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ON BASES, COMPACTNESS AND WEAK CONVERGENCE IN THE BANACH SPACE A_p

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1. Introduction. Let A_p $(1 \leqslant p < \infty)$ be the Banach space of all holomorphic functions $f(z)$ in the unit disc $D = \{z|\ |z| < 1\}$ such that $\int\limits_D |f(z)|^p d\mu(z) < \infty$, with norm

$$\|f\| = \Big[\int\limits_D |f(z)|^p d\mu(z)\Big]^{1/p},$$

where μ is the planar Lebesgue measure in D. It turns out that A_p is a closed linear subspace of the Banach space $L_p(\mu)$ of the set of all equivalence classes of p-th-power integrable complex functions on D. The usual Hardy spaces H_p [3] are the Banach spaces of all elements of A_p for which the norm

$$\|f\| = \sup\Big\{\Big[(2\pi)^{-1}\int\limits_0^{2\pi} |f(re^{i\varphi})|^p d\varphi\Big]^{1/p} \,|\, 0 \leqslant r < 1\Big\}$$

is finite. A_p, as a set is distinct of H_p, i.e. for $1 \leqslant p < \infty$, the functions in H_p form a proper subset of A_p. This is easily seen by choosing, for example, a suitable branch of the function $(1-z)^{-1/p}$, which is in A_p but not in H_p. Next, a sequence $\{x_j\}$ in a Banach space X is called a *basis* for X if every x in X has the unique series expansion $\lim\limits_n \sum\limits_{j\leqslant n} a_j x_j$ with scalar coefficients a_j, where the convergence is in the strong topology of X. It is known that the sequence $\{x_j\}$ defined by

$$x_j(z) = \big((j+1)/\pi\big)^{1/2} z^j, \quad z \in D,\ j = 0, 1, \ldots,$$

forms an orthonormal basis for the Hilbert space A_2 and that the same sequence constitutes a basis for H_p $(1 < p < \infty)$ [3].

In this paper we state the result that $\{x_j\}$ is a basis for A_p $(1 < p < \infty)$. This gives an affirmative answer to the question, whether the Taylor series expansion at $z = 0$ for each function f in A_p converges to f in the topology of A_p. However, it remains an open problem whether $\{x_j\}$ forms a basis for A_1. Next, as in the case of the spaces $C(S)$ and L_p, it is possible to specify weak convergence and conditionally compact sets in A_p. Indeed, necessary and sufficient conditions can be given for the weak convergence of a sequence in A_p and also for the elements of conditionally compact sets in A_p, both relating the abstract concepts with the special form of the elements as functions of the complex variable z. Finally, it is shown that the shift operator T in A_p, as usual defined by $(Tf)(z) = zf(z)$, $z \in D$, has the following interesting spectral properties: The spectrum is $\bar{D}$, the point spectrum is empty, the residual spectrum is D and the continuous spectrum is the unit circle.

2. A basis for the Banach space A_p. The proof that A_p $(1 \leqslant p < \infty)$ is a Banach space is, in principle, based on the following estimate. It is easy to see that every f in A_p satisfies the mean value equation

$$f(z) = \frac{1}{\pi(1-|z|)^2} \int\limits_{|\lambda - z| < 1-|z|} f(\lambda)\, d\mu(\lambda), \qquad z \in D,$$

from which one obtains

$$|f(z)| \leqslant [\pi(1-|z|)^2]^{-1/p} \|f\|, \qquad z \in D. \tag{1}$$

We omit this proof, since it follows directly the lines of that for the case $p = 2$ given in [2], where Hölder's inequality is used instead of Schwarz's.

The following lemma is substantial for proving the theorem on the existence of the basis $\{x_j\}$ for A_p:

LEMMA 1. *If $1 \leqslant p < \infty$, the sequence $\{x_j\}$ is total in A_p.*

Proof. If f is any element of A_p with Taylor series $\sum\limits_{j=0}^{\infty} a_j z^j$ in D, we define the function f_t, $0 < t < 1$, on D by $f_t(z) = f(tz)$, $z \in D$. Since the Taylor series for $f(z)$ converges absolutely and uniformly for $|z| \leqslant t$, the Taylor series $\sum\limits_{j=0}^{\infty} a_j t^j z^j$ for $f_t(z)$ converges absolutely and uniformly in D. Hence f_t is an element of $\overline{\mathrm{sp}}\{x_j\}$ in A_p for each $t \in (0, 1)$. We show that, given $\varepsilon > 0$, there exists a t in $(0, 1)$ for which $\|f - f_t\| < \varepsilon$.

Since $f \in L_1(\mu)$, there is a $\delta \in (0, \frac{1}{2})$ such that

$$\Big[\int\limits_{1-\delta<|z|<1} |f(z)|^p d\mu(z)\Big]^{1/p} < \varepsilon/5.$$

Taking $r = 1 - \delta/2$, we thus have

$$\Big[\int\limits_{r<|z|<1} |f_t(z)|^p d\mu(z)\Big]^{1/p} = \Big[t^{-2} \int\limits_{tr<|z|<t} |f(z)|^p d\mu(z)\Big]^{1/p} < 2\varepsilon/5$$

for $t\epsilon(r, 1)$. The uniform and absolute convergence of the Taylor series for $f(z)$ for $|z| \leqslant r$ implies the existence of an integer n for which

$$\sum_{j=n+1}^{\infty} |a_j(1-t^j)z^j| < \pi^{-1/p}\varepsilon/5,$$

$|z| \leqslant r$ and $t\epsilon(r,1)$. Furthermore, there is a fixed $t\epsilon(r,1)$, depending on n, such that

$$\sum_{j=0}^{n} |a_j(1-t^j)z^j| < \pi^{-1/p}\varepsilon/5, \qquad |z| \leqslant r.$$

Therefore, one has

$$\|f-f_t\| \leqslant \Big[\int\limits_{0<|z|<r} |f(z)-f_t(z)|^p d\mu(z)\Big]^{1/p} + \Big[\int\limits_{r<|z|<1} |f(z)|^p d\mu(z)\Big]^{1/p} +$$

$$+\Big[\int\limits_{r<|z|<1} |f_t(z)|^p d\mu(z)\Big]^{1/p}$$

$$<\Big[\int\limits_{0<|z|<r} \Big|\sum_{j=0}^{n} a_j(1-t^j)z^j\Big|^p d\mu(z)\Big]^{1/p} +$$

$$+\Big[\int\limits_{0<|z|<r} \Big|\sum_{j=n+1}^{\infty} a_j(1-t^j)z^j\Big|^p d\mu(z)\Big]^{1/p} + \varepsilon/5 + 2\varepsilon/5 < \varepsilon,$$

which is the desired result.

There is a more general concept than that of a basis [5]: A biorthogonal system $\{x_j, x_j^*\}$ is called a *Markuševič basis* for X, if $\{x_j\}$ is total in X and $\{x^*\}$ in X^* is such that for every $x \epsilon X$, $x_j^*(x) = 0$, all j, implies $x = 0$. It is clear that the functionals $x_j^* \epsilon A_p^*$, defined by

$$x_j^*(f) = \int\limits_D f(z)\overline{x_j(z)}\, d\mu(z),$$

are biorthogonal to the x_j's used in the above lemma. Also, $x_j^*(f)$ is proportional to the j-th coefficient of the Taylor series for f at $z = 0$. From this it follows that $x_j^*(f) = 0$, $f\epsilon A_p$, $j = 0, 1, \ldots$, implies $f = 0$. Thus, as an immediate consequence of Lemma 1, one obtains

COROLLARY 2. *If* $1 \leqslant p < \infty$, $\{x_j\}$ *is a Markuševič basis for* A_p.

One of the most important theorems in the theory of bases is the theorem of Grinblyum-Nikol'skiĭ [6] which states that a sequence $\{x_j\}$ in X is a basis for $\overline{\mathrm{sp}}\{x_j\}$ if and only if there is a constant $K \geqslant 1$ such that

$$\Big\|\sum_{j\leqslant m} a_j x_j\Big\| \leqslant K \Big\|\sum_{j\leqslant n} a_j x_j\Big\|$$

for every pair of integers m, n with $m \leqslant n$ and any scalars a_j. Now, using the fact that for $1 < p < \infty$ the trigonometrical system forms [7] a basis for $L_p[0, 2\pi]$ and by a twofold application of the Grinblyum-Nikol'skiĭ theorem, it is possible to prove

THEOREM 3. *If $1 < p < \infty$, the sequence $\{x_j\}$ is a basis for A_p and the associated biorthogonal set to $\{x_j\}$ is $\{x_j^*\}$.*

3. Compact sets and weak convergence in A_p. It is of special interest to investigate the conditionally compact sets in A_p; as has been done for many other important spaces, such as $C(S)$ (theorem of Arzelà-Ascoli) or L_p (cf. [1]).

THEOREM 4. *Let $1 \leqslant p < \infty$. A set K in A_p is conditionally compact if and only if*

(i) *K is bounded,*

(ii) *the functions in K are equicontinuous on each compact subset of D and*

(iii) $\lim\limits_{r \to 1} \int\limits_{r<|z|<1} |f(z)|^p d\mu(z) = 0$ *uniformly for all f in K.*

Proof. Let S be any compact set in D and let $\delta = 1 - \sup\{|z| \mid z \in S\}$. If K is conditionally compact, then K is bounded and for every $\varepsilon > 0$ there exist functions $f_1, \ldots, f_n$ in K such that

$$\inf_{i \leqslant n} \|f - f_i\| < (\pi\delta^2)^{1/p}\varepsilon/3$$

for each f in K (i.e. K is totally bounded). Then, by estimate (1) we have on S, $|f(z)| \leqslant (\pi\delta^2)^{-1/p}\|f\|$. Given any z_0 in S we can choose a neighborhood N of z_0 in D such that

$$\sup_{i \leqslant n} |f_i(z) - f_i(z_0)| < \varepsilon/3, \qquad z \in N.$$

Thus for $f \in K$ and some $i \leqslant n$ we have on $N \cap S$

$$\begin{aligned} |f(z) - f(z_0)| &\leqslant |f(z) - f_i(z)| + |f_i(z) - f_i(z_0)| + |f_i(z_0) - f(z_0)| \\ &< 2(\pi\delta^2)^{-1/p}\|f - f_i\| + \varepsilon/3 < \varepsilon, \end{aligned}$$

showing that the functions in K are equicontinuous on S.

In a similar manner we can prove the last statement of the theorem. Let $r < 1$ be such that $\sup\limits_{i \leqslant n} \|\chi_r f_i\| < \varepsilon/2$, where χ_r is the characteristic function of the set $\{z \in D \mid |z| > r\}$. By Minkowski's inequality one obtains for some $i \leqslant n$,

$$\|\chi_r f\| \leqslant \|\chi_r(f - f_i)\| + \|\chi_r f_i\| < \|f - f_i\| + \varepsilon/2 < 2\varepsilon$$

and so $\lim\limits_{r \to 1} \|\chi_r f\| = 0$ uniformly for all f in K.

To prove the converse we assume (i), (ii) and (iii) to be true. Let $r < 1$. By (ii) the subset $K_r = \{(1 - \chi_r)f \mid f \in K\}$ of the Banach space

$C(D_r)$, $D_r = \{z \in D \mid |z| \leqslant r\}$ (the norm in D_r given by $\|g\|_\infty = \sup\{|g(z)| \mid z \in D_r\}$, $g \in C(D_r)$) is equicontinuous and by (i),

$$\sup\{\|g\|_\infty \mid g \in K_r\} \leqslant [\pi(1-r)^2]^{-1/p} \sup\{\|f\| \mid f \in K\} < \infty.$$

Thus the theorem of Arzelà-Ascoli ([1], p. 266) applies and K_r is conditionally compact, hence totally bounded in $C(D_r)$. Therefore, given $\varepsilon > 0$, there exist functions $f_1, \dots, f_n$ in K such that

$$\inf_{i \leqslant n} \|(1-\chi_r)(f-f_i)\|_\infty < \pi^{-1/p}\varepsilon/3$$

for every $f \in K$. According to (iii) we now take r such that $\sup\{\|\chi_r f\| \mid f \in K\} < \varepsilon/3$ and have for every f in K and some $i \leqslant n$,

$$\begin{aligned}\|f-f_i\| &\leqslant \|(1-\chi_r)(f-f_i)\| + \|\chi_r(f-f_i)\| \\ &< \pi^{1/p}\|(1-\chi_r)(f-f_i)\| + 2\varepsilon/3 < \varepsilon.\end{aligned}$$

Consequently, K is totally bounded in A_p. Since $\bar{K}$ is complete, it follows that K is a conditionally compact subset of A_p (or equivalently, since A_p is a metric space, K is sequentially compact).

In the case $1 < p < \infty$, A_p is reflexive and it is possible to characterize weak convergence in A_p. One observes that from estimate (1) it immediately follows that for any fixed z in D, the functional x_z^* on A_p defined by $x_z^*(f) = f(z)$, $f \in A_p$, belongs to A_p^* so that the weak convergence of a sequence $\{f_n\}$ in A_p implies simple convergence of $f_n(z)$ in D.

THEOREM 5. *A sequence $\{f_n\}$ in A_p, $1 < p < \infty$, converges weakly to zero if and only if it is bounded and for each $j \geqslant 0$ the coefficient a_{nj} of the Taylor series $\sum_{j=0}^{\infty} a_{nj} z^j$ for $f_n(z)$ converges to zero with n.*

Proof. If $\{f_n\}$ converges weakly to zero, $\{f_n\}$ must be bounded. Moreover, since

$$a_{nj} = \big((j+1)/\pi\big)^{1/2} x_j^*(f_n),$$

where $\{x_j^*\}$ is the biorthogonal sequence in A_p^* which belongs to the basis $\{x_j\}$ of Theorem 3, it follows that

$$\lim_n a_{nj} = \big((j+1)/\pi\big)^{1/2} \lim_n x_j^*(f_n) = 0, \quad j = 0, 1, \dots$$

This is the necessary condition.

Conversely, suppose that $\lim_n a_{nj} = 0$ for all j. Since A_p is reflexive, the basis $\{x_j\}$ for A_p has, by a known theorem of James ([4], p. 519), the following property:

For any $x^* \in A_p^*$ one has

$$\lim_m \sup\{|x^*(x)|\,|x \in \overline{\mathrm{sp}}\{x_m, x_{m+1}, \ldots\}, \|x\| = 1\} = 0$$

(i.e. the basis is shrinking). Now, due to the principle of uniform boundedness, there is a constant $K \geqslant 1$ such that

$$\sup_m \sup\Big\{\Big\|\sum_{j=m}^{\infty} x_j^*(x)x_j\Big\|\,\Big|\,\|x\| \leqslant 1\Big\} \leqslant K.$$

Hence

$$\sup\Big\{\Big|x^*\Big(\sum_{j=m}^{\infty} x_j^*(x)x_j\Big)\Big|\,\Big|\,x \in A_p,\ \|x\| \leqslant 1\Big\}$$
$$\leqslant \sup\big\{|x^*(x)|\,|x \in \overline{\mathrm{sp}}\{x_m, x_{m+1}, \ldots\},\ \|x\| \leqslant K\big\}.$$

Thus

$$\lim_m \sup\Big\{\Big|x^*\Big(\sum_{j=m}^{\infty} x_j^*(x)x_j\Big)\Big|\,\Big|\,x \in A_p,\ \|x\| \leqslant 1\Big\} = 0$$

and we have for every $\varepsilon > 0$ an m such that

$$\sup_n \Big|x^*\Big(f_n - \sum_{j\leqslant m} x_j^*(f_n)x_j\Big)\Big| < \varepsilon/2,$$

where without loss of generality one may take x^* and all f_n's of norm one. Because there is an n_ε, depending on m, for which

$$\Big\|\sum_{j\leqslant m} x_j^*(f_n)x_j\Big\| = \Big\|\sum_{j\leqslant m} \big(\pi/(j+1)\big)^{1/2} a_{nj}x_j\Big\| < \varepsilon/2, \qquad n \geqslant n_\varepsilon,$$

one gets

$$|x^*(f_n)| \leqslant \Big|x^*\Big(f_n - \sum_{j\leqslant m} x_j^*(f_n)x_j\Big)\Big| + \Big\|\sum_{j\leqslant m} x_j^*(f_n)x_j\Big\| < \varepsilon, \qquad n \geqslant n_\varepsilon,$$

and the theorem follows.

4. The spectrum of the shift operator. Let $T\colon A_p \to A_p$ be the linear operator defined by $(Tf)(z) = zf(z)$, $z \in D$. It is immediate that T, usually called *shift operator*, is bounded, with norm $\|T\| \leqslant 1$. Therefore the spectrum $\sigma(T)$ of T must be contained in the closed unit disc $\bar{D}$. If $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ donote the point spectrum, residual spectrum and continuous spectrum respectively, and if δD denotes the unit circle $\bar{D} - D$, we can determine the partition of $\bar{D}$ into the mutually exclusive sets $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$.

THEOREM 6. *The shift operator T has the following spectral properties:*

(i) $\sigma(T) = \bar{D}$,

(ii) $\sigma_p(T)$ *is empty,*

(iii) $\sigma_r(T) = D$,
(iv) $\sigma_c(T) = \delta D$.

Proof. Assume $\lambda \in D$ but $\lambda \notin \sigma(T)$. Then $(\lambda I - T)^{-1}$ would exist and would be bounded in A_p. Thus $(\lambda I - T)^{-1} f$ would be in A_p for any f in A_p which is a contradiction since $(\lambda - z)^{-1} f(z)$ is not holomorphic in D. This shows that $D \subset \sigma(T)$ and, since $\sigma(T)$ is closed, (i) follows. Next, because $(\lambda I - T)f = 0$, $f \in A_p$, implies $(\lambda - z)f(z) = 0$ and thus $f(z) = 0$, $z \neq \lambda$, it is clear that $\sigma_p(T)$ is empty.

To determine whether a point λ of $\sigma(T)$ is in $\sigma_r(T)$ or in $\sigma_c(T)$, we look at the range of the operator $\lambda I - T$. Let first λ be in D. Suppose that the range of $\lambda I - T$ is dense in A_p. Then, given f in A_p with $f(\lambda) \neq 0$, there would exist a sequence $\{g_n\}$ in A_p such that $\lim_n (\lambda I - T) g_n = f$. But by estimate (1) this would imply that

$$\lim_n (\lambda - z) g_n(z) = f(z), \qquad z \in D,$$

and hence that $f(\lambda) = 0$ which is impossible. Thus $D \subset \sigma_r(T)$. On the, other hand, if λ is of modulus 1, then the range of $\lambda I - T$ is dense in A_p (it is always assumed that $1 \leqslant p < \infty$), i.e. for every $f \in A_p$ there exists a sequence $\{g_n\}$ in A_p with

$$\lim_n \|f - (\lambda I - T) g_n\| = 0.$$

Let $\{g_n\}$ be defined by

$$g_n(z) = f(z) \sum_{j=0}^{n} (z^j / \lambda^{j+1}), \qquad z \in D.$$

Clearly, $g_n \in A_p$. By

$$(\lambda - z) g_n(z) = f(z) \Big[\sum_{j=0}^{n} (z/\lambda)^j - \sum_{j=1}^{n+1} (z/\lambda)^j \Big] = f(z) \big[1 - (z/\lambda)^{n+1} \big],$$

it is apparent that

$$\|f - (\lambda I - T) g_n\| = \Big[\int_D |f(z) z^{n+1}|^p \, d\mu(z) \Big]^{1/p}$$

$$\leqslant r^{n+1} \Big[\int_{0<|z|<r} |f(z)|^p \, d\mu(z) \Big]^{1/p} + \Big[\int_{r<|z|<1} |f(z)|^p \, d\mu(z) \Big]^{1/p}, \qquad 0 < r < 1.$$

Given $\varepsilon > 0$, it is now possible to choose r such that the last term on the right-hand side of the above inequality is smaller than $\varepsilon/2$. On the other hand, the first term on the right is dominated by $r^{n+1} \|f\|$ so that $\|f - (\lambda I - T) g_n\| < \varepsilon$ for n large enough. Hence $\overline{(\lambda I - T)(A_p)} = A_p$, implying that $\delta D \subset \sigma_c(T)$. Since the sets $\sigma_r(T)$ and $\sigma_c(T)$ are disjoint, one obtains the results (iii) and (iv).

COROLLARY 7. *T is not compact.*

The result is an immediate consequence of the fact that the non-zero points in the residual spectrum of a compact linear operator are isolated.

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