscr{H}$ are unitary and anti-unitary operators, respectively, and $a, \beta > 0$. Moreover, by (3),

$$\tilde{V}^+a,\ \tilde{V}^-a \in \pi([a]),\ a \in [e']^\perp, \quad \text{and} \quad \tilde{V}^+e \perp \tilde{V}^-e.$$

Hence, by (B) and (C), case (c) holds. Moreover, the subunitary operation $V = aV^+ + \beta V^-$ satisfies (10) and, by (C), is uniquely determined by the vector Ve on the whole space H. Hence the vector $f = aV^+e + \beta V^-e$ determines uniquely (cf. Proposition 1) the operations V^+ and V^- on the whole space H.

From Proposition 4 we obtain immediately the following (cf. [2], Proposition 5)

PROPOSITION 5. *If for some non-zero projection $P_0 \in S_H$ the value of the spectral measure $\pi(P_0)$ equals zero, then $\pi(P) = 0$ for any $P \in S_H$.*

Proof. Indeed, suppose that $\pi(P) \neq 0$ for some operator $P \in S_H$. Then $\pi([e]) \neq 0$ for some one-dimensional projection $[e]$ ($e \in H$, $\|e\| = 1$). Thus there is a vector $f \in \pi([e])$, $\|f\| = 1$, and hence there is at least one unitary or anti-unitary operator V such that $[V] \leqslant \pi$ and $[V](P_0) \neq 0$, which contradicts the assumption.

3. We shall now prove the following

THEOREM 2. *If $\pi: S_H \to S_{\mathscr{H}}$ is a spectral measure (H and $\mathscr{H}$ are separable complex Hilbert spaces) and* $\dim H \geqslant 3$, *then*

$$\pi = \sum_i [V_i^+] + \sum_j [V_j^-], \tag{15}$$

where $V_1^+, V_2^+, \ldots$ and $V_1^-, V_2^-, \ldots$ are sequences (finite or not) of unitary and anti-unitary operators, respectively, and all measures $[V_i^+]$ and $[V_j^-]$ $(i, j = 1, 2, \ldots)$ are mutually orthogonal.

Let us observe that, making use of the definition of the simple sum of Hilbert spaces and that of the simple sum of operators, we then can write (in case where the measure π is normed)

$$\mathscr{H} = (\bigoplus_i \mathscr{H}_i^+) \oplus (\bigoplus_j \mathscr{H}_j^-),$$

$$\pi(P) = \big(\bigoplus_i V_i^+ P(V_i^+)^{-1}\big) \oplus \big(\bigoplus_j V_j^- P(V_j^-)^{-1}\big), \quad P \in S_H,$$

where $\mathscr{H}_i^+$ and $\mathscr{H}_j^+$ are the images of the operations V_i^+ and V_j^-, respectively.

Proof of Theorem 2. Let $\pi: S_H \to S_{\mathscr{H}}$ be a spectral measure. Consider a vector $e \in H$, $\|e\| = 1$. Let a sequence $f_1, f_2, \ldots$ (finite or not) be an orthonormal basis in $\pi([e])$. We find a sequence of mutually orthogonal measures $\pi_1, \pi_2, \ldots$ (finite or not) such that $\pi_i \leqslant \pi$,

$$f_1, \ldots, f_n \in \sum_{i=1}^{n} \pi_i([e]) \tag{16}$$

and for every $n = 1, 2, \ldots$ either $\pi_n = 0$ or one of the following cases holds (V_n^+ stands for a unitary operation, V_n^- — for an anti-unitary one):

(a) $\pi_n = [V_n^+]$,

(b) $\pi_n = [V_n^-]$,

(c) $\pi_n = [V_n^+] + [V_n^-]$, where $[V_n^+] \perp [V_n^-]$.

The existence of π_1 follows directly from Proposition 4 if we put $f = f_1$.

Assume that there are measures $\pi_1, \ldots, \pi_n$ of the required form, contained in π and satisfying (16). If f_{n+1} exists and

$$f_{n+1} \in \sum_{i=1}^{n} \pi_i([e]),$$

then it suffices to put $\pi_{n+1} = 0$. On the other hand, if

$$f_{n+1} \notin \sum_{i=1}^{n} \pi_i([e]),$$

then putting

$$f = \left\|f_{n+1} - \Big(\sum_{i=1}^{n} \pi_i\Big) f_{n+1}\right\|^{-1} \left(f_{n+1} - \Big(\sum_{i=1}^{n} \pi_i\Big) f_{n+1}\right)$$

in Proposition 4 and substituting π by the measure $\pi - \sum_{i=1}^{n} \pi_i$ we obtain a measure

$$\pi_{n+1} \leqslant \pi - \sum_{i=1}^{n} \pi_i$$

of the required form and such that

$$f_{n+1} \in \sum_{i=1}^{n+1} \pi_i([e]).$$

Observe that $\pi = \sum_i \pi_i$. Indeed,

$$\sum_i \pi_i([e]) = [f_1, f_2, \ldots] = \pi([e]).$$

Thus $\big(\pi - \sum_i \pi_i\big)([e]) = 0$ and, by Proposition 5,

$$\pi - \sum_i \pi_i = 0.$$

After suitable renumerating of operations V_n^+ and V_n^- (occurring when $\pi_n \neq 0$) we obtain formula (15).

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