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A GENERALISED MAZURKIEWICZ–SIERPIŃSKI THEOREM WITH AN APPLICATION TO ANALYTIC SETS

BY

R. M. SHORTT (MIDDLETOWN, CONNECTICUT)

0. Introduction. A classical result of Mazurkiewicz and Sierpiński [8] says that if A is an analytic subset of the product of Polish spaces $S_1 \times S_2$, then the set of points s in S_1 such that the section of A over s is uncountable is an analytic subset of S_1. We find and prove a higher dimensional analogue of this result in Theorem 2 below. If A is a subset of a product $S_0 \times S_1 \times S_2$, we consider its "two-dimensional" sections over points s in S_0. As a criterion of size, "uncountable" is replaced with "non-reticulate", a condition suited to the two-dimensional character of the sections. Recent selection theorems of Graf and Mauldin [1] facilitate a characterisation of non-reticulate sets (Theorem 1) and a reasonably straightforward proof of Theorem 2 and its corollary on the order of values of a measurable function.

Theorem 2 is applied to the problem of determining Borel-isomorphism types of "Borel-dense" analytic sets, i.e., those with totally imperfect complement. Earlier work in [10] on Borel-density produced the following result:

(1) If X is Borel-dense and is isomorphic with $X \times X$, then X is a standard (i.e., absolute Borel) space (Lemma 8). In passing to higher dimensional products, it is natural to replace "Borel-dense" with "Borel-dense of order 2" (defined below). Theorem 2 then enables us to prove:

(2) If X is Borel-dense of order 2, and $X \times X$ is isomorphic with $X \times X \times X$, then X is an analytic space (Theorem 3).

1. Definitions and preliminaries. We assume that the reader is familiar with the elements of descriptive set theory and the study of Borel spaces. For the former, we refer one to the texts of Kuratowski [3] and Hoffmann-Jørgensen [2]; for the latter, to the monograph of Rao and Bhaskara Rao [9]. By and large, our usage will follow theirs; see also Shortt [10].

LEMMA 1. *Let A_1 and A_2 be subsets of the standard spaces S_1 and S_2, respectively. Suppose that f is a Borel isomorphism of A_1 onto A_2. Assuming that $S_1 \setminus A_1$ and $S_2 \setminus A_2$ are not totally imperfect (i.e., they contain uncountable* *members of* $\mathscr{B}(S_1)$ *and* $\mathscr{B}(S_2)$), *then* f *extends to an isomorphism of* S_1 *onto* S_2.

Proof. The lemma is easily derived from the Lavrentiev–Kuratowski extension theorem in [3], Section 36, VII, and the fact that any two uncountable standard spaces are Borel-isomorphic.

In what follows, we shall often be working with subsets of the product of (usually uncountable standard) spaces S_1 and S_2. By a *slice* of $S_1 \times S_2$ we mean a set of the form $A_1 \times A_2$, where one of the sets A_i is singleton and the other is non-empty. If $A_i = \{s\}$, call $A_1 \times A_2$ a *slice over the point* s. If $A \subset S_1 \times S_2$, then by a *section* of A we mean the intersection of A with a slice of $S_1 \times S_2$. Likewise, we speak of a *section of* A *over the point* s. If S_1 and S_2 are separable spaces, then a *thread* T of $S_1 \times S_2$ is an uncountable standard subset of $S_1 \times S_2$, each of whose sections contains at most one point; equivalently, T is the graph of a Borel isomorphism between uncountable standard subsets of S_1 and S_2. A subset R of $S_1 \times S_2$ is *reticulate* if it is contained in a countable union of slices of $S_1 \times S_2$.

LEMMA 2. *Let* E *and* F *be analytic spaces and let* A *be an analytic subset of* $E \times F$. *If* $A(y) = \{x \in E\colon (x, y) \in A\}$ *denotes the section of* A *over the point* y, *then* $\{y \in F\colon A(y)$ *is uncountable*$\}$ *is an analytic subset of* F.

Proof. This result is originally due to Mazurkiewicz and Sierpiński [8]; proofs may be found in [3], Section 39, VII, and [2], III.6.1.

LEMMA 3. *Let* A *be an analytic subset of the product of analytic spaces* E *and* F. *If the sections* $A(x) = \{y \in F\colon (x, y) \in A\}$ *are countable for all* x *in* E, *then there exist analytic subsets* B_n *of* E $(n = 1, 2, \ldots)$ *and measurable mappings* $f_n\colon B_n \to F$ *whose (analytic) graphs satisfy*:

(1) $G(f_n) \cap G(f_m) = \emptyset$ *for* $n \neq m$,

(2) $A = \bigcup_{n=1}^{\infty} G(f_n)$.

Proof. This is essentially a result of Lusin [4], p. 243. Proofs may be found in [2], III.6.7, and [5].

Let E and F be analytic spaces and let A be an uncountable analytic subset of $E \times F$. Given $x_0 \in E$ and $y_0 \in F$, define the sections

$$A_1(x_0) = \{y \in F\colon (x_0, y) \in A\}$$

and

$$A_2(y_0) = \{x \in E\colon (x, y_0) \in A\}.$$

LEMMA 4. *Suppose that, for each* $x \in E$ *and* $y \in F$, *the sections* $A_1(x)$ *and* $A_2(y)$ *are countable. Then* A *is a countable union of analytic graphs of one-one measurable functions*; *in particular,* A *contains a thread.*

Proof. Using Lemma 3, we find (for $n = 1, 2, \ldots$) analytic subsets $B_n \subset E$ and measurable $f_n\colon B_n \to F$ so that $A = \bigcup G(f_n)$. Apply Lemma 3 to the sets $G(f_n)$, using the fact that the "horizontal" sections of $G(f_n)$ are countable. For $m = 1, 2, \ldots$ there are analytic sets $C_m \subset F$ and measurable mappings $g_m\colon C_m \to E$ such that

$$G(f_n) = \bigcup_{m=1}^{\infty} G(g_m);$$

each g_m is one-one. Repeating the process for each n yields the lemma.

Continuing in the context of Lemma 4 with E, F, and $A \subset E \times F$ analytic, and assuming E to be uncountable, we have

LEMMA 5. *Suppose that for each $x \in E$ the section $A_1(x)$ is uncountable. Then A contains a thread.*

Proof. It suffices to consider the case where A is standard, since any analytic set with uncountably many uncountable sections contains a standard set with the same property: see [6], Theorem 1. If A is standard, the result follows from Theorem 4.1 of [1].

Say that a separable space X is *Borel-dense of order* 1 if X is contained in some standard space S with $S \setminus X$ totally imperfect, i.e., such that $S \setminus X$ contains no uncountable members of $\mathscr{B}(S)$. X is *Borel-dense of order* 2 if X is contained in some standard space S in such a way that all members of $\mathscr{B}(S \times S)$ contained in $(S \times S) \setminus (X \times X)$ are reticulate in $S \times S$. A space Borel-dense of order 2 is easily seen to be Borel-dense of order 1.

LEMMA 6. *If a separable space X can be written as a countable union $X = X_1 \cup X_2 \cup \ldots$ of sets X_i, each Borel-dense of order* 1, *then X is Borel-dense of order* 1.

Proof. Embed X in some standard space S. Then there are sets $S_1, S_2, \ldots$ in $\mathscr{B}(S)$ with $X_i \subset S_i$ and $S_i \setminus X_i$ totally imperfect. $S_0 = S_1 \cup S_2 \cup \ldots$ is standard, contains X, and is such that $S_0 \setminus X \subset \bigcup (S_i \setminus X_i)$ is totally imperfect.

LEMMA 7. *If a separable space X is Borel-dense of order* 1, *then so is any member of $\mathscr{B}(X)$.*

Proof. Embed X in a standard space S with $S \setminus X$ totally imperfect. Then each A in $\mathscr{B}(X)$ is $A = B \cap X$ for some B in $\mathscr{B}(S)$. The fact that $B \setminus A \subset S \setminus X$ implies the lemma.

LEMMA 8. *If X_1 and X_2 are uncountable separable spaces such that $X_1 \times X_2$ is Borel-dense of order* 1, *then X_1 and X_2 are standard.*

This is Proposition 13 of [10].

2. Main results. Very recent work of Graf and Mauldin, combined with some classical methods (used to prove Lemma 2), enables us to obtain a characterisation of non-reticulate subsets of a two-dimensional product. The idea: simply that a thread is never reticulate and that non-reticulate analytic sets contain a thread.

THEOREM 1. *Let S_1, S_2, and P be Polish spaces. Suppose that $f\colon P \to S_1 \times S_2$ is a continuous function whose image $A = f(P)$ is an (analytic) subset of $S_1 \times S_2$. Then the following three statements are equivalent*:

(I) *A contains a thread of $S_1 \times S_2$.*

(II) *There is a dense-in-itself sequence of points of P on which each of the component functions f_1, f_2 of f is one-one.*

(III) *A is not reticulate in $S_1 \times S_2$.*

Proof. (I)$\Rightarrow$(II). Suppose that $T \subset A$ is a thread of $S_1 \times S_2$. Then f is continuous from $f^{-1}(T)$ onto T, and the implication follows from [3], Section 36, V, Corollary 3, applied to each of f_1 and f_2.

(II)$\Rightarrow$(III). We assume that $E \subset P$ is such a countable dense-in-itself set and look for a contradiction. Suppose that A is contained in $C_1 \cup C_2 \cup \ldots$, where each C_j is some slice of $S_1 \times S_2$. Put $X_j = f^{-1}(C_j)$ for $j = 1, 2, \ldots$ Then $P = X_1 \cup X_2 \cup \ldots$ expresses P as the union of a sequence of closed sets. Since each of the components f_1, f_2 is one-one on E, it follows that, for each j, $E \cap X_j$ contains at most one point and is therefore a scattered set. By [3], Section 34, IV, Corollary 5, E is scattered, a contradiction.

(III)$\Rightarrow$(I). Suppose that $A = f(P)$ is not reticulate in $S_1 \times S_2$ and consider the sections $A(S_1)$ and $A(S_2)$ of A over points $s_1 \in S_1$ and $s_2 \in S_2$. Define sets

$$A_1 = \{s_1 \in S_1\colon A(s_1) \text{ is uncountable}\},$$

$$A_2 = \{s_2 \in S_2\colon A(s_2) \text{ is uncountable}\}.$$

By Lemma 2, A_1 and A_2 are analytic sets. If either A_1 or A_2 is uncountable, Lemma 5 will imply that A contains a thread of $S_1 \times S_2$.

We are left with the case where both A_1 and A_2 are countable. By removing the slices of $S_1 \times S_2$ over points in A_1 and A_2, we may assume that every section of A is countable. Since A is not reticulate, it is uncountable. Lemma 4 now applies to produce a thread of $S_1 \times S_2$ contained in A.

Theorem 1 leads to a two-dimensional extension of the original Mazurkiewicz–Sierpiński result. Although these extensions are stated for Polish spaces, the equivalence of conditions (I) and (III) in Theorem 1 and the conclusion of Theorem 2 are of course valid for standard measurable spaces.

THEOREM 2. *Let S_0, S_1, S_2 be Polish spaces and let B be an analytic subset of $S = S_0 \times S_1 \times S_2$. Then the set*

$$A = \{s \in S_0\colon B(s) \text{ is not reticulate in } S_1 \times S_2\}$$

is an analytic subset of S_0.

Proof. Since B is analytic, there are a Polish space P and a continuous function $f\colon P \to S$ mapping P onto B. Let f_0, f_1, f_2 be the components of the function f. Let Z be the (Polish) space of all sequences in $P^{\infty} = P \times P \times \ldots$

that are dense-in-themselves. (Cf. [3], Section 30, XII, or [2], I.5.23.) Define A_0 to be the set of all s in S_0 such that there is a dense-in-itself sequence of points of P on which:

(1) f_0 is identically equal to s,

(2) each of the functions f_1, f_2 is one-one.

A_0 is the projection on S_0 of the following subset of $S_0 \times Z$:

$$\bigcap_{k=1}^{\infty} \{(s, z):\ s = f_0(z(k))\} \cap \bigcap_{i=1}^{2} \bigcap_{k \neq l} \{(s, z):\ f_i(z(k)) \neq f_i(z(l))\}.$$

Since this last is a G_δ-set, its projection A_0 is analytic.

Seeing that for each s in S_0 the set $f_0^{-1}(s)$ is closed, and hence Polish, we infer from Theorem 1 that $A = A_0$.

COROLLARY. *Let S_0, S_1, S_2 be Polish spaces and let f be a measurable function from $S_1 \times S_2$ to S_0. Then*

$$\{s \in S_0:\ f^{-1}(s) \textit{ is not reticulate in } S_1 \times S_2\}$$

is an analytic subset of S_0.

Proof. Since f is measurable, the graph of f is a Borel subset of $S_0 \times S_1 \times S_2$; the sets $f^{-1}(s)$ are sections of this graph over points s in S_0. Theorem 2 now applies.

As is, Theorem 2 might strike one as a minor footnote to the classical Mazurkiewicz–Sierpiński Theorem (our Lemma 2). However, it has a most intriguing application to an open problem in the descriptive theory of sets: the determination of the isomorphism classes of analytic and co-analytic sets. Mauldin [6] showed that if A is a Borel-dense analytic space, then A is not isomorphic with any of its powers A^n $(n > 1)$ or with $A^n \times S$, where S is standard. In [10], Proposition 13, the result was somewhat improved (see Lemma 8 above). The question of whether A^2 could be isomorphic with A^3 is still open **(P 1328)**(¹).

Approaching the problem "in reverse", we ask for which Borel-dense spaces X the product $X \times X$ is isomorphic with $X \times X \times X$. Under the assumption that X is Borel-dense of order 2, such an X must be analytic! Of course, it might be that such spaces are always standard. The assumption of second-order Borel-density is not very severe in view of the following facts, proved in [10] and [11]:

(1) If X is Borel-dense of order 1 and is universally measurable, then X is Borel-dense of order 2. (Such spaces are precisely the complements of universally null sets in a standard space.)

(2) If X is Borel-dense of order 1, then X has the Blackwell property if and only if X is Borel-dense of order 2.

(¹) The answer is affirmative (see *Problèmes*, p. 165).

THEOREM 3. *Let X be a separable space, Borel-dense of order 2. If $X \times X$ is Borel-isomorphic with a product $X \times A_1 \times A_2$, where A_1 and A_2 are uncountable separable spaces, then X is analytic.*

Proof. Case 1. At least one of the spaces A_1, A_2 is not standard. Suppose that X is Borel-dense of order 2 in the uncountable standard space S (if X is countable, the result is trivial). Suppose also that

$$g: X \times X \to X \times A_1 \times A_2$$

is a Borel isomorphism. By Lemma 1, g extends to an isomorphism f of $S \times S$ onto a product of standard spaces $S_0 \times S_1 \times S_2$. Let the components of f be denoted by f_0, f_1, f_2 and consider the set

$$A = \{s \in S_0: f_0^{-1}(s) \text{ is not reticulate in } S \times S\}.$$

By Theorem 2, this set is analytic; in addition, since X is Borel-dense of order 2 in S, we have $A \subset X$. We claim that $A = X$, which will yield the desired result. If $x \in X$, then $f_0^{-1}(x)$ contains the set

$$Y = f^{-1}(\{x\} \times A_1 \times A_2),$$

a member of $\mathscr{B}(X \times X)$ isomorphic with $A_1 \times A_2$. Now, if $f_0^{-1}(x)$ is reticulate in $S \times S$, so also is Y. In this event, Y may be written as $Y = Y_1 \cup Y_2 \cup \ldots$, where each Y_j is a member of $\mathscr{B}(X \times X)$ contained in a single slice of $X \times X$. This means that each Y_j is an isomorph of a measurable subset of x. An application of Lemmas 6 and 7 shows that Y is Borel-dense of order 1. By Lemma 8, $A_1 \times A_2$ must be standard, a contradiction. So $f_0^{-1}(x)$ is not reticulate, and $A = X$ as claimed.

Case 2. Both of the spaces A_1 and A_2 are standard. Let S be an uncountable standard space. In this case, $X \times X$ and $X \times S \times S$ are isomorphic. Since S and $S \times S$ are isomorphic, we have the isomorphisms

$$X \times X \simeq X \times S \simeq X \times S \times S \simeq X \times X \times S.$$

Case 1 now applies to show that X is analytic.

I do not know whether the product $X \times A_1 \times A_2$ in Theorem 3 may be replaced by the product $A_0 \times A_1 \times A_2$ of any three uncountable spaces (**P 1329**). Higher dimensional versions would depend on the strengthening of uniformisation theorems such as Lemma 5.

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