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A CHAINABLE CONTINUUM NOT HOMEOMORPHIC TO AN INVERSE LIMIT ON [0, 1] WITH ONLY ONE BONDING MAP*

BY

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1. Introduction. Mahavier has exhibited [1] a chainable continuum not homeomorphic to an inverse limit on $[0, 1]$ with only one bonding map. Here we present such a continuum which is not homeomorphic to Mahavier's example and is of a simpler nature than his example.

2. Definitions, notation and a theorem. If each term of the sequence $f_1, f_2, f_3, \ldots$ maps $[0, 1]$ onto $[0, 1]$, then the *inverse limit* of the sequence $f_1, f_2, f_3, \ldots$, denoted by $\operatorname{invlim}([0, 1], f_i)$, is the subspace of the Cartesian product $\prod_{i=1}^{\infty} [0, 1]_i$ to which the number sequence $x_1, x_2, x_3, \ldots$ belongs only in case $f_n(x_{n+1}) = x_n$ for each positive integer n. If $f_1, f_2, f_3, \ldots$ is a constant sequence, say g is f_i for each positive integer i, then $\operatorname{invlim}([0, 1], g)$ denotes $\operatorname{invlim}([0, 1], f_i)$. If $x_1, x_2, x_3, \ldots$ is a constant number sequence, then (x_1) denotes the point $(x_1, x_2, x_3, \ldots)$ in $\prod_{i=1}^{\infty} [0, 1]_i$. For each positive integer j, π_j denotes the projection mapping from $\prod_{i=1}^{\infty} [0, 1]_i$ onto the j-th factor space. By *continuum* we mean a non-degenerate, compact, connected metric space. The metric d on $\prod_{i=1}^{\infty} [0, 1]_i$ is defined as

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|\pi_i(x) - \pi_i(y)|}{2^i}$$

for each x and y belonging to $\prod_{i=1}^{\infty} [0, 1]_i$. If T denotes an arc with non-separating points a and b, we write T as $[a, b]$.

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[1] W. S. Mahavier, *A chainable continuum not homeomorphic to an inverse limit on* [0, 1] *with only one bonding map*, Proceedings of the American Mathematical Society 18 (1967), p. 284-286.

THEOREM. *Suppose that k maps $[0, 1]$ onto $[0, 1]$ in such a way that $A = \operatorname{invlim}([0, 1], k)$ is the sum of two mutually exclusive connected sets M and N, where M is a topological ray, N is an arc, and M is dense in A. Then there exists a proper subcontinuum $[a, b]$ of $[0, 1]$ such that N is homeomorphic to $\operatorname{invlim}([a, b], k^2|[a, b])$ and each of the non-separating points of N corresponds to a constant number sequence in $\operatorname{invlim}([a, b], k^2|[a, b])$.*

Proof. Since k is not the identity, the function h defined as

$$h(x_1, x_2, x_3, \ldots) = \big(k(x_1), k(x_2), k(x_3), \ldots\big) = \big(k(x_1), x_1, x_2, \ldots\big)$$

for each point $(x_1, x_2, x_3, \ldots)$ in A is a non-trivial homeomorphism from A onto A (op. cit.). For each positive integer j we have $\pi_j N = \pi_j hN$, since h maps N onto N. Also, by the definition of h we have $\pi_j N = \pi_{j+1} hN$. Thus $\pi_j N = \pi_{j+1} N$ and if $[a, b]$ denotes $\pi_j N$, then $[a, b]$ is a proper subcontinuum of $[0, 1]$ such that N is homeomorphic to $\operatorname{invlim}([a, b], k^2|[a, b])$.

The two non-separating points of N must either be fixed points of h or be switched by h. So we infer that any projection of a non-separating point of N is a fixed point of k^2, thus the non-separating points of $\operatorname{invlim}([a, b], k^2|[a, b])$ are constant number sequences.

3. Example. In this section we give an example of a chainable continuum not homeomorphic to an inverse limit on $[0, 1]$ with only one bonding map.

Let f be the mapping from $[0, 1]$ onto $[0, 1]$ defined by

$$f(x) = \begin{cases} 4x & \text{if } x \text{ is in } [0, 1/4], \\ -2x+3/2 & \text{if } x \text{ is in } [1/4, 1/2], \\ x & \text{if } x \text{ is in } [1/2, 1]. \end{cases}$$

Let g be the mapping from $[0, 1]$ onto $[0, 1]$ defined by

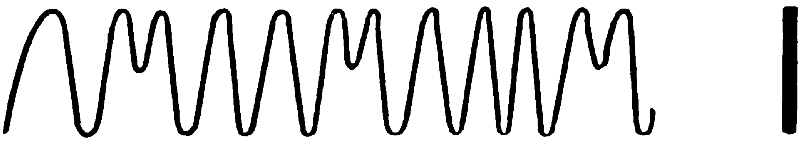
$$g(x) = \begin{cases} 4x & \text{if } x \text{ is in } [0, 1/4], \\ -3x+7/4 & \text{if } x \text{ is in } [1/4, 1/3], \\ 3x-1/4 & \text{if } x \text{ is in } [1/3, 5/12], \\ -6x+7/2 & \text{if } x \text{ is in } [5/12, 1/2], \\ x & \text{if } x \text{ is in } [1/2, 1]. \end{cases}$$

Let $N_1, N_2, N_3, \ldots$ denote the positive integer sequence such that $N_1 = 2$, and if n is a positive integer greater than 1, then $N_n = N_{n-1} + n + 1$. Let $t_1, t_2, t_3, \ldots$ denote the function sequence defined as

$$t_n = \begin{cases} g & \text{if } n = N_j \text{ for some positive integer } j, \\ f & \text{otherwise}. \end{cases}$$

Let A denote $\operatorname{invlim}([0, 1], t_i)$.

The continuum A is the union of a topological ray and an arc, say M and N, respectively, such that M and N are mutually exclusive and M is dense in A. The arc N is $\operatorname{invlim}([1/2, 1], t_i|[1/2, 1])$ with non-separating points (1/2) and (1), and $t_n | [1/2, 1]$ is the identity for each positive integer n. Note here that a point x in A is a constant number sequence if and only if x belongs to N or $\pi_1(x) = 0$. The reader will likely recognize that the continuum A is a sinusoid homeomorphic to



4. Proof. We prove here that *the continuum A is not homeomorphic to an inverse limit on* $[0, 1]$ *using only one bonding map.*

Assume that there exists a mapping k' from $[0, 1]$ onto $[0, 1]$ such that $\operatorname{invlim}([0, 1], k')$ is homeomorphic to A. Let k denote $(k')^2$, A_k the continuum $\operatorname{invlim}([0, 1], k)$, and F a homeomorphism from A onto A_k. By the Theorem, there exists a proper subcontinuum of $[0, 1]$, say $[a, b]$, such that $F[N]$ is $\operatorname{invlim}([a, b], k | [a, b])$. Let h denote the non-trivial homeomorphism from A_k onto A_k defined by

$$h(x_1, x_2, x_3, \ldots) = \big(k(x_1), k(x_2), k(x_3), \ldots\big) = \big(k(x_1), x_1, x_2, \ldots\big)$$

for each point $(x_1, x_2, x_3, \ldots)$ in A_k. From the Theorem we also infer that the non-separating points $F\big((1/2)\big)$ and $F\big((1)\big)$ in $F[N]$ are constant sequences, say (c) and (d), respectively. The non-separating point $F\big((0)\big)$ in $F[M]$ is a fixed point of h, thus a constant sequence, say (p), and p belongs to $[0, 1] \setminus [a, b]$.

Let x_0 be a point in $[0, 1]$ such that $k(x_0) = c$ and such that x_0 belongs to the component of $[0, 1] \setminus [a, b]$ containing p. That there is such a point follows from the irreducibility of A_k from (p) to any point of $F[N]$ together with the intermediate value theorem. Let y be a point in the ray $F[M]$ such that $\pi_1(y) = x_0$ and let $y_1, y_2, y_3, \ldots$ denote the sequence of points in $F[M]$ such that $y_m = h^m(y)$ for each positive integer m. We observe that, for each positive integer m, if j is a positive integer not greater than m, then $\pi_j(y_m) = c$. Thus $y_1, y_2, y_3, \ldots$ converges to (c).

For each point x in the ray M there is a positive integer m such that $\pi_m(x) < 1/2$. Let J be the function from M onto the positive integers such that if x is in M, then $J(x)$ is the least positive integer i such that $\pi_i(x) < 1/2$.

The continuum A is ordered with respect to the following meaning of the word "precedes":

(1) if each of u and v is a point on the ray M, then u precedes v provided $J(u) < J(v)$ or $J(u) = J(v)$ and $\pi_{J(u)}(u) \leqslant \pi_{J(v)}(v)$;

(2) if each of u and v belongs to N, then u precedes v provided $\pi_1(u) \leqslant \pi_1(v)$;

(3) if one of u and v belongs to M and the other to N, then u precedes v provided u belongs to M.

We observe that, for points in the ray M, the above-defined meaning of "precedes" is equivalent to the usual order on a ray. Since F is order-preserving, x precedes y in the ray $F[M]$ provided x is (p) or x is a separating point of the arc $[(p), y]$.

It follows from the order on A_k, the definition of the homeomorphism h and the convergence of the sequence $y_1, y_2, y_3, \dots$ to a point of $F[N]$ that if i denotes a positive integer, then y_i precedes y_{i+1}. Thus, for each positive integer i, if C_i denotes the arc $[y_i, y_{i+1}]$ and D_i the arc $F^{-1}[C_i]$, then $h[C_i] = C_{i+1}$ and $F^{-1}hF[D_i] = D_{i+1}$. Let G denote $F^{-1}hF$.

We now show the following:

(1) *There exist positive integers L and Q such that if s is an integer greater than L, then the subset V_s of D_s to which v belongs only in case $\pi_1(v) = 1$ contains only Q elements.*

To see this we first show:

(2) *There exists a positive integer L' such that if m is an integer greater than L', then D_m contains a point u such that $\pi_1(u) = 1$.*

Since $y_1, y_2, y_3, \dots$ converges to (c), $F^{-1}(y_1)$, $F^{-1}(y_2)$, $F^{-1}(y_3), \dots$ converges to $(1/2)$. Let $\varepsilon > 0$ be such that $\varepsilon < 1/8$. Let W denote a positive integer such that if v is an integer greater than W, then

$$d\big(F^{-1}(y_v), (1/2)\big) < \varepsilon.$$

Thus $\pi_1 F^{-1}(y_v) < 3/4$, for if x belongs to M and $\pi_1(x) \geqslant 3/4$, then

$$d\big(x, (1/2)\big) = \sum_{i=1}^{\infty} \frac{|\pi_i(x) - 1/2|}{2^i} \geqslant \frac{1}{8} + \sum_{i=2}^{\infty} \frac{|\pi_i(x) - 1/2|}{2^i} > \varepsilon.$$

Assuming that statement (2) is not true, we let T denote the set to which the integer s belongs if and only if $s > W$ and 1 is not in $\pi_1[D_s]$. Let $T_1, T_2, T_3, \dots$ denote the increasing integer sequence with final set T. If each of m and n denotes a positive integer, then $\pi_n[D_{T_m}]$ is a subset of the half-open interval $[0, 3/4)$. This is a consequence of the order of the elements in $F^{-1}(y_1), F^{-1}(y_2), F^{-1}(y_3), \dots$ together with the fact that, given a positive integer j, $d\big(F^{-1}(y_{T_j}), (1/2)\big) < \varepsilon$ but no point z in D_{T_j} is such that $\pi_1(z) = 1$. Let $Q_1, Q_2, Q_3, \dots$ denote an infinite, increasing, positive integer sequence such that if j denotes a positive integer, then D_{Q_j} contains a point y such that $\pi_1(y) = 1$ and $D_{Q_j - 1} = D_{T_n}$ for some positive integer n. Let $z_1, z_2, z_3, \dots$ denote a sequence such that if m is a positive integer, then z_m belongs to D_{Q_m} and $\pi_1(z_m) = 1$. The sequence $z_1, z_2, z_3, \dots$ converges to (1), and so does the sequence $G^{-1}(z_1), G^{-1}(z_2)$, $G^{-1}(z_3), \dots$ However, if j is a positive integer, then $G^{-1}(z_j)$ is a point of D_{T_m} for some integer m, so $\pi_n G^{-1}(z_j)$ is not greater than $3/4$ for any positive integer n, and thus $G^{-1}(z_1), G^{-1}(z_2), G^{-1}(z_3), \ldots$ cannot converge to (1). This is a contradiction from which it follows that statement (2) is true.

Let L' denote a positive integer as in statement (2) and such that $L' > W$. For each positive integer i greater than L', let V_i denote the set to which v belongs if and only if v is a point of D_i and $\pi_1(v) = 1$; let $p(i)$ denote the number of elements belonging to V_i. We write V_i as $\{v_{i1}, v_{i2}, \ldots, v_{ip(i)}\}$ and note that if v_{js} precedes v_{kt}, where v_{js} and v_{kt} are points of $\bigcup_{i>L'} V_i$, then either $j < k$ or $j = k$ and $s < t$. Let $a_1, a_2, a_3, \ldots$ denote the sequence of points in M with final set $\bigcup_{i>L'} V_i$ and such that if a_j precedes a_k, then $j < k$. The sequence $a_1, a_2, a_3, \ldots$ converges to (1) as do the sequences $G(a_1), G(a_2), G(a_3), \ldots$ and $G^{-1}(a_1), G^{-1}(a_2), G^{-1}(a_3), \ldots$ Let $b > 0$ be such that $b < 1/16$. Let R denote an integer such that if s is an integer greater than R, then

$$d\big(a_s, (1)\big) < b, \quad d\big(G(a_s), (1)\big) < b \quad \text{and} \quad d\big(G^{-1}(a_s), (1)\big) < b.$$

Thus we have

$$\pi_1(a_s) > 7/8, \quad \pi_1 G(a_s) > 7/8 \quad \text{and} \quad \pi_1 G^{-1}(a_s) > 7/8,$$

for if z belongs to M and $d\big(z, (1)\big) < b$, then

$$\frac{|\pi_1(z) - 1|}{2} < b < \frac{1}{16},$$

so $\pi_1 z > 7/8$.

Assume now that statement (1) is not true. Let $m_1, m_2, m_3, \ldots$ denote an increasing, positive integer sequence such that $m_1 > L'$ and either $p(m_j) < p(m_j+1)$ for each positive integer j or $p(m_j) > p(m_j+1)$ for each positive integer j.

Suppose first that $p(m_j) < p(m_j+1)$ for each positive integer j. Let K denote a positive integer such that whenever u denotes a positive integer greater than K and i denotes a positive integer not greater than $p(m_u)$, then $v_{m_u i}$ is a_s for some positive integer s greater than R. Now let u denote a positive integer greater than K; there exist points $x(u)$ and $y(u)$ in V_{m_u+1} such that $x(u)$ precedes $y(u)$ and such that each point z in the arc $G^{-1}\big[[x(u), y(u)]\big]$ has the property that

$$\pi_1(z) \geqslant \min\big\{\pi_1 G^{-1}\big(x(u)\big), \pi_1 G^{-1}\big(y(u)\big)\big\}.$$

Thus $\pi_1 z > 7/8$. Let α_{m_u+1} denote the arc $[x(u), y(u)]$ and let α_{m_u} denote $G^{-1}\big[[x(u), y(u)]\big]$; i.e., α_{m_u} is the arc $\big[G^{-1}\big(x(u)\big), G^{-1}\big(y(u)\big)\big]$. Let $w_1, w_2, w_3, \ldots$ denote a sequence of points such that w_j belongs to α_{m_j} for each positive integer j. The sequence $w_1, w_2, w_3, \ldots$ has the sequential limit point (1) and $G(w_j)$ belongs to α_{m_j+1} for each positive integer j.

For each positive integer j, a_{m_j+1} contains a point, say d_j, such that $\pi_1(d_j) = 3/4$, and so the sequence $d_1, d_2, d_3, \ldots$ converges to $(3/4)$. The sequence $G^{-1}(d_1), G^{-1}(d_2), G^{-1}(d_3), \ldots$ converges to (1) but $G^{-1}\big((3/4)\big) \neq (1)$, a contradiction from which it follows that $p(m_j) > p(m_j+1)$ for each positive integer j. A similar argument contradicts $p(m_j) > p(m_j+1)$ for each positive integer j from which it follows that statement (1) is true.

Let L and Q denote positive integers which satisfy statement (1). There exists a positive integer greater than L, say L_0, such that if s denotes a positive integer greater than L_0 and t denotes the least positive integer i such that $N_i \leqslant J(z)$ for each point z in D_s, then $N_{t+1} - N_t \geqslant 3Q$.

There exists an increasing, positive integer sequence, say $s_1, s_2, s_3, \ldots$, such that $s_1 > L_0$ and if i is a positive integer, then D_{s_i} contains an arc, say γ_{s_i}, with non-separating points in V_{s_i}, and such that if z is a point of γ_{s_i}, then $J(z) = N_t$ for some positive integer t. So $\pi_1(z) \geqslant 3/4$, and if $z_1, z_2, z_3, \ldots$ denotes a sequence of points such that z_n is a point of γ_{s_n} for each positive integer n, then any limit point of $\{z_1, z_2, z_3, \ldots\}$ is a point of the arc N and has the first projection not less than $3/4$. Let (q) denote $G\big((3/4)\big)$; q is a number such that $1/2 < q < 1$.

If i is a positive integer, then $G(\gamma_{s_i})$ is a subset of D_{s_i+1}, call it γ_{s_i+1}. Since $s_i > L_0$, D_{s_i+1} does not contain a point u such that $J(u) = N_t$ for any positive integer t. For each positive integer i, let $x(s_i)$ and $y(s_i)$ denote the non-separating points of γ_{s_i} with $x(s_i)$ preceding $y(s_i)$. It follows from an argument similar to that in proving statement (1) that there do not exist more than finitely many integers m such that if z is a point of the arc $G\big[[x(s_m), y(s_m)]\big]$, then

$$\pi_1(z) \geqslant \min\big\{\pi_1 G\big(x(s_m)\big), \pi_1 G\big(y(s_m)\big)\big\}.$$

So there is a positive integer T such that if n is a positive integer greater than T, then $JG\big(x(s_n)\big) \neq JG\big(y(s_n)\big)$. Let B denote a positive integer such that if n is a positive integer greater than B, then γ_{s_n+1} contains a point z with $\pi_1(z) = 1/2$. Let $h_1, h_2, h_3, \ldots$ denote a sequence of points in M such that if i is a positive integer, then h_i belongs to $\gamma_{s_{(B+i)}+1}$ and $\pi_1(h_i) = 1/2$. The sequence $h_1, h_2, h_3, \ldots$ converges to $(1/2)$ and so does $G^{-1}(h_1), G^{-1}(h_2), G^{-1}(h_3), \ldots$ But the sequence $G^{-1}(h_1), G^{-1}(h_2), G^{-1}(h_3), \ldots$ converges to a point with the first projection not less than $3/4$. Thus G is not a homeomorphism, a contradiction from which it follows that the continuum A is not an inverse limit on $[0, 1]$ with only one bonding map.

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