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ON FACTOR COVERINGS OF GROUPS

BY

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1. Introduction. The notion of a factor cover of a group seems to be due to Durbin [3] and Ayoub [1]. In this paper we study factor covers in a somewhat wider context.

Let ψ be a relative group property. If $H \triangleleft G$, we write $H \psi G$ to denote that H has the prescribed property relative to G.

Definition 1.1. Let ψ be a relative group property. A *ψ-factor cover* of a group G is a collection of pairs $\{(U_a, L_a) \mid a \in \Gamma\}$ such that

(i) $\forall a \in \Gamma,\ U_a \triangleleft G,\ L_a \triangleleft G$,
(ii) $\forall a \in \Gamma,\ L_a \subset U_a$,
(iii) $G-\{1\} = \bigcup\{U_a - L_a \mid a \in \Gamma\}$,
(iv) $\forall a \in \Gamma,\ U_a/L_a \,\psi\, G/L_a$.

A collection of pairs $\{(U_a, L_a) \mid a \in \Gamma\}$ that satisfies (i)-(iii) of Definition 1.1 will be called an *(invariant) factor cover* of G. The class of groups having a ψ-factor cover will be denoted by ψC. We note that if ψ is the property "is in the center of", then ψC is the class of "residually central" groups studied by Durbin [3] (called property (**) by Ayoub [1]). Also, if ψ is the property "is an abelian subgroup of", ψC is the class denoted $\mathfrak{A}^+$ by Durbin [4] ((*) by Ayoub [1]).

We let ψI denote the class of groups having an invariant ψ-series (see Robinson [13], p. 9). Obviously, $\psi I \subset \psi C$, but the reverse inclusion is, in general, unresolved. In papers [1], [3], [4], [15] and [16], it has been shown that $\psi C \cap \Sigma = \psi I \cap \Sigma$ for certain properties ψ and certain finiteness conditions Σ.

It is our intent here to study factor by methods similar to those utilized by Mal'cev (see [11], p. 89) in studying series. Mal'cev showed that the existence of types of series in a group is equivalent to the existence of a binary relation satisfying certain algebraic properties. (These results can be found in Robinson [14], p. 94-99.) In Section 3 we show that, for certain classes ψ, the existence of a ψ-factor cover can be characterized in teims of a pair of binary relations. This characterization appears to make the logical structure of ψC somewhat more illuminating than the usual group-theoretic characterizations. This approach also seems to unify the known results that deal with the question $\psi C \cap \Sigma = \psi I \cap \Sigma$, Σ being a finiteness condition. This is illustrated in Theorem 4.1 and its corollaries.

The results in Sections 5 and 6 give new characterizations of series. These characterizations both involve the notion of a factor cover.

2. Preliminary lemmas and notation. Let $\mathscr{L}$ be the class of pairs (H, G), where G is a group and $H \lhd G$. A relative group property ψ is a subclass of $\mathscr{L}$. Sometimes $H \psi G$ will be used instead of $(H, G) \in \psi$.

A class of groups is a collection Σ of groups such that $\{1\} \in \Sigma$ and isomorphic copies of groups in Σ are again in Σ. If Σ is a class of groups and ψ is a relative property given by $H \psi G \Leftrightarrow H \in \Sigma$, then ψ will be called an *absolute property*.

Let $\mathscr{C} = \{(U_a, L_a) \mid a \in \Gamma\}$ be a factor cover of a group G. For each $x \neq 1$, select a pair $(U_x, L_x) \in \mathscr{C}$ such that $x \in U_x - L_x$. We obtain then the factor cover $\mathscr{C}_1 = \{(U_x, L_x) \mid x \in G - \{1\}\}$; clearly, $\mathscr{C}_1 \subset \mathscr{C}$. It is not difficult to show that, by making further deletions, we may assume that

(2.1) $\forall x \in G - \{1\}$, $U_{x^{-1}} = U_x$ and $L_{x^{-1}} = L_x$, and

(2.2) $\forall x \in G - \{1\}$ and $\forall g \in G$, $U_{x^g} = U_x$ and $L_{x^g} = L_x$.

In the sequel, factor covers will be denoted by $\{(U_x, L_x) \mid x \in G - \{1\}\}$ and it is always assumed that conditions (2.1) and (2.2) hold. The U_x's are called the *upper sections* of the cover and the L_x's the *lower sections*.

Amongst the properties that a factor cover may have, one that plays an essential role in the sequel is what we have termed a monotone property.

Definition 2.1. A factor cover $\mathscr{L} = \{(U_x, L_x) \mid x \in G - \{1\}\}$ is *monotone* if

(i) $\forall x \in G - \{1\}$, $y \in L_x$ implies $U_y \subset L_x$, and

(ii) $\forall x \in G - \{1\}$, $y \in U_x$ implies $U_y \subset U_x$.

We now associate a pair of binary relations with a monotone factor cover.

Definition 2.2. Let $\mathscr{L} = \{(U_x, L_x) \mid x \in G - \{1\}\}$ be a monotone factor cover of G. Define binary relations σ and τ on G by

(i) $1 \tau x$ $\forall x \in G$; if $y \neq 1$, $y \tau x \Leftrightarrow U_y \subset L_x$;

(ii) $1 \sigma x$ $\forall x \in G$; if $y \neq 1$, $y \sigma x \Leftrightarrow U_y \subset U_x$.

LEMMA 2.1. *If $\mathscr{L}$ is a monotone factor cover and σ and τ are defined as above, then*

(i) $\forall x, y \in G$, $x \tau y \Rightarrow x \sigma y$;

(ii) $\forall x, y, g \in G$, $x \tau y \Rightarrow x^g \tau y$ *and* $x \sigma y \Rightarrow x^g \sigma y$;

(iii) $\forall x, y, g \in G$, $x\tau y$ *and* $g\tau y \Rightarrow x^{-1}g\tau y$, *and* $x\sigma y$ *and* $g\sigma y \Rightarrow x^{-1}g\sigma y$;

(iv) $\forall x \in G-\{1\}$, x non$\tau\, x$ *and* $x\sigma x$.

The proof of Lemma 2.1 is easy and we omit it. From a given factor cover it is always possible to construct a monotone cover. This is illustrated by

LEMMA 2.2. *Let* $\mathscr{L} = \{(U_x, L_x) \mid x \in G-\{1\}\}$ *be a factor cover of* G. *For each* $x \in G-\{1\}$, *write*

$$\bar{U}_x = \bigcap\{U_y \mid x \in U_y\} \cap \bigcap\{L_y \mid x \in L_y\} \quad \text{and} \quad \bar{L}_x = \bar{U}_x \cap L_x.$$

Then $\bar{\mathscr{L}} = \{(\bar{U}_x, \bar{L}_x) \mid x \in G-\{1\}\}$ *is a monotone factor cover of* G.

In Lemma 2.2, we note that $\bar{U}_x/\bar{L}_x$ is isomorphic to a subgroup of U_x/L_x. Using this fact it is easy to prove

LEMMA 2.3. *If* ψ *is an absolute property closed under normal subgroups and* G *has a* ψ*-factor cover, then* G *has a monotone* ψ*-factor cover.*

In order to apply Lemma 2.2 to relative properties, we need

LEMMA 2.4. *Let* ψ *be a relative property satisfying the following property: if* H, L *and* K *are normal subgroups of* G *with* $H \subset K$ *and* $L \subset K$, *then*

$$K/H \;\psi\; G/H \Rightarrow L/L\cap H \;\psi\; G/L\cap H.$$

If G *has a* ψ*-factor cover, then* G *has a monotone* ψ*-factor cover.*

Since monotone factor covers are of central interest, we adopt

Definition 2.3. If ψ is a relative property, $m\psi C$ is the class of groups G that have a monotone ψ-factor cover.

3. Some model-theoretic aspects of factor covers. Our standard references for logic and model theory are [2] and [8].

We choose a first-order language L_Γ (with equality) for groups. The logical symbols of L_Γ are $=$, $\neg$, $\vee$, $\forall$, and variables $x_1, x_2, \ldots$ The non--logical symbols are: a binary operation $\circ$ (written as juxtaposition), a unary operation $^{-1}$, and a constant 1. The non-constant terms of L_Γ are just words in the variables (here we ignore the occurrences of 1).

The following series of definitions is intended to describe the relative group properties to be studied in this section.

An admissible formula of L_Γ is a formula built from atomic formulas of the form $W(x_1, \ldots, x_n) = 1$, where W is a word in the x_i's, using $\neg$, $\vee$ and $\forall$. In the first-order theory of groups each formula of L_Γ is equivalent to an admissible formula. Since we treat only admissible formulas, we drop the word "admissible". Further, to make our formulas more readable, we introduce the connectives $\wedge$, $\rightarrow$ and the quantifier $\exists$ as abbreviations for certain expressions using $\vee$, $\neg$ and $\forall$ (see [8], p. 18).

Let G be an L_Γ-structure, φ a formula, and a and b sequences of elements of G. We alter the usual notation and write $G \models \varphi[a, b]$ to indicate satisfaction, where a and b are substituted for the odd-indexed and even--indexed free variables in φ, respectively.

We define L_π to be the two-sorted language (see [8], p. 12) which is obtained from L_Γ as follows. Split the variables of L_Γ into two sets $\{x_1, x_3, \dots\}$ and $\{x_2, x_4, \dots\}$. An L_π-structure has the form

$$(M, N, 0_M, {}^{-1}M, 1_M),$$

where $N \subset M$, and the x_{2i+1}'s vary over M while the x_{2i}'s vary over N. For $\varphi \in L_\pi$, we write

$$\varphi(x_{i_1}, \dots, x_{i_m};\ x_{j_1}, \dots, x_{j_n}),$$

where the odd-indexed free variables of φ are the first sequence.

Definition 3.1. A relative group property ψ is *elementary* if there is a set Λ of sentences of L_π such that $H\psi G$ if and only if $(G, H) \models \varphi$ for each $\varphi \in \Lambda$.

In this section we study groups that have monotone ψ-factor covers for elementary ψ.

We expand the language L_Γ to L_Γ^+ by adding binary relation symbols S and T and a new constant c. Further we define a transformation * from the formulas of L_π into the formulas of L_Γ^+ as follows (by induction on logical complexity):

(i) if φ is an atomic formula $W(x_i;\ x_j) = 1$, then φ^* is $W(x_i;\ x_j)Tc$;

(ii) if φ^* and ψ^* have been defined, then

$\neg(\varphi)^*$ is $\neg\varphi^*$,

$(\varphi \vee \psi)^*$ is $\varphi^* \vee \psi^*$,

$(\forall x_{2i+1}\varphi)^*$ is $\forall x_{2i+1}\varphi^*$,

$(\forall x_{2i}\varphi)^*$ is $\forall x_{2i}\,(x_{2i}Sc \to \varphi^*)$ (this is just the relativization of the quantifier to the S predecessors of c).

Let $\{(U_g, L_g) \mid g \in G - \{1\}\}$ be a monotone factor cover of G. Define relations τ, σ on G as before (see Definition 2.2). In addition, we write $U_1 = \{1\}$ and $L_1 = \{1\}$.

LEMMA 3.1. *Let φ be a formula of L_π, $g \in G$. Let a and b be sequences from G and U_g, respectively ($\bar{a}$ and $\bar{b}$ are the corresponding sequences of cosets from G/L_g and U_g/L_g). Then*

$$(G/L_g, U_g/L_g) \models \varphi[\bar{a}, \bar{b}] \quad \textit{iff} \quad (G, \sigma, \tau, g) \models \varphi^*[a, b]$$

(*here c is interpreted by g*).

The Lemma is proved by induction on the logical complexity of φ.

COROLLARY 3.1. *Let φ be a sentence of L_π. Then, for $g \in G$,*

$$(G/L_g, U_g/L_g) \models \varphi \quad \textit{iff} \quad (G, \sigma, \tau, g) \models \varphi^*.$$

In L_{Γ^+} consider the following set of sentences:

I (group axioms):

(i) $\forall x_1 \forall x_2 \forall x_3 \ \big((x_1(x_2x_3))\ ((x_1x_2)x_3)^{-1} = 1\big)$;

(ii) $\forall x_1 \ \big((x_1 1)\ (1x_1)^{-1} = 1\big)$;

(iii) $\forall x_1 \ \big((x_1 1)x_1^{-1} = 1\big)$;

(iv) $\forall x_1 \ (x_1x_1^{-1} = 1)$;

(v) $\forall x_1 \ (x_1^{-1}x_1 = 1)$.

II (T, S axioms):

(i) $\forall x_1 \ [x_1 S x_1]$;

(ii) $\forall x_1 \ [1 T x_1]$;

(iii) $\forall x_1 \ [x_1 T x_1 \rightarrow x_1 = 1]$;

(iv) $\forall x_1 \forall x_2 \ [x_1 T x_2 \rightarrow x_1 S x_2]$;

(v) $\forall x_1 \forall x_2 \forall x_3 \ [x_1 S x_3 \wedge x_2 S x_3 \rightarrow x_1^{-1} x_2 S x_3]$;

(vi) $\forall x_1 \forall x_2 \forall x_3 \ [x_1 T x_3 \wedge x_2 T x_3 \rightarrow x_1^{-1} x_2 T x_3]$;

(vii) $\forall x_1 \forall x_2 \forall x_3 \ [x_1 S x_3 \rightarrow x_1^{x_2} S x_3]$;

(viii) $\forall x_1 \forall x_2 \forall x_3 \ [x_1 T x_3 \rightarrow x_1^{x_2} T x_3]$;

(ix) $\forall x_1 \forall x_2 \forall x_3 \ [x_1 T x_3 \wedge x_2 S x_1 \rightarrow x_2 T x_3]$;

(x) $\forall x_1 \forall x_2 \forall x_3 \ [x_1 S x_3 \wedge x_2 S x_1 \rightarrow x_2 S x_3]$.

Let φ be a sentence of L_π, and x a variable not occurring in φ^*. Then $\varphi^* \langle {c \atop x} \rangle$ denotes the formula obtained from φ^* by replacing each occurrence of c by x. Let ψ be an elementary property determined by Λ, and assume that Σ_ψ and Σ'_ψ are the following sets of L_Γ^+-sentences:

$$\Sigma_\psi = \{\mathrm{I}\} \cup \{\mathrm{II}\} \cup \{\forall x \varphi^* \langle {c \atop x} \rangle \mid \varphi \in \Lambda\},$$

$$\Sigma'_\psi = \{\mathrm{I}\} \cup \{\mathrm{II}\} \cup \{\forall x \, [x \neq 1 \rightarrow \varphi^* \langle {c \atop x} \rangle] \mid \varphi \in \Lambda\}.$$

THEOREM 3.1. *G has a monotone ψ-factor cover if and only if there exist binary relations σ and τ on G such that $(G, \sigma, \tau) \models \Sigma_\psi$ (here we assume that ψ is a property such that $\{1\} \ \psi \ K$ for all groups K; otherwise we use Σ'_ψ).*

Proof. Let $\{(U_g, L_g) \mid g \in G - \{1\}\}$ be a monotone ψ-factor cover, and let σ and τ be given as in Definition 2.2. We interpret S and T by σ and τ, respectively. It is easy to show that the sentences of axioms I and II hold in (G, σ, τ) (see Lemma 2.1). Since we have a ψ-factor cover, for each $g \in G$ and each $\varphi \in \Lambda$ we obtain $(G/L_g, U_g/L_g) \models \varphi$. Thus, by Corollary 3.1,

$$(G, \sigma, \tau) \models \forall x \varphi^* \langle {c \atop x} \rangle \quad \text{for each } \varphi \in \Lambda,$$

and so $(G, \sigma, \tau) \models \Sigma_\psi$.

Conversely, suppose $(G, \sigma, \tau) \models \Sigma_\psi$. By the sentences of axioms I, G is a group. For $g \in G - \{1\}$, write

$$U_g = \{h \in G \mid h\sigma g\} \quad \text{and} \quad L_g = \{h \in G \mid h\tau g\}.$$

The sentences of axioms II make $\{(U_g, L_g) \mid g \in G - \{1\}\}$ a monotone factor cover of G: for example, (i), (ii), (v) and (vi) imply that U_g and L_g are subgroups; U_g and L_g are normal by (vii) and (viii); $L_g \subset U_g$ by (iv); and $g \in U_g - L_g$ for $g \in G - \{1\}$ by (i) and (iii); monotonicity follows from (ix) and (x).

If $\bar{\sigma}$ and $\bar{\tau}$ are defined for this cover as in Definition 2.2, it is easy to show that $\bar{\sigma} = \sigma$ and $\bar{\tau} = \tau$. Again using Corollary 3.1, $(G/L_g, U_g/L_g) \models \varphi$ for each $g \in G$ and for each $\varphi \in \Lambda$. Thus we have a ψ-factor cover.

The case where Σ'_ψ is used is treated similarly.

COROLLARY 3.2. *If ψ is an elementary property, then $m\,\psi\,C$ is a pseudo-elementary class. In particular, $m\,\psi\,C$ is closed under ultraproducts* (see [2], p. 154).

Further information on $m\,\psi\,C$ can be obtained from the logical nature of Λ (a set of elementary sentences) as it is reflected in Σ_ψ (Σ'_ψ). In this direction we note that the sentences in axioms I and II are universal Horn sentences (see [5] for the definition of Horn sentence). If the sentences of Λ are also universal Horn sentences, then so are those of Σ_ψ (and sometimes Σ'_ψ). In this case, $m\,\psi\,C$ is a pseudo-universal Horn class and is closed under reduced products and subgroups [5]. If the sentences of Λ are just universal not necessarily Horn, then $m\,\psi\,C$ is a pseudo-universal class (using either Σ_ψ or Σ'_ψ) and, by a theorem of Tarski and Łoś (see [9], p. 18), $m\,\psi\,C$ is a universally axiomatizable class of groups; i.e., there is a set χ of universal sentences of L_Γ such that $G \in m\,\psi\,C \Leftrightarrow G \models \chi$.

Examples. 3.1. Let V be a variety or quasivariety of groups, and let ψ be such that $H\,\psi\,G \Leftrightarrow H \in V$. Then $m\,\psi\,C$ is a pseudo-universal Horn class.

3.2. Let V be a variety, $V \neq \{1\}$. Let ψ be such that $H\,\psi\,G \Leftrightarrow H \subseteq V$-marginal subgroup of G. Then $m\,\psi\,C$ is a pseudo-universal Horn class. The necessary facts concerning marginal subgroups may be found in [13], p. 8.

3.3. Let ψ be such that $H\,\psi\,G \Leftrightarrow H \subset$ center of G. Then $m\,\psi\,C$, the class of groups with monotone factor covers with residually central factors, is a pseudo-universal Horn class. This is also a special case of 3.2.

3.4. Let ψ be such that $H\,\psi\,G \Leftrightarrow |H| < n$. Then $m\,\psi\,C$ is a pseudo-universal class.

If the sentences of Λ are universal ($\forall$) or universal existential ($\forall\exists$), then so are those of Σ_ψ (Σ'_ψ), and we infer that $m\,\psi\,C$ is a local class [10]. This is the case for all the above-given examples and, in addition, 3.5. Let ψ be such that $H \psi G \Leftrightarrow H$ has one conjugacy class. Then $m \psi C$ is a local class.

3.6. Let ψ be such that $H \psi G \Leftrightarrow [G: H] \leqslant n$. Then $m \psi C$ is a local class.

3.7. Let ψ be such that $H \psi G \Leftrightarrow |H| = \infty$. Then $m \psi C$ is a local class.

Definition 3.2. A word $W(x_i; x_j)$ of L_π is *acceptable* if, for all groups H and G with $H \lhd G$ and for all $a_i \in G$ and $b_j \in H$, we have $W(a_i; b_j) \in H$.

LEMMA 3.2. *If $W(x_i; x_j)$ is acceptable, and H, L and K are normal subgroups of G with*

$$H \subset K, \quad L \subset K \quad \text{and} \quad (G/H, K/H) \models \forall x_i \forall x_j \; [W(x_i; x_j) = 1],$$

then

$$(G/L \cap H,\ L/L \cap H) \models \forall x_i \forall x_j \; [W(x_i; x_j) = 1].$$

Proof. Assume the conclusion is false. Then, for some $g_i \in G$ and $h_j \in L \subset K$, we have $W(g_i; h_j) \notin L \cap H$. But $W(g_i; h_j) \in L$, since W is acceptable. Thus $W(g_i; h_j) \notin H$, that is, in $(G/H, K/H)$, $W(\bar{g}_i, \bar{h}_j) \neq 1$, a contradiction with the hypothesis that

$$(G/H, K/H) \models \forall x_i \forall x_j \; [W(x_i; x_j) = 1].$$

COROLLARY 3.3. *If the Λ which determines ψ consists of universal quantifications of acceptable words, then $m \psi C = \psi C$.*

For the proof use Lemmas 3.2 and 2.4.

This corollary applies to Examples 3.1 (where V is a variety), 3.2 and 3.3. Hence previous remarks hold for ψC in these cases. Example 3.5, where U_g/L_g has one conjugacy class, is not covered by the corollary, yet in this case $m \psi C = \psi C$.

4. Imposition of finiteness conditions. In this section we study the influence of certain finiteness conditions on groups which have a monotone ψ-factor cover.

Definition 4.1. Let τ be as in Definition 2.2. We say that τ *satisfies the minimum condition* (min-τ) if, for each subset $H \neq \emptyset$ of G, $\exists x \in H$ such that $\forall y \in H$, y non τ x.

Definition 4.2. Let ψ be a relative property. G is a *hyper-ψ-group* if G has an ascending invariant series $1 = G_0 \subset \ldots \subset G_\lambda = G$ such that $G_{p+1}/G_p \; \psi \; G/G_p$ for all $p < \lambda$.

THEOREM 4.1. *Let ψ be a relative property satisfying*

(a) *if M, N and Y are normal subgroups of G such that $N \subset M$ and $N \subset Y$, then $M/N \; \psi \; G/N \Rightarrow MY/Y \; \psi \; G/Y$.*

Suppose G has a monotone ψ-factor cover. If the binary relation τ satisfies the minimum condition, then G is a hyper-ψ-group.

Proof. It suffices to show that, for each normal subgroup H of G, $H \neq G$, there is a normal subgroup K of G such that $H \lhd K$ and $K/H\ \psi\ G/H$ (see [13], p. 14). Let $H \lhd G$, $H \neq G$, and let x be τ-minimal with respect to $x \notin H$. Then $L_x \subset H$ and, by the conditions on ψ, we have $U_x H/H\ \psi\ G/H$.

COROLLARY 4.1. *If ψ is an absolute property closed under normal subgroups and quotients, $G \in \psi C$, and the monotone ψ-factor cover of G* (see Lemma 2.3) *has* min-τ, *then G is a hyper-ψ-group.*

Proof. Let $G \in \psi C$. Since ψ is closed under normal subgroups, G has a monotone ψ-cover by Lemma 2.3. The quotient closure of ψ implies condition (a) of Theorem 4.1.

We note that min-τ is a considerably weaker condition than min-N, the minimum condition for normal subgroups. It is routine to show that min-N implies min-τ. The fact that min-τ does not imply min-N is illustrated in the following easy example:

Let $\{A_n \mid n < \omega\}$ be a collection of non-trivial groups. Let

$$G = \sum_{n=1}^{\infty} A_n, \quad G_0 = \{1\}, \quad G_k = \sum_{n=1}^{k} A_n \text{ for } k \geqslant 1, \quad \text{and} \quad G_\omega = G.$$

The set of normal subgroups $\{G_k\}_{k=0}^{\omega}$ forms an ascending series, and hence a monotone factor cover. The induced binary relation τ satisfies the minimum condition. Evidently, G does not have min-N.

With this in mind, we note that Corollary 4.1 has, as consequences, several results in the literature; notably Durbin [4], Theorem 1, and Ayoub [1], p. 220.

We now consider some applications of Theorem 4.1 to non-absolute properties. It has been noted in Section 3 following Corollary 3.3 (see the remarks) that if ψ is the property "is in the V-marginal subgroup of", then it satisfies the conditions of Lemma 2.4. It is not difficult to show that ψ also satisfies condition (a) of Theorem 4.1. Thus, we have

COROLLARY 4.2. *Let V be a variety, $V \neq \{1\}$, and let ψ be the property "is in the V-marginal subgroup of". If $G \in \psi C$ and a monotone ψ-cover G has* min-τ, *then G is a hyper-ψ-group.*

For the variety $V = \{[x, y]\}$ Corollary 4.2 reduces to the class of residually central groups. Thus Corollary 4.2 gives the results of Durbin [3], Theorem 1, and Ayoub [1], p. 226.

Another class of non-absolute properties has been studied by Stanley [17]. Let Σ be a class of groups closed under subgroups, finite direct products, and quotients. Following Stanley [17], let G be a group and let

$$\Sigma(G) = \{x \in G \mid G/C_G(x^G) \in \Sigma\}.$$

It is easy to show that $\Sigma(G)$ is a normal subgroup of G.

COROLLARY 4.3. *Let Σ be a class of groups closed under subgroups, finite direct products, and quotients. Let ψ be the property $H \ \psi \ G \Leftrightarrow H \subset \Sigma(G)$. If $G \in \psi \ C$, then*

(i) *G has a monotone ψ-cover $\mathscr{L}$, and*

(ii) *if $\mathscr{L}$ has* min-τ, *G is a hyper-ψ-group.*

Proof. To prove (i) we show that the conditions of Lemma 2.4 hold. Let H, L and K be normal subgroups of G with $H \subset K$ and $L \subset K$. Suppose $K/H \subset \Sigma(G/H)$. Let $x \in L$, and let

$$R/L\cap H = C_{G/L\cap H}\big(x(L\cap H)\big)^{G/L\cap H} \quad \text{and} \quad S/H = C_{G/H}(xH)^{G/H}.$$

Let $s \in S$ and $g \in G$. Then $[x^g, s] \in H \cap L$ which forces $s \in R$. Thus $S \subset R$. Now G/R is a quotient of G/S and, by the hypotheses, $G/S \in \Sigma$. Thus

$$G/R \in \Sigma \quad \text{and} \quad (G/L\cap H)/(R/L\cap H) \in \Sigma.$$

Thus $L/L\cap H \subset \Sigma(G/L\cap H)$; i.e., $L/L\cap H \ \psi \ G/L\cap H$.

The proof of (ii) consists of establishing condition (a) of Theorem 4.1. The argument is similar to that given in the preceding section and will be omitted.

Corollary 4.3 encompasses some results of Stanley [16], p. 3.

Although the following argument has been used repeatedly above, we feel it important enough to isolate as

COROLLARY 4.4. *Let G be a ψC-group with* min-N *and let ψ satisfy the following conditions:*

(*) *If H, K and L are normal subgroups of G with $H \subset K$ and $L \subset K$, then*

$$K/H \ \psi \ G/H \Rightarrow L/L\cap H \ \psi \ G/L\cap H.$$

(**) *If M, N and Y are normal subgroups of G with $N \subset M$ and $N \subset Y$, then*

$$M/N \ \psi \ G/N \Rightarrow MY/Y \ \psi \ G/Y.$$

Then G is a hyper-ψ-group.

In paper [15], Slotterbeck studies factor covers $\mathscr{L}$ with only finitely many elements. Suppose that G has such a factor cover $\mathscr{L}$. Then the induced monotone cover $\overline{\mathscr{L}}$ (Lemma 2.2) has only finitely many elements and, accordingly, has min-τ. Further, if Theorem 4.1 is applied to $\overline{\mathscr{L}}$, the resulting series will have finite length. We have essentially proved

COROLLARY 4.5. *If G has a ψ-cover $\mathscr{L}$ consisting of only finitely many elements, and ψ satisfies the conditions of Lemma 2.4 and Theorem 4.1 (or Corollary 4.4), then G has a ψ-series of finite length.*

Theorem 2 of [15] is related to Corollary 4.5.

5. In this section we are concerned solely with relative properties ψ of the following type: V is a variety, $V \neq \{1\}$, and ψ is the property "is in the V-marginal subgroup of".

The methods here make an essential use of a local technique of Mal'cev [11]. Exposition of the method may be found in Robinson [14], p. 94-99, or Hickin and Plotkin [7]. We use the terminology of Robinson's book [14]. As a cautionary note, we shall apply the Mal'cev method to the finite subsets of a group, whereas the applications in [12] apply the method to subgroups. The use of subsets will require some additional argument, but the method remains valid.

Our first lemma shows that any group with a ψ-factor cover has a chain of normal subsets with ψ-factors.

LEMMA 5.1. *If G has a ψ-factor cover, then there is a complete chain of normal subsets $\{G_\alpha \mid 1 \leqslant \alpha \leqslant \varrho\}$ such that, for each $\lambda < \varrho$ and each defining word $W(x_1, \ldots, x_n)$ of V,*

$$x \in G_{\lambda+1} \Rightarrow W(x_1 \ldots xx_j \ldots x_n)\, W(x_1, \ldots, x_n)^{-1} \in G_\lambda$$

for $1 \leqslant j \leqslant n$ and each n-tuple $(x_1, \ldots, x_n) \in G^n$.

Proof. We take a monotone ψ-cover of G and define the binary relation τ as in Definition 2.2. Let F be a finite subset of G with $1 \in F$. We define a series of subsets $1 = F_0 \subset F_1 \subset \ldots \subset F_k = F$ of F as follows:

$F_0 = 1$ and

$F_i = F_{i-1} \cup \{x \in F \mid y\tau x$ and $y \in F \Rightarrow y \in F_{i-1}\}$ for $i > 1$ (i.e., $F_i = F_{i-1} \cup$ the set of all τ-minimal elements of $F - F_{i-1}$).

The finiteness of F insures the existence of a k such that $F_k = F$. The F_i have the following properties for $0 \leqslant i \leqslant k$:

(a) if $x, g, x^g \in F$ and $x \in F_i$, then $x^g \in F_i$;

(b) if $W(x_1, \ldots, x_n)$ is a defining word of a variety V, and the words $x, x_1, \ldots, x_n$, $W(x_1, \ldots, xx_j, \ldots, x_n)$, $W(x_1, \ldots, x_n)$, $W(x_1, \ldots, x_n)^{-1}$, $W(x_1, \ldots, xx_j, \ldots, x_n)\, W(x_1, \ldots, x_n)^{-1}$ $(1 \leqslant j \leqslant n)$ belong to F, and $x \in F_i$ $(i \geqslant 1)$, then

$$W(x_1, \ldots, xx_j, \ldots, x_n)\, W(x_1, \ldots, x_n)^{-1} \in F_{i-1}.$$

Prove (a) by induction on i. If $x \in F_{i-1}$, the conclusion follows from the induction hypothesis. Suppose $x \in F_i - F_{i-1}$. If $y \in F$ and $y\tau x^g$, it follows that $y\tau x$ (see (2.2)). Hence $y \in F_{i-1}$ and we have $x^g \in F_i$.

The proof of (b) is similar; induct on i. If $x \in F_{i-1}$, the conclusion follows from the induction hypothesis. If $x \in F_i$, we have

$$z = W(x_1, \ldots, xx_j, \ldots, x_n)\, W(x_1, \ldots, x_n)^{-1} \tau x.$$

Since $z \in F$, $z \in F_{i-1}$.

Let $\mathscr{C}$ be the local system of all finite subsets F of G that contain 1. With each $F \in \mathscr{C}$ we associate a series of subsets satisfying (a) and (b).

The Mal'cev limit of these series induces a series of the same type in G (see [14], p. 97-99, for the details).

Lemma 5.1 gives seemingly little information about the existence of series of subgroups. The most that we can deduce is that if G has only a finite number of conjugacy classes, and satisfies the conditions of Lemma 5.1, then G has a ψ-series. This fact however, is subsumed in the results in Section 4 on min-N.

We now discuss properties of the factor cover which insure that the series constructed in the proof of Lemma 5.1 will be a series of subgroups. This is essentially requiring that the F_i's of the preceding proof enjoy some sort of subgroup properties; specifically we need

(c) If $x, y, xy \in F$ and $x, y \in F_i$, then $xy \in F_i$.

This property is a condition on τ, which we formally state as

THEOREM 5.1. *If G has a monotone ψ-factor cover and the induced binary relation τ satisfies*

$$(*) \qquad \forall x, y, z \in G, z\tau xy \Rightarrow z\tau x \text{ or } z\tau y,$$

then G has a ψ-series.

This theorem gives another characterization of a ψ-series, i.e.,

COROLLARY 5.1. *G has a ψ-series if and only if G has a monotone ψ-factor cover satisfying* (*).

6. Factor covers with directed upper or lower sections. Let V be a variety of groups, $V \neq \{1\}$. In this section we admit properties ψ of the following type:

(i) ψ stands for "is in the V-marginal subgroup of",

(ii) ψ stands for "is a normal V-subgroup of".

Our main result here is

THEOREM 6.1. *If G has a ψ-cover $\mathscr{L} = \{(U_x, L_x) \mid x \in G - \{1\}\}$ and either the U_x's or L_x's are directed, then G has a ψ-series.*

COROLLARY 6.1. *G has a ψ-series if and only if G has a ψ-factor cover $\mathscr{L} = \{(U_x, L_x) \mid x \in G - \{1\}\}$ such that either*

(a) *the U_x's are directed, or*

(b) *the L_x's are directed.*

We need some special terminology for the proof of Theorem 6.1.

Definition 6.1. Let V be a variety of groups and $K \triangleleft G$. Let $V_K(G) = \bigcap\{Y \mid Y \triangleleft G, Y \subset K$ and K/Y is contained in the V-marginal subgroup of $G/Y\}$.

It is easy to show that $K/V_K(G)$ is the V-marginal subgroup of $G/V_K(G)$. Further, if $K \subset A \triangleleft B$, then $V_K(A) \subset V_K(B)$.

Definition 6.2. If V is a variety of groups, $V(G)$ denotes the corresponding *verbal subgroup* of G.

The following lemma is due to Hickin and Phillips [6]:

LEMMA 6.1. *Let V be a variety of groups.*

(a) *If ψ is of type* (i), *then G has a ψ-series if and only if, for every finitely generated subgroup $\{1\} \neq K$ of G, $V_{K^G} \neq K^G$.*

(b) *If ψ is of type* (ii), *then G has a ψ-series if and only if, for every finitely generated subgroup $\{1\} \neq K$ of G, $V(K^G) \neq K^G$.*

Proof of Theorem 6.1. Suppose that ψ is of type (i), and that the L_x's are directed. Let $K = \langle x_1, \ldots, x_n \rangle$ be a finitely generated subgroup of G. We may suppose without loss of generality that $L_{x_1} \subset \ldots \subset L_{x_n}$. Then, for each i $(1 \leqslant i \leqslant n)$ and each $g \in G$, $x_i^g L_{x_n}$ is in the V-marginal subgroup of G/L_{x_n}. An easy argument shows that $K^G/K^G \cap L_{x_n}$ is in the V-marginal subgroup of $G/K^G \cap L_{x_n}$, so that $V_{K^G}(G) \neq K^G$. The application of Lemma 6.1 (a) shows that G has a ψ-series.

Now suppose the U_x's are directed. Let $L_x^* = V_{U_x}(G)$. Then $L_x^* \neq U_x$ and U_x/L_x^* is in the V-marginal subgroup of G/L_x^*. Further, the directedness of the U_x's implies the directedness of the L_x^*'s. By the first part of this proof, G has a ψ-series.

Suppose ψ is of type (ii), and the L_x's are directed. Let $K = \langle x_1, \ldots, x_n \rangle$ be a finitely generated subgroup of G, and suppose $L_{x_1} \subset \ldots \subset L_{x_n}$. Now

$$K^G \subset R = U_{x_1} U_{x_2} \ldots U_{x_n}$$

and, for each i $(1 \leqslant i \leqslant n)$,

$$V(U_{x_i}) \subset L_{x_i} \subset L_{x_n}.$$

Thus R/L_{x_n} is generated by normal V-subgroups. It follows that R/L_{x_n} has an ascending V-series. Consequently, $K^G/K^G \cap L_{x_n} \cong K^G L_{x_n}/L_{x_n}$ has a V-series (a subgroup of a group with an ascending V-series again has a V-series). Since $K^G/K^G \cap L_{x_n}$ is generated by the normal closure of a finite set,

$$V(K^G/K^G \cap L_{x_n}) \neq K^G/K^G \cap L_{x_n}.$$

Thus $V(K^G) \neq K^G$, and the application of Lemma 6.1 shows that G has a V $(= \psi)$ series.

For the U_x's directed, let $L_x^* = V(U_x)$; then $\{(U_x, L_x^*) \mid x \in G - \{1\}\}$ is a ψ-factor cover with directed lower sections. By the first part of this case, G has a ψ-series.

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