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ON F-CONNECTIONS

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Introduction. In this paper we consider a C^∞-manifold V of dimension $\dim V = m \geqslant 3$, equipped with a $(1,1)$-tensor field F and a linear connection ∇. Duggal [2] and Mishra [3] have introduced the following definitions:

A. ∇ is called an *F-connection* if $\nabla F = 0$.

B. ∇ is called a *semi-F-connection* if $\operatorname{div} F = 0$.

C. ∇ is called an *M-connection* if $(\nabla_X F)(Y) + (\nabla_Y F)(X) = 0$ for any vector fields X and Y on V.

In Section 1 we introduce a mapping L which is defined by F and by a mapping of the connection ∇ (cf. [1]). In Section 2 we give some conditions equivalent to definitions A, B and C. In Section 3 we consider the following question:

When does the Levi-Cività connection on V remain an F-connection, semi-F-connection or an M-connection after a conformal change of Riemannian metric?

1. Mapping L. Let $K\colon TTV \to TV$ be a mapping of the connection ∇. Then

$$\nabla_X Y = K(Y_* X) \tag{1}$$

for any vector fields X and Y on V. We put

$$L = K \circ F_* - F \circ K, \tag{2}$$

where F_* denotes the differential of $F\colon TV \to TV$. Of course, $L\colon TTV \to TV$, and if $a \in (TV)_Z$, then $L(a) \in V_{\pi(Z)}$ for any $Z \in TV$.

LEMMA 1.1. *The equality*

$$L(Y_* X) = (\nabla_X F)(Y) = \nabla_X(FY) - F(\nabla_X Y) \tag{3}$$

holds for any vector fields X and Y on V.

This equality follows by the comparison of (1) and (2).

Let $x = (x^1, \ldots, x^m)$ be a coordinate system on V and put

$$y^i = x^i \circ \pi, \quad y^{m+i} = dx^i \quad \text{for} \quad i = 1, \ldots, m.$$

Then the functions $y^1, \ldots, y^{2m}$ form a coordinate system on TV. We introduce the notation

$$X_i = \frac{\partial}{\partial x_i} \quad \text{and} \quad A_k = \frac{\partial}{\partial y_k} \quad \text{for } i = 1, \ldots, m \text{ and } k = 1, \ldots, 2m.$$

For any $a = \sum_{k=1}^{2m} a^k A_{k|Z}$, where $Z = z^i X_{i|p}$ [1] and $p \in V$, we obtain

$$K(a) = \left(a^{n+i} + \Gamma^i_{jk}(p) a^j z^k\right) X_{i|p}, \tag{4}$$

where $\Gamma^i_{jk} = dx^i(\nabla_{X_j} X_k)$ (see [1], p. 76).

Writing $f^i_j = dx^i\left(F(X_j)\right)$, we have

LEMMA 1.2. *If*

$$a = \sum_{k=1}^{2m} a^k A_{k|Z} \quad \textit{and} \quad Z = z^i X_{i|p},$$

then

$$L(a) = a^k z^l Z_{kl}(p), \tag{5}$$

where

$$Z_{kl} = (X_k f^i_l + f^j_l \Gamma^i_{kj} - f^i_j \Gamma^j_{kl}) X_i. \tag{6}$$

Proof. The statement follows by the standard computation of the coordinate expression for K (formula (4)) and from the equality

$$F_*(a) = a^i A_{i|Z} + \left(z^j a^k X_{k|p} f^i_j + f^i_j(p) a^{n+j}\right) A_{n+i|Z}.$$

PROPOSITION 1.3. $L|\mathrm{v}TV = L|\mathrm{o}TV = 0$, *where* $\mathrm{v}TV$ *is the vertical subbundle of* TTV *and*

$$\mathrm{o}TV = \bigcup_{p \in V} (TV)_{o_p}.$$

2. F-connections.

PROPOSITION 2.1. *∇ is an F-connection iff $L|\mathrm{h}TV = 0$, where $\mathrm{h}TV$ is the set of horizontal vectors of TTV.*

Proof. The statement follows from Lemma 1.1 and Proposition 1.3, since the set of all vectors $Y_* X$, where Y is a vector field on V and $X \in TV$, contains the set of horizontal vectors.

If X is a vector field on V, then LX_* is an endomorphism of TV, and we can consider its trace.

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(1) We use the Einstein summation convention only for indices which change from 1 to m.

PROPOSITION 2.2. *∇ is a semi-F-connection iff* $\operatorname{tr} LX_* = 0$ *for any vector field X on V.*

Proof. $(\operatorname{div} F)(X) = 0$ iff $\operatorname{tr}\big(Y \mapsto (\nabla_Y F)(X)\big) = 0$. But $(\nabla_Y F)(X) = (LX_*)Y$.

For an arbitrary vector field X on V, we denote by X^c the complete lift of X. X^c is defined by the equalities

$$\pi_* X^c = X \quad \text{and} \quad X^c(\omega) = L_X \omega$$

for any 1-form ω on V (see [4], p. 496).

LEMMA 2.3. *If $[X, Y] = 0$, then $X^c \circ Y = Y_* X$.*

Proof. Obviously, $\pi_*(Y_* X) = X$. For any function $f \in C^\infty(V)$, we have

$$Y_* X(df) = X\langle df | Y\rangle$$

and

$$X^c \circ Y(df) = \langle L_X df | Y\rangle = L_X\langle df | Y\rangle - \langle df | L_X Y\rangle = X\langle df | Y\rangle,$$

since $L_X Y = 0$.

PROPOSITION 2.4. *The following conditions are equivalent:*

(i) *∇ is an M-connection;*

(ii) *$L(Y_* X) = L(X_* Y)$ for any vector fields X and Y on V;*

(iii) *$L(X^c \circ Y) = L(Y^c \circ X)$ for any vector fields X and Y on V such that $[X, Y] = 0$.*

Proof. Lemma 1.1 shows that (i) $\Leftrightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is a consequence of Lemma 2.3. Finally, if condition (iii) holds, then vector fields Z_{kl} defined by (6) satisfy the equalities $Z_{kl} = Z_{lk}$ for any $k, l \leqslant m$, since $Z_{kl} = L(A_k \circ X_l)$, $A_k = X_k^c$ and $[X_k, X_l] = 0$. Now formula (5) shows that condition (ii) holds.

3. Conformal change of a Riemannian metric. Let us suppose that ∇ and $\tilde{\nabla}$ are the Levi-Cività connections of conformal equivalent Riemannian tensors g and $\tilde{g}$ on V. Put $\tilde{g} = e^\psi \cdot g$. Denoting by K and $\tilde{K}$ the mappings of connections ∇ and $\tilde{\nabla}$, respectively, and putting

$$L = K \circ F_* - F \circ K \quad \text{and} \quad \tilde{L} = \tilde{K} \circ F_* - F \circ \tilde{K},$$

we can immediately verify that

$$(\tilde{L} - L)(a) = \langle \nabla\psi|_p, FX\rangle Y - \langle \nabla\psi|_p, X\rangle FY - \langle FX, Y\rangle \nabla\psi|_p + \langle X, Y\rangle F\nabla\psi|_p, \tag{7}$$

where $a \in (TV)_X$, $\pi(X) = p$, $\pi_*(a) = Y$, $\nabla\psi$ denotes the gradient of ψ, and $\langle X_1, X_2\rangle = g(X_1, X_2)$ for any pair $(X_1, X_2) \in TV \oplus TV$.

PROPOSITION 3.1. *If* $\tilde{L}-L = 0$ *and* $\nabla\psi|_p \neq 0$, *then vectors* Y *and* FY *are parallel for any vector* $Y \in V_p$ *such that*

$$\langle Y, \nabla\psi|_p\rangle = \langle Y, F\nabla\psi|_p\rangle = 0. \tag{8}$$

Proof. Taking an arbitrary vector $b \in TTV$ such that $\pi_1(b) = \nabla\psi|_p$ and $\pi_*(b) = Y$, we infer from (7) that

$$0 = (\tilde{L}-L)(b) = \langle \nabla\psi|_p, F\nabla\psi|_p\rangle Y - |\nabla\psi|_p|^2 FY.$$

We say that F *satisfies condition* $(*)$ if vectors X and FX are linearly independent for any $X \neq 0$. For instance, almost complex structures satisfy $(*)$.

COROLLARY 3.2. *If* ∇ *is an* F*-connection and* F *satisfies* $(*)$, *then* $\tilde{\nabla}$ *is an* F*-connection iff* $\psi = \text{const}$.

Using formula (7), we obtain

$$\begin{aligned} \mathbf{L}(X, Y) &\overset{\text{def}}{=} (\tilde{L}-L)(X_* Y) - (\tilde{L}-L)(Y_* X) \\ &= \langle FX, \nabla\psi\rangle Y - \langle FY, \nabla\psi\rangle X - \langle X, \nabla\psi\rangle FY + \\ &\quad + \langle Y, \nabla\psi\rangle FX - (\langle FX, Y\rangle - \langle X, FY\rangle)\nabla\psi \end{aligned} \tag{9}$$

for any vector fields X and Y on V.

PROPOSITION 3.3. *If* $\mathbf{L} = 0$ *and* $\nabla\psi|_p \neq 0$, *then, for any* Y *satisfying equalities* (8), *vectors* Y *and* FY *are parallel.*

The proof follows by putting $X = \nabla\psi$ in (9).

COROLLARY 3.4. *If* ∇ *is an* M*-connection and* F *satisfies* $(*)$, *then* $\tilde{\nabla}$ *is an* M*-connection iff* $\psi = \text{const}$.

For any vector field X on V, we have

$$\operatorname{tr}(\tilde{L}-L)\circ X_* = m\langle FX, \nabla\psi\rangle - \operatorname{tr}F\langle X, \nabla\psi\rangle - \langle FX, \nabla\psi\rangle + \langle X, F\nabla\psi\rangle.$$

In particular,

$$\operatorname{tr}(\tilde{L}-L)\circ \nabla\psi_* = m\langle \nabla\psi, F\nabla\psi\rangle - \operatorname{tr}F|\nabla\psi|^2.$$

This proves the following

PROPOSITION 3.5. *If* $\operatorname{tr}(\tilde{L}-L)\circ(\nabla\psi)_* = 0$, *then*

$$\langle \nabla\psi, F\nabla\psi\rangle = \frac{1}{m}\operatorname{tr}F|\nabla\psi|^2.$$

We say that a *pair* (F, g) *satisfies condition* $(**)$ if, for any $X \in TV$, vectors X and FX are g-orthogonal. Of course, in this case F satisfies condition $(*)$. For instance, if F is an almost complex structure on V and g is a Hermitian metric, then (F, g) satisfies $(**)$.

COROLLARY 3.6. *If* ∇ *is a semi-*F*-connection,* (F, g) *satisfies* $(**)$ *and* $\operatorname{tr}F \neq 0$, *then* $\tilde{\nabla}$ *is a semi-*F*-connection iff* $\psi = \text{const}$.

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