COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

SEPARABLE VARIABLES ALGEBRAS

BY

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In this paper we adopt the definitions and notation given by Marczewski in [2] and [3]. Let $(A, \boldsymbol{F})$ be an *algebra*, i.e. a set A of elements and a class $\boldsymbol{F}$ of fundamental operations consisting of A-valued functions of several variables running over A. We denote by $\boldsymbol{A}$ the class of all algebraic operations, i.e. the smallest class containing trivial operations

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k \quad (k = 1, 2, \ldots, n; n = 1, 2, \ldots)$$

and closed under the composition with the fundamental operations. The subclass of all algebraic n-ary operations will be denoted by $\boldsymbol{A}^{(n)}$ $(n = 1, 2, \ldots)$. Further, by $\boldsymbol{A}^{(0)}$ we shall denote the set of values of all constant algebraic operations. Elements of $\boldsymbol{A}^{(0)}$ will be called *algebraic constants*. If the class of algebraic operations on $(A, \boldsymbol{F}_1)$ is contained in the class of algebraic operations on $(A, \boldsymbol{F}_2)$, then we say that $(A, \boldsymbol{F}_1)$ is a *reduct* of $(A, \boldsymbol{F}_2)$.

We say that $(A, \boldsymbol{F})$ is an *algebra with separable k variables* $(k \geqslant 1)$ if for every pair $f, g \in \boldsymbol{A}^{(n)}$ $(n > k)$ there exist operations $f_0 \in \boldsymbol{A}^{(k)}$ and $g_0 \in \boldsymbol{A}^{(n-k)}$ such that the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$$

is equivalent in A to the equation

$$f_0(x_1, x_2, \ldots, x_k) = g_0(x_{k+1}, x_{k+2}, \ldots, x_n).$$

Algebras with separable k variables for all $k = 1, 2, \ldots$ will be called briefly *separable variables algebras*. This concept was introduced by Marczewski who investigated the exchange of independent sets in separable variables algebras (see [1]).

Now we shall introduce for a time a notion of a quasi-linear algebra. However, it will soon turn out that it coincides with that of a separable variables algebra (Theorems 1 and 2).

An algebra $(A, \boldsymbol{F})$ is said to be *quasi-linear* if the following conditions hold:

(i) the set A is a subset of an Abelian group G,

(ii) for any operation $f \in A^{(n)}$ $(n = 1, 2, \ldots)$ there exist unary operations $f_1, f_2, \ldots, f_n$ on A (not necessarilly algebraic) such that

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j),$$

where the summation is the group-operation in G,

(iii) there exists a one-to-one unary algebraic operation q such that the binary operation $r(x, y) = q(x) - q(y)$ is algebraic.

Since, by (iii), $r(x, x) = 0$, the zero-element of G is always an algebraic constant in A.

Now we shall give some examples of quasi-linear algebras.

1. Let A be an Abelian group and $\boldsymbol{F}$ the class of all operations f defined as

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} h_j(x_j) + a,$$

where $n = 1, 2, \ldots,$ $a \in A$ and $h_1, h_2, \ldots, h_n$ are endomorphisms of A. Each reduct of $(A, \boldsymbol{F})$ containing a binary operation $r(x, y) = v(x) - v(y)$, where v is an isomorphism from the group A onto its subgroup, is a quasi-linear algebra.

2. Suppose that G is an Abelian group containing two different but isomorphic proper subgroups G_1 and G_2 satisfying the condition $G_1 \supset G_2$. Let B be an arbitrary subset of $G \setminus G_1$ whose cardinal number is not greater than that of $G_1 \setminus G_2$. Further, let q_1 be an isomorphism from G_1 onto G_2 and q_2 a one-to-one mapping from B into $G_1 \setminus G_2$. Put $A = B \cup G_1$, $q(x) = q_1(x)$ if $x \in G_1$, $q(x) = q_2(x)$ if $x \in B$, and $r(x, y) = q(x) - q(y)$. Taking into account the formulas $q(A) \subset G_1$ and $q\big(q(x) - q(y)\big) = q\big(q(x)\big) - q\big(q(y)\big)$ one can prove that the algebra (A, r) is quasi-linear. It should be noted that the set A need not be a group.

The aim of the present paper is to prove the following theorems.

THEOREM 1. *Quasi-linear algebras are separable variables algebras.*

THEOREM 2. *Algebras with separable k variables are quasi-linear.*

It is easy to see that algebras with separable $k+1$ variables are algebras with separable k variables. The converse implication is a consequence of Theorems 1 and 2. Consequently, each algebra with separable one variable is a separable variables algebra. It would be interesting to find a direct proof of this statement. Further, we note that Theorem 2 can be regarded as a representation theorem for separable variables algebras.

THEOREM 3. *Let (A, F) be an algebra with separable variables satisfying one of the following conditions:*

(*) *A is finite,*

(**) *$g(A) = A$ for each non-constant unary algebraic operation g.*

Then A is an Abelian group and each operation f algebraic in (A, F) is of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} h_j(x_j) + a,$$

where $a \in A^{(0)}$ and $h_1, h_2, \ldots, h_n$ are endomorphisms of the group A.

THEOREM 4. *If a semigroup is a separable variables algebra, then it is an Abelian group whose elements have the order bounded in common.*

Before proving Theorem 1 we shall prove a simple lemma.

LEMMA 1. *Let (A, F) be a quasi-linear algebra. If $f \in A^{(1)}$, $g \in A^{(n)}$,*

$$g(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} g_j(x_j)$$

and $b = g(0, 0, \ldots, 0)$, then

$$f\big(g(x_1, x_2, \ldots, x_n)\big) = \sum_{j=1}^{n} f\big(g_j(x_j) - g_j(0) + b\big) - (n-1) f(b).$$

Proof. Setting

$$f\big(g(x_1, x_2, \ldots, x_n)\big) = \sum_{j=1}^{n} h_j(x_j), \tag{1}$$

we have the formula

$$f(b) = \sum_{j=1}^{n} h_j(0). \tag{2}$$

Consequently,

$$f\big(g(0, 0, \ldots, 0, x_j, 0, \ldots, 0)\big) = h_j(x_j) - h_j(0) + f(b).$$

On the other hand, we have the formula

$$f\big(g(0, 0, \ldots, 0, x_j, 0, \ldots, 0)\big) = f\big(g_j(x) - g_j(0) + b\big)$$

which implies the equation

$$h_j(x_j) = f\big(g_j(x_j) - g_j(0) + b\big) - f(b) + h_j(0).$$

Hence and from (1) and (2) the assertion of the Lemma follows.

Proof of Theorem 1. Suppose that $n > k \geqslant 1$ and $f, g \in A^{(n)}$. Let q be the algebraic unary operation satisfying condition (iii) of the definition of quasi-linear algebras. Since the operation q is one-to-one, the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \tag{3}$$

is equivalent to the equation

$$q^2\big(f(x_1, x_2, \ldots, x_n)\big) = q^2\big(g(x_1, x_2, \ldots, x_n)\big), \tag{4}$$

where $q^2(x) = q\big(q(x)\big)$. Setting

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j), \qquad g(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} g_j(x_j),$$

we have, by Lemma 1, the equations

$$q^2\big(f(x_1, x_2, \ldots, x_n)\big) = \sum_{j=1}^{n} q^2\big(f_j(x_j) - f_j(0) + a\big) - (n-1)q^2(a), \tag{5}$$

$$q^2\big(g(x_1, x_2, \ldots, x_n)\big) = \sum_{j=1}^{n} q^2\big(g_j(x_j) - g_j(0) + b\big) - (n-1)q^2(b), \tag{6}$$

where $a = f(0, 0, \ldots, 0)$ and $b = g(0, 0, \ldots, 0)$. Put

$$\begin{aligned} &f_0(x_1, x_2, \ldots, x_k) \\ &= r\Big(q\big(f(x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0)\big), q\big(g(x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0)\big)\Big), \\ &g_0(x_{k+1}, x_{k+2}, \ldots, x_n) \\ &= r\Big(q\big(g(0, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots, x_n)\big), q\big(f(0, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots, x_n)\big)\Big), \end{aligned}$$

where $r(x, y) = q(x) - q(y)$. Obviously, the operations f_0 and g_0 are algebraic. Moreover, by Lemma 1, we have the equations

$$\begin{aligned} f_0(x_1, x_2, \ldots, x_k) &= \sum_{j=1}^{k} q^2\big(f_j(x_j) - f_j(0) + a\big) - (k-1)q^2(a) - \\ &\quad - \sum_{j=1}^{k} q^2\big(g_j(x_j) - g_j(0) + b\big) + (k-1)q^2(b), \\ g_0(x_{k+1}, x_{k+2}, \ldots, x_n) &= \sum_{j=k+1}^{n} q^2\big(g_j(x_j) - g_j(0) + b\big) - (n-k-2)q^2(b) - \\ &\quad - \sum_{j=k+1}^{n} q^2\big(f_j(x_j) - f_j(0) + a\big) + (n-k-2)q^2(a). \end{aligned}$$

Hence and from (5) and (6) it follows that the equation

$$f_0(x_1, x_2, \ldots, x_k) = g_0(x_{k+1}, x_{k+2}, \ldots, x_n)$$

is equivalent to equation (4) and, consequently, to equation (3). Thus $(A, \boldsymbol{F})$ is a separable variables algebra, which completes the proof.

Now we shall prove some Lemmas for algebras with separable one variable. In all further considerations we shall assume that the algebra in question is an algebra with separable one variable.

LEMMA 2. *Let* $f, g \in \boldsymbol{A}^{(n)}$ $(n \geqslant 3)$. *If the equation*

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \tag{7}$$

holds whenever $x_1 = x_2$ *or* $x_1 = x_3$, *then* $f = g$.

Proof. Equation (7) is equivalent to the equation

$$f_0(x_1) = g_0(x_2, x_3, \ldots, x_n),$$

where $f_0 \in \boldsymbol{A}^{(1)}$ and $g_0 \in \boldsymbol{A}^{(n-1)}$. Since (7) holds whenever $x_1 = x_2$ or $x_1 = x_3$, we infer that $f_0(x_2) = g_0(x_2, x_3, \ldots, x_n) = f_0(x_3)$ for all $x_2, x_3, \ldots, x_n \in A$. Thus both operations f_0 and g_0 are constant and $f_0 = g_0$. Consequently, equation (7) holds for all $x_1, x_2, \ldots, x_n \in A$, which completes the proof.

LEMMA 3. *There exists a ternary algebraic operation* s *such that*

$$s(x, y, y) = s(y, x, y) = q(x), \tag{8}$$

where q *is a one-to-one operation having a fixed point in* $\boldsymbol{A}^{(0)}$.

Proof. The equation $e_1^{(3)}(x, y, z) = e_2^{(3)}(x, y, z)$ is equivalent to the equation $f_0(z) = g_0(x, y)$, where $f_0 \in \boldsymbol{A}^{(1)}$ and $g_0 \in \boldsymbol{A}^{(2)}$. Since $e_1^{(3)}(x, x, z) = e_2^{(3)}(x, x, z)$, we infer that $f_0(z) = g_0(x, x)$. Thus f_0 is a constant operation. Denoting its value by c_0 we have $g_0(x, y) = c_0$ if and only if $e_1^{(3)}(x, y, z) = e_2^{(3)}(x, y, z)$, i.e. $x = y$. Further, the equation

$$g_0(x, y) = g_0(z, t) \tag{9}$$

is equivalent to the equation

$$f_1(x) = g_1(y, z, t), \tag{10}$$

where $f_1 \in \boldsymbol{A}^{(1)}$ and $g_1 \in \boldsymbol{A}^{(3)}$. Since (9) holds whenever $x = z$ and $y = t$ or $x = y$ and $z = t$, we have the formulas

$$f_1(x) = g_1(y, x, y) = g_1(x, y, y). \tag{11}$$

Suppose that $f_1(a) = f_1(b)$. Then, by (11), $f_1(b) = g_1(a, y, y)$ which is equivalent to the equation $g_0(b, a) = g_0(y, y)$. Thus $g_0(b, a) = c_0$ and, consequently, $a = b$ which shows that the operation f_1 is one-to-one.

Let c be an element of A. We shall prove that the unary operation $g_0(x, c)$ is one-to-one. Suppose that $g_0(a, c) = g_0(b, c)$. Then, by the equivalence of (9) and (10), $f_1(a) = g_1(c, b, c)$. Hence, by (11), $f_1(a) = f_1(b)$ which implies $a = b$, because the operation f_1 is one-to-one.

Put $q(x) = g_0(f_1(x), f_1(c_0))$ and $s(x, y, z) = g_0(g_1(x, y, z), f_1(c_0))$. Of course, the operations q and s are algebraic. Moreover, the operation q, being a composition of one-to-one operations, is one-to-one too, and $q(c_0) = g_0(f_1(c_0), f_1(c_0)) = c_0$, i.e. the algebraic constant c_0 is a fixed point of q. Finally, formula (11) implies (8) which completes the proof.

LEMMA 4. *For each algebraic ternary operation s satisfying the condition*

$$s(x, y, y) = s(y, x, y) = q(x) \tag{12}$$

the following equations are true:

$$s(x, y, z) = s(y, x, z), \tag{13}$$

$$\begin{aligned} &q(f(x_1, x_2, \ldots, x_n)) \\ &= s(f(x_1, x_1, x_3, x_4, \ldots, x_n), f(x_1, x_2, x_1, x_4, \ldots, x_n), f(x_1, x_1, x_1, x_4, \ldots, x_n)) \\ &\qquad\qquad \textit{for} \quad f \in A^{(n)} \ (n \geqslant 3), \end{aligned} \tag{14}$$

$$q(s(x, y, z)) = s(q(x), q(y), q(z)), \tag{15}$$

$$q(g(x, y)) = s(g(z, y), g(x, z), g(z, z)) \quad \textit{for} \quad g \in A^{(2)}, \tag{16}$$

$$s(s(x, y, t), q(z), q(t)) = s(q(x), s(y, z, t), q(t)), \tag{17}$$

$$q(s(x, y, z)) = s(s(x, y, t), s(t, t, z), q(t)). \tag{18}$$

Proof. From (12) it follows that equation (13) holds whenever $z = x$ or $z = y$. Thus, by Lemma 2, it holds for all $x, y, z \in A$. Further, by (12), equation (14) holds whenever $x_1 = x_2$ or $x_1 = x_3$. Consequently, by Lemma 2, it holds for all $x_1, x_2, \ldots, x_n \in A$ and $f \in A^{(n)}$ $(n \geqslant 3)$. Setting $f(x_1, x_2, x_3) = s(x_3, x_2, x_1)$ into (14) and taking into account (12) we get formula (15). Setting $f(x_1, x_2, x_3) = g(x_2, x_3)$ into equation (14) we obtain formula (16).

From (12) it follows that (17) holds if $t = y$. Moreover, by (12) and (15), it holds if $t = z$. Thus, by Lemma 2, equation (17) holds for all $x, y, z, t \in A$.

According to (12), equation (18) holds if $t = z$. Moreover, by (12), (13) and (14),

$$\begin{aligned} q(s(x, y, z)) = q(s(y, x, z)) &= s(s(y, y, z), q(x), q(y)) \\ &= s(q(x), s(y, y, z), q(y)), \end{aligned}$$

which shows that (18) holds if $t = y$. Thus, by Lemma 2, equation (18) holds for all $x, y, z, t \in A$ which completes the proof of the Lemma.

In the sequel q^n $(n \geqslant 0)$ will denote the n-th composition of the unary operation q, i.e. $q^0(x) = x$ and $q^{n+1}(x) = q(q^n(x))$ $(n = 0, 1, \ldots)$.

Let N be the set of all non-negative integers. We define a congruence relation in the Cartesian product $A \times N$ as follows: $\langle a, k\rangle \sim \langle b, r\rangle$ if and only if $q^{n-k}(a) = q^{n-r}(b)$ for all $n \geqslant \max(k, r)$, where q is a one-to-one algebraic unary operation determined by Lemma 3. The equivalence class containing the element $\langle a, k\rangle$ will be denoted by $[a, k]$. Let us denote by G the set of all equivalence classes $[a, k]$ ($\langle a, k\rangle \in A\times N$). It is very easy to verify the equation

$$[a, k] = [q(a), k+1]. \tag{19}$$

Moreover, if $a\neq b$, then $[a, k]$ and $[b, k]$ are disjoint because the operation q is one-to-one. Consequently, we may identify the elements a and $[a, 0]$ ($a\in A$), i.e. the set A can be regarded as a subset of the set G. Further, by (19), each finite system of elements of G can be represented in the form $[a_1, k], [a_2, k], \ldots, [a_n, k]$, where k is a sufficiently large integer.

LEMMA 5. *If* $q(A) = A$, *then* $A = G$.

Proof. For any $b\in A$ and $k\geqslant 1$ there exists an element $a\in A$ such that $q^k(a) = b$. Thus, by (19),

$$[b, k] = [q^k(a), k] = [a, 0] = a,$$

which completes the proof.

LEMMA 6. *The set G is an Abelian group under the addition*

$$[a, k]+[b, k] = [s(a, b, 0), k+1], \tag{20}$$

where the operation s is determined by Lemma 3 and 0 is a fixed point of q in $A^{(0)}$. Moreover,

$$-[a, k] = [s(0, 0, a), k+1]. \tag{21}$$

Proof. First of all we shall prove that the definition (20) does not depend upon the choice of the representation of elements of G. Suppose that $[a, k] = [c, r]$ and $[b, k] = [d, r]$. Consequently, $q^{n-k}(a) = q^{n-r}(c)$ and $q^{n-k}(b) = q^{n-r}(d)$ for $n\geqslant \max(k, r)$. Hence, by (15), we get the equation

$$\begin{aligned} q^{(n+1)-(k+1)}\big(s(a, b, 0)\big) &= q^{n-k}\big(s(a, b, 0)\big) = s\big(q^{n-k}(a), q^{n-k}(b), 0\big) \\ &= s\big(q^{n-r}(c), q^{n-r}(d), 0\big) = q^{n-r}\big(s(c, d, 0)\big) = q^{(n+1)-(r+1)}\big(s(c, d, 0)\big). \end{aligned}$$

Thus $[s(a, b, 0), k+1] = [s(c, d, 0), r+1]$ which shows that the definition (20) does not depend upon the choice of the representation of elements of G.

From (13) the commutative law follows. Further, by (17) and (19), we have the associative law:

$$\begin{aligned} ([a, k]+[b, k])+[c, k] &= [s(a, b, 0), k+1]+[q(c), k+1] \\ &= \big[s\big(s(a, b, 0), q(c), 0\big), k+2\big] = \big[s\big(q(a), s(b, c, 0), 0\big), k+2\big] \\ &= [q(a), k+1]+[s(b, c, 0), k+1] = [a, k]+([b, k]+[c, k]). \end{aligned}$$

The element 0 of A is the zero-element of G. In fact, by (19), $0 = [0,0] = [q^k(0), k] = [0, k]$ and, consequently,

$$[a,k]+0 = [a,k]+[0,k] = [s(a,0,0), k+1] = [q(a), k+1] = [a,k].$$

Finally, from (18) for $t = y = 0, x = z = a$ and (19) we get the equation

$$\begin{aligned}&[a,k]+[s(0,0,a),k+1] = [q(a),k+1]+[s(0,0,a),k+1]\\&= \big[s\big(q(a), s(0,0,a), 0\big), k+2\big] = [s(a,0,a), k+2] = [q(0), k+2)]\\&= [0, k+1] = 0,\end{aligned}$$

which implies (21). The Lemma is thus proved.

LEMMA 7. *The ternary operation s satisfies the equation*

$$s(x,y,z) = q(x)+q(y)-q(z)$$

for all $x, y, z \in A$.

Proof. By (18) and (19) we have the equation

$$\begin{aligned}s(x,y,z) &= [s(x,y,z),0] = \big[q\big(s(x,y,z)\big),1\big]\\&= \big[s\big(s(x,y,0), s(0,0,z), 0\big),1\big] = [s(x,y,0),0]+[s(0,0,z),0].\end{aligned}$$

Further, by (15) and (19),

$$\begin{aligned}[s(x,y,0),0] &= \big[q\big(s(x,y,0)\big),1\big] = \big[s\big(q(x),q(y),0\big),1\big]\\&= [q(x),0]+[q(y),0] = q(x)+q(y)\end{aligned}$$

and, by (15), (19) and (21),

$$[s(0,0,z),0] = \big[q\big(s(0,0,z)\big),1\big] = \big[s\big(0,0,q(z)\big),1\big] = -[q(z),0] = -q(z),$$

whence the assertion of the Lemma follows.

LEMMA 8. *Each n-ary algebraic operation f is of the form*

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} f_j(x_j). \tag{22}$$

Proof. We shall prove the Lemma by induction with respect to n. For $n=1$ it is obvious. Suppose that $n \geqslant 2$ and that for all $(n-1)$-ary algebraic operations the Lemma is true. Let $f \in A^{(n)}$. By (14) and (16) for $z = 0$ we have the formula

$$\begin{aligned}&q\big(f(x_1,x_2,\ldots,x_n)\big)\\&\qquad= s\big(g_1(x_1,x_2,\ldots,x_n), g_2(x_1,x_2,\ldots,x_n), g_3(x_1,x_2,\ldots,x_n)\big),\end{aligned}$$

where the algebraic operations g_1, g_2 and g_3 depend on at most $n-1$

variables and, consequently, by the inductive assumption, are of the form (22). Further, by Lemma 7,

$$q\big(f(x_1, x_2, \ldots, x_n)\big)$$
$$= q\big(g_1(x_1, x_2, \ldots, x_n)\big) + q\big(g_2(x_1, x_2, \ldots, x_n)\big) - q\big(g_3(x_1, x_2, \ldots, x_n)\big).$$

Consequently, by (19),

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= [f(x_1, x_2, \ldots, x_n), 0] = \big[q\big(f(x_1, x_2, \ldots, x_n)\big), 1\big] \\ &= \big[q\big(g_1(x_1, x_2, \ldots, x_n)\big), 1\big] + \big[q\big(g_2(x_1, x_2, \ldots, x_n)\big), 1\big] - \\ &\quad - \big[q\big(g_3(x_1, x_2, \ldots, x_n)\big), 1\big] = [g_1(x_1, x_2, \ldots, x_n), 0] + \\ &\quad + [g_2(x_1, x_2, \ldots, x_n), 0] - [g_3(x_1, x_2, \ldots, x_n), 0] \\ &= g_1(x_1, x_2, \ldots, x_n) + g_2(x_1, x_2, \ldots, x_n) - g_3(x_1, x_2, \ldots, x_n), \end{aligned}$$

whence it follows that f is also of the form (22). The Lemma is thus proved.

Proof of Theorem 2. Let $(A, \boldsymbol{F})$ be an algebra with separable k variables and, consequently, with separable one variable. By Lemma 6 the set A is contained in the Abelian group G and, by Lemma 8, all algebraic operations are of the form (22). The unary algebraic operation q determined by Lemma 3 is one-to-one. Since $0 \in \boldsymbol{A}^{(0)}$, $s \in \boldsymbol{A}^{(3)}$ and, by Lemma 7, $q(x) - q(y) = s(x, 0, y)$, the binary operation $q(x) - q(y)$ is algebraic. Thus the algebra $(A, \boldsymbol{F})$ is quasi-linear.

Proof of Theorem 3. If the set A is finite, then the one-to-one algebraic unary operation q determined by Lemma 3 maps A onto itself. Consequently, in both cases (*) and (**) we have the equation $q(A) = A$. Thus, by Lemmas 5 and 6, the set A is an Abelian group under the addition (20). Let $f \in \boldsymbol{A}^{(n)}$ $(n \geqslant 1)$. By Theorem 2 the operation f is of the form (22). Put $a = f(0, 0, \ldots, 0)$ and $h_j(x_j) = f(0, 0, \ldots, 0, x_j, 0, \ldots, 0) - a = f_j(x_j) - f_j(0)$. Of course, $a \in \boldsymbol{A}^{(0)}$ and

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} h_j(x_j) + a.$$

To prove the Theorem it suffices to prove that $h_1, h_2, \ldots, h_n$ are endomorphisms of the group A. By Lemma 7 the binary operation $q(x) + q(y) = s(x, y, 0)$ is algebraic. Moreover, the operation $h_j(x) + a$ is also algebraic. Since, by Theorem 2, the algebra $(A, \boldsymbol{F})$ is quasi-linear, we have, according to Lemma 1, the equation

$$h_j\big(q(x) + q(y)\big) + a = h_j\big(q(x)\big) + h_j\big(q(y)\big) + a$$

and, consequently, the equation

$$h_j\big(q(x) + q(y)\big) = h_j\big(q(x)\big) + h_j\big(q(y)\big).$$

Since $q(A) = A$, the last formula implies the equation $h_j(x+y) = h_j(x)+h_j(y)$ for all $x, y \in A$. Thus h_j are endomorphisms of A which completes the proof.

Proof of Theorem 4. Suppose that (A, f), where $f(x, y) = xy$ denotes a semigroup multiplication, is a separable variables algebra. By Theorem 2, (A, f) is a quasi-linear algebra. In particular, the set A is a subset of an Abelian group G and the zero-element 0 of G is an algebraic constant in A. Since each unary algebraic operation in (A, f) is of the form x^n $(n = 1, 2, \ldots)$, we infer that there exists a positive integer p such that

$$x^p = 0 \qquad (x \in A). \tag{23}$$

The element 0 is an idempotent in the semigroup A. In fact, by (23),

$$0^2 = (x^p)^2 = (x^2)^p = 0. \tag{24}$$

Moreover, if $c \in A^{(0)}$, then $x^m = c$ for a positive integer m and for all $x \in A$. In particular, $0 = 0^m = c$. Thus 0 is the only algebraic constant in A.

Now we shall prove that

$$xy = yx \tag{25}$$

for all $x, y \in A$. Since A is a separable variables algebra, equation (25) is equivalent to an equation

$$x^r = y^s, \tag{26}$$

where r and s are positive integers. Equation (25) holds, by (23), if $y = 0$. Consequently, by (24), $x^r = 0$ for all $x \in A$. Similarly we obtain the formula $y^s = 0$ for all $y \in A$. Thus (26) and, consequently, (25) holds for all $x, y \in A$.

Since 0 is the only algebraic constant and the semigroup A is commutative, the operation xy can be written, in view of Theorem 2, in the form

$$xy = h(x)+h(y), \tag{27}$$

where $h(0) = 0$. Hence and from (23) we get the formula

$$h(x) = x0 = x^{p+1} \qquad (x \in A), \tag{28}$$

which shows that the operation h is algebraic.

Let A_a be the subalgebra of A generated by an arbitrary element $a \in A$, i.e. a subsemigroup generated by a. Of course, by (28), we have the inclusion

$$h(A_a) \subset A_a. \tag{29}$$

Moreover, by (23), the subsemigroup A_a is finite. Further, by (27), we have the equation

$$h(h(x)) = (x0)0 = x(00) = x0 = h(x). \tag{30}$$

Hence and from (27) if follows that

$$x^n = nh(x) \quad (n \geqslant 2). \tag{31}$$

By Theorem 2 there exists a one-to-one algebraic unary operation q such that the binary operation $q(x)-q(y)$ is also algebraic. Consequently, $q(0) = 0$ and $q(x)-q(y) = x^r y^s$, where r and s are positive integers. Thus, by (23), $q(x) = x^r 0 = x^{r+p}$ and, consequently, by (31), $q(x) = (r+p)h(x)$, because $r+p \geqslant 2$. Hence it follows that the operation h is one-to-one which, by (29) and the finiteness of A_a, implies the equation $h(A_a) = A_a$. This equation and (30) yield the formula $h(x) = x$ for $x \in A_a$ and, consequently, for all $x \in A$. Thus, by (27), $xy = x+y$. Moreover, by (23), $px = 0$ for all $x \in A$, which implies that A is a subgroup of the group G and all elements of A have the order not greater than p. The Theorem is thus proved.

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Reçu par la Rédaction le 29. 8. 1965