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# A NOTE ON ALMOST s-TANGENT STRUCTURES

BY

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**1.** In a previous paper [3], we have defined a new kind of $G$-structure on odd-dimensional manifolds, which we have called an almost s-tangent structure. In some way, this structure may be regarded as an odd-dimensional version of the almost tangent structure (see [1], [2], [6]), just as the almost contact structure may be considered as the odd-dimensional version of the almost complex structure.

In this note, we study the relationship between an almost s-tangent structure and a couple of an almost contact structure and an almost paracontact structure.

**2.** Let $V$ be a differentiable manifold ($C^\infty$, connected, second-countable) of dimension $2n+1$. An *almost s-tangent structure* on $V$ is a triple $(I, \omega, \xi)$, where $I$ is a tensor field of type $(1, 1)$, $\omega$ is a differential 1-form, and $\xi$ is a vector field on $V$ such that

$$\omega(\xi) = 1, \qquad I^2 = \omega \otimes \xi, \qquad \operatorname{rank}(I) = n+1.$$

Then $I\xi = \lambda\xi$ and $\omega I = \lambda\omega$, where $\lambda = \omega(I\xi)$, $\lambda^2 = 1$.

A differentiable manifold with an almost s-tangent structure is called an *almost s-tangent manifold*.

Let $V(I, \omega, \xi)$ be an almost s-tangent manifold of dimension $2n+1$, $P$ the one-dimensional distribution determined by $\xi$, $Q = \operatorname{Ker}\omega \subset T(V)$ and $K = \operatorname{Ker} I \subset Q$. Then there exist a subbundle $S$ of $Q$ and a Riemannian metric $g$ on $V$ such that $K$, $S$ and $P$ are mutually orthogonal and

$$g(X, \xi) = \omega(X) \quad \text{for any vector field } X \text{ on } V,$$

$$g_x(X, Y) = g_x(IX, IY) \quad \text{for any } X, Y \in S_x,\ x \in V.$$

We call such a metric $g$ a *compatible metric* and we say that $(I, \omega, \xi, g)$ is an *almost s-tangent metric structure* on $V$.

Now, let us consider a manifold $V$, of dimension $2n+1$, with an almost s-tangent metric structure $(I, \omega, \xi, g)$. We can take a local orthonormal basis $\{X_\alpha, X_{\alpha^*}, \xi\}_{\alpha=1,\dots,n}$ in a coordinate neighborhood $U$, where $\{X_\alpha\}$ is a basis for $S$ on $U$, and $\{X_{\alpha^*} = IX_\alpha\}$ is a basis for $K$ on $U$. Let $\{\eta^\alpha, \eta^{\alpha^*}, \omega\}$ be the dual basis of $\{X_\alpha, X_{\alpha^*}, \xi\}$. In terms of these bases we can write (locally)

$$I = \sum_{\alpha=1}^{n} \eta^\alpha \otimes X_{\alpha^*} + \lambda\omega \otimes \xi$$

and

$$IX_\alpha = X_{\alpha^*}, \quad IX_{\alpha^*} = 0, \quad I\xi = \lambda\xi \ (\lambda = \pm 1).$$

Putting

$$L_1 = \sum_\alpha \eta^{\alpha^*} \otimes X_\alpha + \lambda\omega \otimes \xi, \quad L_2 = \sum_\alpha \eta^{\alpha^*} \otimes X_\alpha - \lambda\omega \otimes \xi,$$

we obtain two well-defined tensor fields $L_1$ and $L_2$ of type (1, 1) on $V$ such that

$$L_1 X_\alpha = 0, \quad L_1 X_{\alpha^*} = X_\alpha, \quad L_1\xi = \lambda\xi,$$
$$L_2 X_\alpha = 0, \quad L_2 X_{\alpha^*} = X_\alpha, \quad L_2\xi = -\lambda\xi.$$

Moreover,

$$L_1^2 = \omega \otimes \xi, \quad L_2^2 = \omega \otimes \xi,$$

so that $(L_1, \omega, \xi)$ and $(L_2, \omega, \xi)$ are also almost s-tangent structures on $V$.

Furthermore,

(1) $$L_1 L_2 = L_2 L_1 = -\omega \otimes \xi.$$

On the other hand,

(2) $$IL_1 = \sum_\alpha \eta^{\alpha^*} \otimes X_{\alpha^*} + \omega \otimes \xi, \quad L_1 I = \sum_\alpha \eta^\alpha \otimes X_\alpha + \omega \otimes \xi,$$

(3) $$IL_2 = \sum_\alpha \eta^{\alpha^*} \otimes X_{\alpha^*} - \omega \otimes \xi, \quad L_2 I = \sum_\alpha \eta^\alpha \otimes X_\alpha - \omega \otimes \xi,$$

whence

$$(I - L_1)(I - L_1) = -1 + \omega \otimes \xi,$$
$$(I + L_2)(I + L_2) = 1 - \omega \otimes \xi.$$

Now, we put

$$\varphi = I - L_1, \quad \psi = I + L_2,$$

so that $(\varphi, \xi, \omega)$ is an almost contact structure [4], and $(\psi, \xi, \omega)$ is an almost paracontact structure [5] on $V$. Obviously,

$$\varphi X_\alpha = X_{\alpha^*}, \quad \varphi X_{\alpha^*} = -X_\alpha, \quad \varphi\xi = 0,$$
$$\psi X_\alpha = X_{\alpha^*}, \quad \psi X_{\alpha^*} = X_\alpha, \quad \psi\xi = 0.$$

Moreover, if we put

$$\tau = IL_1 - L_1 I = IL_2 - L_2 I,$$

then

$$\tau X_\alpha = -X_\alpha, \quad \tau X_{\alpha^*} = X_{\alpha^*}, \quad \tau\xi = 0$$

and

$$\tau^2 = 1 - \omega \otimes \xi,$$

whence $(\tau, \xi, \omega)$ is another almost paracontact structure on $V$.

From (1)–(3) we deduce

$$\varphi\psi = -\psi\varphi = \sum_\alpha (\eta^{\alpha^*} \otimes X_{\alpha^*} - \eta^\alpha \otimes X_\alpha) = \tau,$$

and hence

$$\tau\varphi = -\varphi\tau = \psi, \quad \tau\psi = -\psi\tau = \varphi.$$

Conversely, suppose that a $(2n+1)$-dimensional differentiable manifold $V$ admits a vector field $\xi$, a differential 1-form $\omega$ and two tensor fields $\varphi$ and $\psi$ of type (1, 1) such that $(\varphi, \xi, \omega)$ is an almost contact structure and $(\psi, \xi, \omega)$ is an almost paracontact structure and, moreover, being $\varphi\psi = -\psi\varphi$. Then, setting $\tau = \varphi\psi$, one easily verifies that $(\tau, \xi, \omega)$ is an almost paracontact structure and that

$$\psi = \tau\varphi = -\varphi\tau, \quad \varphi = \tau\psi = -\psi\tau$$

hold true. If we put

$$k = \tfrac{1}{2}(1+\tau-\omega \otimes \xi), \quad s = \tfrac{1}{2}(1-\tau-\omega \otimes \xi), \quad p = \omega \otimes \xi,$$

then

$$kk = k, \quad ss = s, \quad pp = p,$$

$$sp = ps = 0, \quad kp = pk = 0, \quad ks = sk = 0,$$

$$k+s+p = 1,$$

and hence $k$, $s$ and $p$ define complementary distributions $K$, $S$ and $P$. The distribution $P$ being always one-dimensional, let us assume that the distributions $K$ and $S$ are both $n$-dimensional and set

$$I = \tfrac{1}{2}(\varphi+\psi)+\omega \otimes \xi, \quad L = \tfrac{1}{2}(-\varphi+\psi)+\omega \otimes \xi.$$

Then

$$I^2 = \omega \otimes \xi, \quad L^2 = \omega \otimes \xi,$$

$$I\xi = \xi, \quad \omega I = \omega, \quad L\xi = \xi, \quad \omega L = \omega,$$

and

$$IL = k + \omega \otimes \xi, \qquad LI = s + \omega \otimes \xi.$$

If $IX = 0$, then $LIX = 0$, so that $sX = -\omega(X)\xi$, and hence

$$kX = X - sX - \omega(X)\xi = X.$$

Then $X \in K$. Conversely, if $X \in K$, then $kX = X$ implies $ILX = X + \omega(X)\xi$, and hence

$$I^2 LX = IX + \omega(X) I\xi = IX + \omega(X)\xi;$$

on the other hand, $I^2 LX = \omega(LX)\xi = \omega(X)\xi$, and thus $IX = 0$. Therefore $\operatorname{Ker} I = K$, so that $\operatorname{rank}(I) = n+1$, and hence $(I, \omega, \xi)$ is an almost s-tangent structure.

Hence we have proved

THEOREM. *A $(2n+1)$-dimensional differentiable manifold $V$ admits an almost s-tangent structure if and only if it admits a vector field $\xi$, a differential 1-form $\omega$ and tensor fields $\varphi$ and $\psi$ of type* $(1, 1)$ *such that*

(a) $(\varphi, \xi, \omega)$ *is an almost contact structure,*

(b) $(\psi, \xi, \omega)$ *is an almost paracontact structure,*

(c) $\varphi\psi = -\psi\varphi$,

(d) $\operatorname{rank}(k) = \operatorname{rank}(s) = n$, *being*

$$k = \tfrac{1}{2}(1 + \varphi\psi - \omega \otimes \xi), \qquad s = \tfrac{1}{2}(1 - \varphi\psi - \omega \otimes \xi).$$

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# LIE DERIVATIVES OF SECTORFORM FIELDS

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White [9] introduced recently a useful concept of sector $r$-form, which generalizes the classical 1-forms to the case of the iterated tangent bundles. In the first part of the present paper, we develop a systematic theory of Lie differentiation of arbitrary sectorform fields (in [9], this question is studied only for those special sectorform fields that coincide with the classical covariant tensors). Our approach is based on a recent definition of the generalized Lie derivative and on the theory of prolongation functors [4]. Another important tool is an original idea of a $T$-natural transformation, which gives a simple construction of the prolongations of vector fields. Our second aim is to show that certain maps analogous to sector $r$-forms can be defined on the fibre bundle of all $k$-dimensional velocities of order $r$. Our construction is based on a reduction procedure dealing with the iterations of a prolongation functor of rather general type. All manifolds and maps are assumed to be infinitely differentiable.

**1. $T$-natural transformations.** Let $\boldsymbol{M}$ be the category of all manifolds and maps, and $\boldsymbol{FM}$ the category of fibred manifolds. A functor $F\colon \boldsymbol{M} \to \boldsymbol{FM}$ transforming any manifold $M$ into a fibred manifold $p_M\colon FM \to M$ and any map $f\colon M \to N$ into a fibred manifold morphism $Ff\colon FM \to FN$ over $f$ is said to be a *prolongation functor* if it satisfies two simple additional conditions of locality and regularity [4]. Any vector field $\xi$ on $M$ is prolonged into a vector field $\boldsymbol{F}\xi$ on $FM$, the flow of which is the prolongation of the flow of $\xi$, i.e.,

$$\exp t(\boldsymbol{F}\xi) = F(\exp t\xi) \tag{1}$$

(see [8]). On the other hand, $\xi$ is a map of $M$ into $TM$ and one can construct $F\xi\colon FM \to FTM$. For $F = T$, we deduced that $\boldsymbol{T}\xi = i_M \circ T\xi$, where $i_M\colon TTM \to TTM$ is the classical canonical involution on $TTM$ (see [5]). From the categorical point of view, $i$ is a natural transformation of the functor $TT$ into itself. We are going to study a generalization of such a situation. Let $\pi_M\colon TM \to M$ be the bundle projection of the tangent bundle.

DEFINITION 1. A natural transformation $i\colon FT \to TF$ satisfying

$$F\pi_M = \pi_{FM} \circ i_M$$

is called a *T-natural transformation* of $F$ if

(2) $$\mathcal{F}\xi = i_M \circ F\xi$$

for any manifold $M$ and any vector field $\xi$ on $M$.

Consider the functor $T_k^r$ of $k$-dimensional velocities of order $r$ (see [1]), i.e., $T_k^r M$ is the fibred manifold of all $r$-jets of $\mathbf{R}^k$ into $M$ with source 0 and

$$(T_k^r f)(j_0^r g) = j_0^r (f \circ g)$$

for any $g\colon \mathbf{R}^k \to M$ and $f\colon M \to N$. Libermann [6] introduced a map

$$i_M\colon T_k^r TM \to TT_k^r M$$

as follows. Any $B \in TM$ is tangent to a curve $\gamma(t)$,

$$B = \frac{\partial}{\partial t}\bigg|_0 \gamma(t),$$

so that any $A \in T_k^r TM$ is of the form

$$A = j_0^r\left(\frac{\partial}{\partial t}\bigg|_0 \gamma(u, t)\right), \qquad u \in \mathbf{R}^k.$$

Then one defines

(3) $$i_M(A) = \frac{\partial}{\partial t}\bigg|_0 (j_0^r \gamma(u, t)) \in TT_k^r M.$$

PROPOSITION 1. *$i\colon T_k^r T \to TT_k^r$ is a T-natural transformation of $T_k^r$.*

Proof. Obviously,

$$\pi_{T_k^r M}(i_M(V)) = j_0^r \gamma(u, 0) = (T_k^r \pi_M)(A).$$

For every $f\colon M \to N$, we have

$$(TT_k^r f)(i_M(A)) = \frac{\partial}{\partial t}\bigg|_0 (j_0^r f(\gamma(u, t))) = i_N(T_k^r Tf(A)).$$

Further, let $\varphi_t(x)$ be the flow of a vector field $\xi$, so that

$$\xi(x) = \frac{\partial}{\partial t}\bigg|_0 \varphi_t(x).$$

For any $X = j_0^r g(u) \in T_k^r M$ we have

$$(T_k^r \xi)(X) = j_0^r\left(\frac{\partial}{\partial t}\bigg|_0 \varphi_t(g(u))\right).$$

On the other hand,

$$\exp t(T_k^r \xi)(X) = (T_k^r \varphi_t)(X) = j_0^r \varphi_t(g(u)).$$

Hence

$$(T_k^r \xi)(X) = \frac{\partial}{\partial t}\Big|_0 \big(j_0^r \varphi_t(g(u))\big).$$

If $j\colon GT \to TG$ is another $T$-natural transformation of a prolongation functor $G$, we have

$$Gi_M\colon GFTM \to GTFM \quad \text{and} \quad j_{FM}\colon GTFM \to TGFM.$$

Hence we can construct

$$k_M := j_{FM} \circ Gi_M\colon GFTM \to TGFM. \tag{4}$$

PROPOSITION 2. *$k$ is a $T$-natural transformation of $GF$.*

Proof. Obviously,

$$F\pi_M = \pi_{FM} \circ i_M \quad \text{and} \quad G\pi_M = \pi_{GM} \circ j_M$$

imply

$$GF\pi_M = G\pi_{FM} \circ Gi_M = \pi_{GFM} \circ j_{FM} \circ Gi_M.$$

Further,

$$j_{FM} \circ Gi_M \circ GF\xi = j_{FM} \circ G(i_M \circ F\xi) = j_{FM} \circ G(F\xi) = GF\xi.$$

By iteration, any $T$-natural transformation $i$ of $F$ induces a $T$-natural transformation $i^r\colon F_r T \to TF_r$ of the $r$-th iterated functor

$$F_r = \underbrace{F \ldots F}_{r\ \text{times}}.$$

If we take $F = T$, then the canonical involution $i\colon TT \to TT$ induces a $T$-natural transformation $i^r\colon T_r T \to TT_r$. (White has shown [9] that the group of all permutations of $r+1$ letters acts naturally on the $(r+1)$-st tangent bundle $T_{r+1} M$. Using (4) we find easily that our $T$-natural transformation $i_M^r\colon T_{r+1} M \to T_{r+1} M$ coincides with one of these permutations.)

**2. Lie derivatives of sectorform fields.** Some recent results suggest that the basic situation for the Lie differentiation is the following one [5]. Given two manifolds $M$ and $N$, a map $f\colon M \to N$, a vector field $\xi$ on $M$ and a vector field $\eta$ on $N$, the *Lie derivative* of $f$ with respect to $\xi$ and $\eta$ is defined by

$$L(\xi, \eta) f := Tf \circ \xi - \eta \circ f\colon M \to TN. \tag{5}$$

This is a so-called *vector field along f*. Using flows [2], we can express $L(\xi, \eta) f$ as

$$(6) \qquad L(\xi, \eta) f = \frac{\partial}{\partial t}\Big|_0 \exp(-t\eta) \circ f \circ \exp t\xi.$$

Given a real-valued map $f: M \to \boldsymbol{R}$, we have

$$Tf: TM \to T\boldsymbol{R} = \boldsymbol{R} \times \boldsymbol{R},$$

and the second component $\delta f: TM \to \boldsymbol{R}$ of $Tf$ is called the *differential* of $f$. Consider a 1-form $\omega$ on a manifold $M$, which can be interpreted as a map $\omega: TM \to \boldsymbol{R}$. Then the classical Lie derivative $L_\xi \omega$ coincides with the second component of $L(\boldsymbol{T}\xi, 0)\omega$, where 0 means the zero vector field on $\boldsymbol{R}$ (see [4]). Using $\boldsymbol{T}\xi = i_M \circ T\xi$, we find

$$L_\xi \omega = (\delta\omega) \circ i_M \circ T\xi.$$

Consider the iterated $r$-th tangent bundle $T_r M$. We have $r$ canonical projections

$$T_k \pi_{T_{r-k-1}M}: T_r M \to T_{r-1} M, \qquad k = 0, \ldots, r-1,$$

and $T_r M$ is a vector bundle with respect to any of these projections. According to White [9], a map $A: (T_r M)_x \to \boldsymbol{R}$ is called a *sector r-form* at $x \in M$ if it is a linear morphism with respect to all $r$ vector bundle structures. Let $T_r^* M \to M$ denote the fibre bundle of all sector $r$-forms on $M$ (see [9]). A *sectorform field* on $M$ is a section $A^r: M \to T_r^* M$ which can be interpreted as a map $A^r: T_r M \to \boldsymbol{R}$. If $\xi$ is a vector field on $M$, then

$$(7) \qquad L(\boldsymbol{T}_r \xi, 0) A^r: T_r M \to T\boldsymbol{R}.$$

Definition 2. The *Lie derivative* $L_\xi A^r$ is the second component of (7).

By Proposition 2 we obtain

Proposition 3. *We have* $L_\xi A^r = (\delta A^r) \circ i_M^r \circ T_r \xi$.

For any subcomplex $K$ of an $(r-1)$-dimensional simplex, White constructs a fibre bundle $T_r[M; K]$ over $M$ and a canonical projection $T_r M \to T_r[M; K]$. A sector $r$-form is called a *sector K-form* if it is projectable with respect to the latter projection. Since this construction has a functorial character, it follows from (6) and (7) that the Lie derivative of a $K$-sectorform field is a $K$-sectorform field. In particular, if $K$ is the subcomplex of all vertices of an $(r-1)$-dimensional simplex, then $T_r[M; K]$ coincides with the Whitney sum $\bigoplus_r TM$, and a $K$-sectorform field is a classical $r$-times covariant tensor field on $M$. In this case, (6) and (7) imply that we get the classical Lie derivative.

**3. Generalized sectorforms.** Let $F$ be a prolongation functor with values in the category $\boldsymbol{VB}$ of vector bundles. Assume further that $F$ satisfies

(i) (linearity axiom) for any vector bundle $q\colon E \to M$, $Fq\colon FE \to FM$ is also a vector bundle, and for any linear morphism

$$f\colon (E \to M) \to (D \to N),$$

$Ff\colon (FE \to FM) \to (FD \to FN)$ is a linear morphism;

(ii) (Pradines' axiom [7]) if $Y \to X$ is a submersion, then $FY$ is a submersion over $Y \oplus FX$ with respect to the pullback map.

As a consequence of Pradines' axiom, if $x^i$, $y^p$ are some fibre coordinates on $Y$, and $x^i$, $z^\alpha$ are some fibre coordinates on $FX$, then there are some fibre coordinates on $FY$ of the form $x^i$, $y^p$, $z^\alpha$, $w^\lambda$.

Consider the $r$-th iteration $F_r$ of $F$. Then we have $r$ canonical projections

$$F_k\, p_{F_{r-k-1}M}\colon F_r M \to F_{r-1} M, \qquad k = 0, \ldots, r-1,$$

and $F_r M$ is a vector bundle with respect to any of these projections. A map $A\colon (F_r M)_x \to \boldsymbol{R}$ is called an *$(F, r)$-form* if $A$ is a linear morphism with respect to all vector bundle structures. We are going to deduce the coordinate expression of an $(F, r)$-form. Given some local coordinates $x^i$ on $M$, let $X_1^{p_1}$ be some additional fibre coordinates on $FM$. On $F_2 M$, we denote by $x^i$, $X_{10}^{p_1}$ the coordinates induced from $F_1 M$ by $p_{FM}\colon F_2 M \to FM$. According to Pradines' axiom, the additional coordinates on $F_2 M$ are $X_{01}^{p_1}$, $X_{11}^{p_2}$, where the superscript $p_2$ corresponds to the fibre dimension of $F_2 M \to FM \oplus FM$. In general, consider a sequence $\gamma = (\gamma_1, \ldots, \gamma_r)$, where $\gamma_i \in \{0, 1\}$, not all $\gamma_i$ zero, and set $|\gamma|$ to be the number of 1's in the sequence, and $e = (1, 1, \ldots, 1)$. By Pradines' axiom, we have a pullback projection

$$F_r M \to F_{r-1} M \oplus_{F_{r-2}M} F_{r-1} M. \tag{8}$$

On $F_r M$, we define local coordinates $x^i$, $X_\gamma^{p|\gamma|}$ for all $\gamma$ as follows. If $\gamma_r = 0$, then $X_\gamma^{p|\gamma|}$ is induced by the corresponding coordinate on the first factor of (8); if $\gamma_{r-1} = 0$, then $X_\gamma^{p|\gamma|}$ is induced by the corresponding coordinate on the second factor of (8) (if both $\gamma_{r-1} = 0 = \gamma_r$, we are on the basis $F_{r-2} M$ of the Whitney sum (8)), and $X_e^{p_r}$ are the additional coordinates in the fibres of (8). According to White [9], if

$$\gamma' = (\gamma_1', \ldots, \gamma_r') \quad \text{and} \quad \gamma'' = (\gamma_1'', \ldots, \gamma_r'')$$

satisfy $\gamma_i' \gamma_i'' = 0$ for all $i$ $(1 \leqslant i \leqslant r)$, then the join $\gamma = \gamma' \cup \gamma''$ is defined by $\gamma_i = \gamma_i' + \gamma_i''$. Any expression of the form

$$\gamma_1 \cup \ldots \cup \gamma_k = e$$

is called a *join decomposition* of $e$. By an induction procedure quite similar to the proof of Theorem 3.2 of [9], one proves

PROPOSITION 4. *A map $A\colon (F_r M)_x \to \mathbf{R}$ is an $(F, r)$-form if and only if it has the coordinate expression*

$$\sum_{\gamma_1 \cup \ldots \cup \gamma_k = e} a^{\gamma_1 \ldots \gamma_k}_{p_{i_1} \ldots p_{i_k}} X^{p_{i_1}}_{\gamma_1} \ldots X^{p_{i_k}}_{\gamma_k}, \tag{9}$$

*where $i_j = |\gamma_j|$ and the sum is taken over all distinct join decompositions of $e$.*

(9) implies that the set of all $(F, r)$-forms at $x \in M$ is a vector space. Hence the space of all $(F, r)$-forms on $M$ is a vector bundle $F_r^* M \to M$. For any map $f\colon N \to M$, $f(y) = x$,

$$(F_r f)_y\colon (F_r N)_y \to (F_r M)_x$$

is a linear morphism with respect to all $r$ vector bundle structures, so that $A \circ (F_r f)_y$ is an $(F, r)$-form at $y \in N$ for any $A \in (F_r M)_x$. By (9) and the linearity axiom, the induced map

$$(F_r^* f)_y\colon (F_r^* M)_x \to (F_r^* N)_y$$

is linear. Consider the dual vector space

$$(F_r' M)_x := (F_r^* M)_x^*$$

and the dual map

$$(F_r' f)_y := (F_r^* f)_y^*\colon (F_r' N)_y \to (F_r' M)_x.$$

In this way, we obtain a prolongation functor $F_r'\colon \mathbf{M} \to \mathbf{VB}$ (the exact proof follows from the basic facts of the theory of prolongation cofunctors explained in [3]).

Having a vector field $\xi$ on $M$, the *Lie derivative $L_\xi A^r$ of an $(F, r)$-form field $A^r$*: $F_r M \to \mathbf{R}$ is defined as the second component of $L(F_r \xi, 0) A^r$.

PROPOSITION 5. *$L_\xi A^r$ is an $(F, r)$-form field for any $\xi$.*

Proof. The prolonged flow $F_r(\exp t\xi)$ transforms any $(F, r)$-form into an $(F, r)$-form. By (9), the derivative of a one-parameter family of $(F, r)$-forms is also an $(F, r)$-form. Then (6) implies our assertion.

If we have a $T$-natural transformation $i\colon FT \to TF$ and we construct the induced $T$-natural transformation $i^r\colon F_r T \to TF_r$, then we deduce in the same way as in Section 2 that

$$L_\xi A^r = (\delta A^r) \circ i^r_M \circ F_r \xi. \tag{10}$$

**4. $T_k^r$-forms.** We have

$$T_k^1 M = \bigoplus_k TM \quad \text{and} \quad T_k^1 f = \bigoplus_k Tf,$$

so that the values of the functor $T_k^1$ are in $\mathbf{VB}$. One verifies easily that $T_k^1$

satisfies conditions (i) and (ii) of Section 3. To study $T_k^r$ by means of $(T_k^1)_r$, we introduce a map

$$h^r\colon\ T_k^r M \to (T_k^1)_r M$$

by the following induction. Let $t_v\colon \mathbf{R}^k \to \mathbf{R}^k$ be the translation $u \mapsto u+v$, $u \in \mathbf{R}^k$. Any $X \in T_k^r M$ is of the form $X = j_0^r g(u)$. Then we set

$$\tilde{g}(v) = j_0^{r-1}(g \circ t_v) \in T_k^{r-1} M \quad \text{and} \quad h^r(X) = j_0^1 h^{r-1}(\tilde{g}(u)), \quad h^1 = \mathrm{id}.$$

Using local coordinates, we find directly that $h^r$ is injective, so that we can write $T_k^r M \subset (T_k^1)_r M$. Moreover, $T_k^r f$ is the restriction of $(T_k^1)_r f$ to $T_k^r M$ for any $f\colon M \to N$. We define a $T_k^r$-form at $x \in M$ to be the restriction of a $(T_k^1, r)$-form $((T_k^1)_r M)_x \to \mathbf{R}$ to $(T_k^r M)_x$. Any local coordinates $x^i$ on $M$ induce fibre coordinates $x^i$, $X_\alpha^i$ on $T_k^r M$, where $\alpha$ are multi-indices $|\alpha| \leqslant r$ corresponding to the partial derivatives on $\mathbf{R}^k$. Comparing the notation, we deduce from (9)

PROPOSITION 6. *A map $(T_k^r M)_x \to \mathbf{R}$ is a $T_k^r$-form if and only if it has the coordinate expression*

$$\sum_{|\alpha_1| + \ldots + |\alpha_s| = r} a_{i_1 \ldots i_s}^{\alpha_1 \ldots \alpha_s} X_{\alpha_1}^{i_1} \ldots X_{\alpha_s}^{i_s}, \tag{11}$$

*where the sum is taken over all distinct sequences of multi-indices satisfying $|\alpha_1| + \ldots + |\alpha_s| = r$.*

From Section 3 it follows that the set $(T_k^r)_x^* M$ of all $T_k^r$-forms at $x \in M$ is a vector space and

$$(T_k^r)^* M = \bigcup_{x \in M} (T_k^r)_x^* M$$

is a vector bundle over $M$. For every $f\colon N \to M$, $f(y) = x$, we obtain an induced linear map $(T_k^r)_x^* M \to (T_k^r)_y^* N$ and the dual constructions lead to a prolongation functor $T_k^{r*}\colon \mathcal{M} \to \mathcal{VB}$.

The *Lie derivative $L_\xi A^r$ of a $T_k^r$-form field $A^r$*: $T_k^r M \to \mathbf{R}$ is defined as the second component of $L(T_k^r \xi, 0) A^r$. Using flows, we deduce that $L_\xi A^r$ is also a $T_k^r$-form field. In the same way as in Section 3, we find

$$L_\xi A^r = (\delta A^r) \circ i_M \circ T_k^r \xi,$$

where $i_M$ is the $T$-natural transformation of Proposition 1.

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