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ON INTERSECTIONS OF SETS OF POSITIVE LEBESGUE MEASURE

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During the VIII Winter School on Abstract Analysis at Moravska Bouda (February 1980), Peter Simon asked if one can prove in ZFC that:

(*) for every family $\{A_\alpha: \alpha < \omega_1\}$ of sets of positive Lebesgue measure on the real line R there exists a set $T \subseteq \omega_1$ of cardinality ω_1 such that $\{A_\alpha: \alpha \in T\}$ has the finite intersection property.

In the present note we show the following

THEOREM. *The following three statements are equivalent*:

(1) *For every family $\{A_\alpha: \alpha < \omega_1\}$ of sets of positive Lebesgue measure on R there is a set $T \subseteq \omega_1$ of cardinality ω_1 such that the family $\{A_\alpha: \alpha \in T\}$ has the finite intersection property.*

(2) *For every family $\{A_\alpha: \alpha < \omega_1\}$ of sets of positive Lebesgue measure on R there is a set $T \subseteq \omega_1$ of cardinality ω_1 such that $\bigcap \{A_\alpha: \alpha \in T\} \neq 0$.*

(3) *There is no partition of R into ω_1 sets of Lebesgue measure zero.*

Proof. (1)$\Rightarrow$(3). Suppose $\neg$(3). Let $\{B_\alpha: \alpha < \omega_1\}$ be a family of pairwise disjoint sets of Lebesgue measure zero such that $\bigcup \{B_\alpha: \alpha < \omega_1\} = R$. Notice that for each $\alpha < \omega_1$ the set $C_\alpha = \bigcup \{B_\beta: \alpha \leqslant \beta < \omega_1\}$ is of full measure. For each $\alpha < \omega_1$ choose a compact subset $F_\alpha \subseteq C_\alpha$ of positive Lebesgue measure. Then for each $T \subseteq \omega_1$, of cardinality ω_1, we have

$$\bigcap \{F_\alpha: \alpha \in T\} \subseteq \bigcap \{C_\alpha: \alpha \in T\} = 0.$$

Thus, by compactness, the family $\{F_\alpha: \alpha < \omega_1\}$ is a counterexample for (1).

(3)$\Rightarrow$(2). Suppose $\neg$(2). Let $\mathscr{A} = \{A_\alpha: \alpha < \omega_1\}$ be a counterexample for (2). We may assume that the union of any subfamily of $\mathscr{A}$ is measurable (if not, then we can remove from each $\mathscr{A}_\alpha$ a set of measure zero to get a family with this property). Let $\mathscr{F}$ be a maximal family of pairwise disjoint sets of positive Lebesgue measure, each of which intersects countably many sets from $\mathscr{A}$ only. Clearly, $\mathscr{F}$ is countable and the set $X = R - \bigcup \mathscr{F}$ is of positive measure. For each $\alpha < \omega_1$ put

$$Z_\alpha = X - \bigcup \{A_\beta: \alpha \leqslant \beta < \omega_1\}.$$

Notice that each Z_α is measurable and intersects at most countably many sets from $\mathscr{A}$. Hence each Z_α is of measure zero. Finally, notice that $X = \bigcup\{Z_\alpha\colon \alpha < \omega_1\}$. Thus some measurable set of positive measure can be splitted into ω_1 sets of measure zero. Consequently, there is a partition of R into ω_1 sets of measure zero, which contradicts (3).

(2) $\Rightarrow$ (1) is obvious.

Remark. C. Ryll-Nardzewski has remarked (oral communication) that each of the statements of the Theorem is equivalent to the following one:

(4) *For each separable Boolean measure algebra* $(\mathscr{B}, \mu)$ *and for every family* $\{a_\alpha\colon \alpha < \omega_1\}$ *of elements of positive measure from* $\mathscr{B}$ *there is a set* $T \in \omega_1$ *of cardinality* ω_1 *such that the family* $\{a_\alpha\colon \alpha \in T\}$ *has the finite intersection property.*

We do not know if the assumption of separability can be omitted from (4).

The following corollary gives an answer to the question of P. Simon mentioned at the beginning.

COROLLARY. (i) $\mathrm{ZFC}+\mathrm{CH} \vdash \neg(*)$.

(ii) $\mathrm{Con(ZFC)} \to \mathrm{Con}(\mathrm{ZFC} + 2^\omega = \omega_2 + (*))$.

(iii) $\mathrm{Con(ZFC)} \to \mathrm{Con}(\mathrm{ZFC} + 2^\omega = \omega_2 + \neg(*))$.

Proof[1]. Obviously, $\mathrm{ZFC}+\mathrm{CH} \vdash \neg(3)$. Thus (i) follows from (1) $\Leftrightarrow$ (3). If we add ω_2 Cohen reals to a model for ZFC+CH, we get a model for $\mathrm{ZFC} + 2^\omega = \omega_2 + \neg(3)$. Thus (iii) also follows from (1) $\Leftrightarrow$ (3).

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(1) See L. Bukovsky, *Random forcing* in: *Set theory and hierarchy theory. V*, Bierutowice, Poland, 1976, Spriner Lecture Notes in Mathematics 619 (1977), p. 101–117.