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ON REGULAR TANGENT COVECTORS, REGULAR DIFFERENTIAL FORMS, AND SMOOTH VECTOR FIELDS ON A DIFFERENTIAL SPACE

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0. Introduction. In [2] Sikorski generalized the notion of differentiable manifold by introducing the notion of differential space. The set of all tangent vectors and the set of all covectors tangent to a given differential space M can be endowed with natural differential structures to form the differential space tangent to M and the differential space cotangent to M, respectively (see [1]). In the present paper the differential space ${}^*TM$ of all regular tangent covectors and, moreover, regular differential forms on M are defined and studied. Some relations between ${}^*TM$ and the above-mentioned differential spaces are also given.

The "regular" notions coincide with the classical ones if considered on a differentiable manifold, and they are surprisingly more intuitive and clear in the case of a general differential space.

Throughout the paper we fix a differential space

$$M = \big(\operatorname{set} M,\ \mathscr{F}(M)\big),$$

where $\mathscr{F}(M)$ denotes the differential structure of M, and $\operatorname{set} M$ its support.

1. Tangent vectors and regular tangent covectors. We recall the notion of tangent vectors.

The standard equivalence relation on $\mathscr{F}(M) \times \operatorname{set} M$ gives rise to the space $G(M)$ of all germs of functions from $\mathscr{F}(M)$. Now, we fix an arbitrary point p of M and denote by $G_0(M, p)$ the ideal of all germs at p whose values are zero. The quotient algebra $G_0(M, p)/G_0(M, p)^2$ is a real vector space and the tangent space T_pM is its dual:

$$T_pM = [G_0(M, p)/G_0(M, p)^2]^*.$$

Since the vector space T_p^*M of all tangent covectors at p is dual to T_pM, there is a natural injection i_p of $G_0(M, p)/G_0(M, p)^2$ into its second dual:

$$(1)\qquad i_p\colon G_0(M,p)/G_0(M,p)^2 \hookrightarrow T_p^*M, \qquad \big(i_p(w)\big)(v) = v(w)$$

for $p \in \operatorname{set} M$.

Definition 1.1. The *vector space* ${}^*T_pM$ of regular covectors tangent to M at a point p of M is the image of i_p:

$${}^*T_pM := \operatorname{im}(i_p) \subset T_p^*M.$$

Obviously, ${}^*T_pM = T_p^*M$ if and only if $\dim T_pM < \infty$.

For any function $a \in \mathscr{F}(M)$ and a point p of M, we define the differential $da_p \in {}^*T_pM$ by the formula

$$da_p = i_p\big([a-a(p), p] + G_0(M,p)^2\big),$$

where $[a-a(p), p]$ denotes the equivalence class (the germ) of the pair $\big(a-a(p), p\big) \in \mathscr{F}(M) \times \operatorname{set} M$. We see that

$$(2)\qquad {}^*T_pM = \{da_p\colon a \in \mathscr{F}(M)\}.$$

Remark 1.1. There is a slight difference between T_pM and the space M_p defined in [2]. For any $p \in \operatorname{set} M$ these two spaces are naturally isomorphic. However, the vector spaces T_pM are always mutually disjoint. Under the dual isomorphism $T_p^*M \cong (M_p)^*$, the covectors da_p correspond to the appropriate values of the standard differential.

We denote the union of all ${}^*T_pM$, $p \in \operatorname{set} M$, by $\operatorname{set}{}^*TM$.

2. The differential space ${}^*TM$. For a set A, let

$$\boldsymbol{R}^{[A]} := \{c \in \boldsymbol{R}^A\colon c = 0 \text{ outside a finite subset of } A\}.$$

We denote by $\Phi = \Phi_M$ the mapping

$$(3)\qquad \boldsymbol{R}^{[\mathscr{F}(M)]} \times \operatorname{set} M \ni (c,p) \mapsto \sum_{a\in\mathscr{F}(M)} c_a da_p \in \operatorname{set}{}^*TM.$$

Now we are able to transfer an appropriate differential structure from $\boldsymbol{R}^{[\mathscr{F}(M)]} \times \operatorname{set} M$ onto $\operatorname{set}{}^*TM$.

For any finite $B \subset \mathscr{F}(M)$, there is a standard product differential structure $\mathscr{F}(\boldsymbol{R}^B \times M) = C^\infty(\boldsymbol{R}^B) \times \mathscr{F}(M)$ on $\boldsymbol{R}^B \times \operatorname{set} M$. There is also a canonical injection $\boldsymbol{R}^B \hookrightarrow \boldsymbol{R}^{[\mathscr{F}(M)]}$ which maps functions $\boldsymbol{R} \to B$ to their zero-extensions onto the whole set $\mathscr{F}(M)$.

Definition 2.1. The *differential space* ${}^*TM$ of all regular covectors tangent to M is the pair $\big(\operatorname{set}{}^*TM, \mathscr{F}({}^*TM)\big)$, where

$$(4)\qquad \mathscr{F}({}^*TM) = \{\xi \in \boldsymbol{R}^{\operatorname{set}{}^*TM}\colon \text{for any finite } B \subset \mathscr{F}(M)$$
$$\text{there is } \xi\circ\Phi|\boldsymbol{R}^B \times \operatorname{set} M \in \mathscr{F}(\boldsymbol{R}^B \times M)\}.$$

Now, we list some special mappings related to the space ${}^*TM$.

(a) The sets ${}^*T_pM$, $p \in \operatorname{set}M$, are mutually disjoint and, therefore, the projection ${}^*\pi$: $\operatorname{set}{}^*TM \to \operatorname{set}M$, ${}^*\pi(da_p) = p$ for $p \in \operatorname{set}M$, is well defined.

(b) For smooth functions $a_1, \ldots, a_k, \beta_1, \ldots, \beta_k \in \mathscr{F}(M)$, where k is any positive integer, the differential gives rise to the mapping

$$\sum_{i\leqslant k} \beta_i da_i\colon \operatorname{set}M \to \operatorname{set}{}^*TM$$

defined by

$$\Big(\sum_{i\leqslant k} \beta_i da_i\Big)(p) = \sum_{i\leqslant k} \beta_i(p) da_{ip} \quad \text{for } p \in \operatorname{set}M.$$

THEOREM 2.1. *The mappings defined in* (a) *and* (b) *determine smooth functions*

$${}^*\pi\colon {}^*TM \to M \quad \textit{and} \quad \sum_{i\leqslant k} \beta_i da_i\colon M \to {}^*TM,$$

respectively.

Proof. (a) For any finite $B \subset \mathscr{F}(M)$, the composition

$${}^*\pi \circ \Phi|\boldsymbol{R}^B \times \operatorname{set}M\colon \boldsymbol{R}^B \times \operatorname{set}M \to \operatorname{set}M$$

is the projection onto the second factor and, therefore, it determines a smooth function.

(b) Suppose that $a_i, \beta_i \in \mathscr{F}(M)$ for $i = 1, \ldots, k$. We may assume that $a_i \neq a_j$ for $i \neq j$ and then define a mapping

$$\psi\colon \operatorname{set}M \to \boldsymbol{R}^{\{a_1, \ldots, a_k\}} \times \operatorname{set}M$$

by putting

$$\psi(p) = \big(\beta_1(p), \ldots, \beta_k(p); p\big) := \big(\eta(p), p\big),$$
$$\text{where } \eta(p)(a_i) = \beta_i(p),\ i = 1, \ldots, k,$$

for $p \in \operatorname{set}M$. We have

$$\sum_{i\leqslant k} \beta_i da_i = (\Phi|\boldsymbol{R}^{\{a_1, \ldots, a_k\}} \times \operatorname{set}M) \circ \psi.$$

Since ψ determines a smooth mapping, so does $\sum_{i\leqslant k} \beta_i da_i$.

THEOREM 2.2. *If M is a differentiable manifold, then the identity mapping j: $\operatorname{set}{}^*TM \to \operatorname{set}T^*M$ yields the diffeomorphism of ${}^*TM$ and the total space T^*M of the cotangent bundle.*

Proof. Since all the tangent spaces are finite dimensional, $\operatorname{set}{}^*TM = \operatorname{set}T^*M$ (i_p is onto for all p). Let n denote $\dim M$.

(i) Smoothness of j. Take an arbitrary function $\xi \in \mathscr{F}(T^*M)$ and choose a chart $x = (x^i)_{i\leqslant n}$ on an open subset U of M. Then there exists an associated chart $\tilde{x}$ on $\pi^{*-1}(U)$, π^* being the natural projection $T^*M \to M$, such that

$$\omega = \sum_{i\leqslant n} \tilde{x}^{n+i}(\omega)\, dx^i_{\pi^*(\omega)} \quad \text{and} \quad \tilde{x}^j = x^j \circ \pi^*$$

for $j \leqslant n$, $\omega \in \pi^{*-1}(U)$.

Let B be any finite subset of $\mathscr{F}(M)$. For some smooth functions $\gamma_{\beta i}\colon U \to \boldsymbol{R}$, $\beta \in B$ and $i = 1, \ldots, n$, we obtain

$$d\beta | U = \sum_{i\leqslant n} \gamma_{\beta i} dx^i \quad \text{for } \beta \in B$$

and, therefore, we have successively

$$\begin{aligned}(\xi\circ j\circ \Phi|\boldsymbol{R}^B \times U)(c, p) &= \xi\Big(\sum_{\beta\in B} c_\beta d\beta_p\Big) = \xi\Big(\sum_{i\leqslant n}\Big(\sum_{\beta\in B} c_\beta \gamma_{\beta i}(p)\Big)\, dx^i_p\Big)\\ &= (\xi\circ\tilde{x}^{-1})\Big(x(p), \Big(\sum_{\beta\in B} c_\beta\gamma_{\beta i}(p)\Big)_{i\leqslant n}\Big).\end{aligned}$$

Thus the function $\xi\circ j\circ\Phi|\boldsymbol{R}^B \times \operatorname{set} M$ is smooth, as the open sets $\boldsymbol{R}^B \times \operatorname{Domain}(x)$, $x \in \operatorname{Atlas}(M)$, cover $\boldsymbol{R}^B \times \operatorname{set} M$ for an arbitrary finite subset B of $\mathscr{F}(M)$. Consequently, the function $\xi\circ j$ is smooth for any $\xi \in \mathscr{F}(T^*M)$, and so is j.

(ii) Smoothness of j^{-1}. Take an arbitrary function $\eta \in \mathscr{F}({}^*TM)$ and choose a chart $x \in (x^i)$ on some $U \subset \operatorname{set} M$ such that $x^i = a^i | U$ for some function $a^i \in \mathscr{F}(M)$, $i = 1, \ldots, n$. Such charts will be called *admissible*. We denote the set $\{a^1, \ldots, a^n\}$ by A.

Consider the diagram

$$\begin{array}{ccccc}\boldsymbol{R}^n \times U & \xrightarrow{a\times \mathrm{id}_U} & \boldsymbol{R}^A \times U \hookrightarrow & \boldsymbol{R}^{[\mathscr{F}(M)]} \times \operatorname{set} M \\ {\scriptstyle\psi}\Big\downarrow{\scriptstyle\cong} & & & \Big\downarrow{\scriptstyle\Phi} \\ \pi^{*-1}(U) \quad\subset & \operatorname{set} T^*M & \xrightarrow{j^{-1}} & \operatorname{set}{}^*TM\end{array}$$

where the diffeomorphisms ψ and a are defined by

$$\psi(c, p) = \tilde{x}^{-1}\big(x(p), c\big) \quad \text{and} \quad a(c)(a^i) = c_i \ (i = 1, \ldots, n),$$

respectively. Since the diagram commutes, we have

$$\eta\circ j^{-1}|\pi^{*-1}(U) = (\eta\circ\Phi|\boldsymbol{R}^A \times U)\circ(a\times\mathrm{id}_U)\circ\psi^{-1},$$

which proves the smoothness of $\eta\circ j^{-1}$, as the domains of the admissible charts cover M. Consequently, j^{-1} is also smooth.

3. Regular differential forms and smooth vector fields. We recall that the differential structure of the tangent differential space TM is generated by the set

$$\{a\circ\pi\colon\ a\in\mathscr{F}(M)\}\cup\{da\colon\ a\in\mathscr{F}(M)\} \tag{5}$$

(cf. [1]), where $\pi\colon TM\to M$ is the projection.

Remark 3.1. For $a\in\mathscr{F}(M)$ we shall use the symbol da in two different meanings corresponding to each other. The mapping $da\colon M\to {}^*TM$ (cf. Theorem 2.1) gives rise to a function

$$da\colon \operatorname{set}TM\to \boldsymbol{R}$$

which is defined by

$$da(v) = da_{\pi(v)}(v) = v\big([a-a(\pi(v)),\ \pi(v)]+G_0(M,\pi(v))^2\big)$$

for $v\in\operatorname{set}TM$. Note that $da\colon TM\to\boldsymbol{R}$ is linear on each T_pM, $p\in\operatorname{set}M$.

Definition 3.1. A *regular differential form* (*1-form*) on M is a function $\omega\in\mathscr{F}(TM)$ such that, for any $p\in\operatorname{set}M$, its restriction $\omega_p := \omega|T_pM$ is linear.

For more general definitions see [1]-[3].

Definition 3.2. A (*smooth*) *vector field* X on M is a smooth mapping $X\colon M\to TM$ such that $\pi\circ X = \mathrm{id}_M$.

This definition is a reformulation of the classical one.

For $p\in\operatorname{set}M$, we shall write X_p instead of $X(p)$.

THEOREM 3.1. *Any vector field X on M defines a function $\tilde{X}\in\mathscr{F}({}^*TM)$ that coincides with $X_p\circ i_p^{-1}$ on each ${}^*T_pM$, $p\in\operatorname{set}M$.*

Conversely, any element of $\mathscr{F}({}^*TM)$ which is linear on each ${}^*T_pM$, $p\in\operatorname{set}M$, is of the form $\tilde{X}$ for some vector field X on M.

Proof. If $X\colon M\to TM$ is a vector field and B is a finite subset of $\mathscr{F}(M)$, then $\tilde{X}\colon \operatorname{set}{}^*TM\to\boldsymbol{R}$ is defined uniquely and

$$(\tilde{X}\circ\Phi|\boldsymbol{R}^B\times\operatorname{set}M)(c,p) = (X\circ i_p^{-1})\Big(\sum_{\beta\in B}c_\beta d\beta_p\Big) = \sum_{\beta\in B}c_\beta(d\beta\circ X)(p),$$

which proves that $\tilde{X}\circ\Phi|\boldsymbol{R}^B\times\operatorname{set}M\in\mathscr{F}(\boldsymbol{R}^B\times M)$, as X is smooth. Thus $\tilde{X}\in\mathscr{F}({}^*TM)$.

Now, let $Z\in\mathscr{F}({}^*TM)$ be linear on each ${}^*T_pM$, $p\in\operatorname{set}M$. Put

$$Y_p = (Z|{}^*T_pM)\circ i_p \quad \text{for } p\in\operatorname{set}M$$

to obtain a mapping

$$Y = (\operatorname{set}M\ni p\mapsto Y_p\in\operatorname{set}TM)$$

such that $\pi \circ Y = \mathrm{id}_{\mathrm{set}M}$. In order to complete the proof it is now sufficient to show the smoothness of Y. This is evident since

$$(a\circ\pi)\circ Y = a \quad \text{and} \quad (da)\circ Y = (Z\circ\Phi|\boldsymbol{R}^{\{a\}}\times \mathrm{set}M)(1,\cdot)$$

for $a \in \mathscr{F}(M)$.

It is natural to ask whether a similar theorem holds for regular differential forms. The problem is that a regular differential form determines covectors in $\mathrm{set}T^*M$ which, generally, is larger than set^*TM. However, we shall prove that all the covectors determined by a regular differential form are in fact regular.

THEOREM 3.2. *Let $\omega \in \mathscr{F}(TM)$ be a regular differential form on M. Then any point p of M has a neighbourhood U such that*

$$\omega = \sum_{i\leqslant k}(\beta_i\circ\pi)\,da_i \quad \text{on } \pi^{-1}(U)$$

for some $a_i, \beta_i \in \mathscr{F}(M)$, $i = 1, \ldots, k$, where k is an appropriate integer.

Proof. Fix an arbitrary point p of M. For $q \in \mathrm{set}M$, denote the zero-vector of T_qM by o_q. Since ω belongs to the differential structure generated by the set (5), there exist a neighbourhood V of o_p in TM and some functions $a_1, \ldots, a_k, \gamma_1, \ldots, \gamma_l \in \mathscr{F}(M)$ and $\varphi \in C^\infty(\boldsymbol{R}^{k+l})$ (k, l are positive integers) such that

$$\omega = \varphi(da_1, \ldots, da_k, \gamma_1\circ\pi, \ldots, \gamma_l\circ\pi) \quad \text{on } V. \tag{6}$$

We may take V from an appropriate base of the topology of TM and assume that

$$V = \bigcap_{a\in A}(d\lambda_a)^{-1}(I_a)\cap\bigcap_{b\in B}(\varkappa_b\circ\pi)^{-1}(J_b),$$

where $\lambda_a, \varkappa_b \in \mathscr{F}(M)$ and I_a, J_b are open subsets of $\boldsymbol{R}$ for $a \in A$, $b \in B$, A and B being finite sets of indices. Since $o_p \in V$, all I_a, $a \in A$, are neighbourhoods of $0 \in \boldsymbol{R}$. We put

$$U = \bigcap_{b\in B}\varkappa_b^{-1}(J_b)$$

to have $o_q \in V$ for $q \in U$. We shall show that

$$\omega = \sum_{i\leqslant k}\varphi_{|i}(0, \ldots, 0, \gamma_1, \ldots, \gamma_l)\circ\pi\cdot da_i \quad \text{on } \pi^{-1}(U),$$

where $|i$ denotes the appropriate partial derivative.

Let $q \in U$. The linear subspace of ${}^*T_qM$ spanned by all da_{iq}, $i \leqslant k$, has a finite basis $\nu_1, \ldots, \nu_r$ and

$$da_{iq} = \sum_{h\leqslant r} c_i^h \nu_h$$

for some real c_i^h, $h \leqslant r$ and $i \leqslant k$.

Choose $v_1, \ldots, v_r \in T_q M$ such that $\nu_h(v_g) = \delta_{hg}$ for $h, g \leqslant r$ (δ_{hg} is the Kronecker symbol) and put

$$w(t) = \sum_{h\leqslant r} t_h v_h \quad \text{for } t = (t_h)_{h\leqslant r} \in \boldsymbol{R}^r.$$

We have

$$da_i(w(t)) = \sum_{h\leqslant r} c_i^h t_h \quad \text{for } i = 1, \ldots, k.$$

Since $d\lambda_a$, $a \in A$, are linear, there exists a real $\varepsilon \neq 0$ such that

$$\varepsilon v_h \in V \tag{7}$$

for all $h \leqslant r$.

We put

$$W = \Big(\boldsymbol{R}^r \ni t \mapsto \Big(\sum_{h\leqslant r} t_h d\lambda_a(v_h)\Big)_{a\in A} \in \boldsymbol{R}^A\Big)^{-1} \Big(\bigtimes_{a\in A} I_a\Big).$$

Then W is an open neighbourhood of $0 \in \boldsymbol{R}^r$ and

$$w(t) \in V \quad \text{for } t \in W. \tag{8}$$

By (6)-(8) and by linearity of ω_q we obtain

$$\omega(w(t)) = \varphi\Big(\sum_{h\leqslant r} c_1^h t_h, \ldots, \sum_{h\leqslant r} c_k^h t_h, \gamma(q)\Big)$$

and, on the other hand,

$$\omega(w(t)) = \varepsilon^{-1} \sum_{h\leqslant r} t_h \omega(\varepsilon v_h) = \varepsilon^{-1} \sum_{h\leqslant r} t_h \varphi(c_1^h \varepsilon, \ldots, c_k^h \varepsilon, \gamma(q))$$

for $t \in W$, where $\gamma(q) = (\gamma_1(q), \ldots, \gamma_l(q))$. Consequently, we have

$$\varphi\Big(\sum_{h\leqslant r} c_1^h t_h, \ldots, \sum_{h\leqslant r} c_k^h t_h, \gamma(q)\Big) = \varepsilon^{-1} \sum_{h\leqslant r} t_h \varphi(c_1^h \varepsilon, \ldots, c_k^h \varepsilon, \gamma(q)) \tag{9}$$

for $t \in W$. Differentiating identity (9) and putting $t = 0$ we get

$$\varepsilon^{-1}\varphi(c_1^h \varepsilon, \ldots, c_k^h \varepsilon, \gamma(q)) = \sum_{i\leqslant k} c_i^h \varphi_{|i}(0, \ldots, 0, \gamma(q)) \tag{10}$$

for $h = 1, \ldots, r$. Identities (9) and (10) give

$$\omega(w(t)) = \sum_{i\leqslant k} \varphi_{|i}(0, \ldots, 0, \gamma(q)) da_i(w(t)) \tag{11}$$

for $t \in W$.

Now, let v be any vector in T_qM. We have

$$da_i(v) = \sum_{h\leqslant r} c_i^h \nu_h(v) = da_i\big(w(t_v)\big) \tag{12}$$

for $i = 1, \dots, k$, $t_v = \big(\nu_h(v)\big)_{h\leqslant r}$. We choose a non-zero $\delta \in \boldsymbol{R}$ so that $\delta v \in V$ and $\delta t_v \in W$.

According to (11) and (12) we get

$$\begin{aligned}\omega(v) &= \delta^{-1}\omega(\delta v) = \delta^{-1}\varphi\big(da_1(\delta v), \dots, da_k(\delta v), \gamma(q)\big)\\ &= \delta^{-1}\varphi\big(da_1\big(w(\delta t_v)\big), \dots, da_k\big(w(\delta t_v)\big), \gamma(q)\big) = \delta^{-1}\omega\big(w(\delta t_v)\big)\\ &= \delta^{-1}\sum_{i\leqslant k}\varphi_{|i}\big(0, \dots, 0, \gamma(q)\big)da_i\big(w(\delta t_v)\big) = \sum_{i\leqslant k}\varphi_{|i}\big(0, \dots, 0, \gamma(q)\big)da_i(v).\end{aligned}$$

We have shown that, for any $v \in \pi^{-1}(U)$,

$$\omega(v) = \sum_{i\leqslant k}\varphi_{|i}\big(0, \dots, 0, (\gamma\circ\pi)(v)\big)da_i(v),$$

which completes the proof of the theorem.

COROLLARY 3.1. *A regular differential form $\omega \in \mathscr{F}(TM)$ determines a smooth mapping $\tilde\omega$: $M \to {}^*TM$ such that ${}^*\pi\circ\tilde\omega = \mathrm{id}_M$ and $\tilde\omega(p) = \omega_p$ for $p \in \mathrm{set}M$.*

Proof. As shown above, $\tilde\omega$ coincides locally with a smooth mapping of the form $\sum_{i\leqslant k}\beta_i da_i$, and so it is smooth (cf. Theorem 2.1 (b)).

It is interesting that some smooth mappings ν: $M \to {}^*TM$, with $\pi\circ\nu = \mathrm{id}_M$, cannot be obtained from any regular differential form on M.

Example 3.1. Put

$$\mathrm{set}M = \{(t^1, t^2) \in \boldsymbol{R}^2\colon t^1t^2 = 0\}, \qquad M = \big(\mathrm{set}M, C^\infty(\boldsymbol{R}^2)_{\mathrm{set}M}\big).$$

(We have localized the differential structure of $\boldsymbol{R}^2$ to $\mathrm{set}M$.)

It is easy to check that, when defining the differential structure of ${}^*TM$ (as in Definition 2.1), we may restrict ourselves to the subset of $\mathscr{F}(M)$ consisting only of all coordinate functions. This means that, for a function ξ: $\mathrm{set}M \to \boldsymbol{R}$, we have

$$\xi \in \mathscr{F}({}^*TM) \text{ if and only if } \xi\circ\varphi \in \mathscr{F}(\boldsymbol{R}^2\times M),$$

where φ: $\boldsymbol{R}^2\times M \to {}^*TM$ is defined by

$$\varphi(c_1, c_2, p) = c_1dx_p^1 + c_2dx_p^2,$$

and $x^i = \big(\mathrm{set}M \ni (t^1, t^2) \mapsto t^i \in \boldsymbol{R}\big)$, $i = 1, 2$. Moreover, $\xi\circ\varphi \in \mathscr{F}(\boldsymbol{R}^2\times M)$ if and only if there exists $\psi \in C^\infty(\boldsymbol{R}^2\times\boldsymbol{R}^2)$ such that

$$\xi\circ\varphi = \psi|\boldsymbol{R}^2\times\mathrm{set}M$$

(see [3]). Consequently, we have

$$\psi(c_1, c_2, t^1, 0) = \xi(c_1 dx^1_{(t^1,0)} + c_2 dx^2_{(t^1,0)}) = \xi(c_1 dx^1_{(t^1,0)}) = \psi(c_1, 0, t^1, 0)$$

for $t^1 \neq 0$, and

$$\psi(c_1, c_2, 0, t^2) = \xi(c_1 dx^1_{(0,t^2)} + c_2 dx^2_{(0,t^2)}) = \xi(c_2 dx^2_{(0,t^2)}) = \psi(0, c_2, 0, t^2)$$

for $t^2 \neq 0$, where c_1 and c_2 are real numbers.

By continuity of ψ we get

$$\psi(c_1, c_2, 0, 0) = \psi(0, 0, 0, 0) \quad \text{for all } c_1, c_2$$

and, therefore, ξ is constant on ${}^*T_{(0,0)}M$ if it is smooth. Hence the value of a smooth mapping $\nu\colon M \to {}^*TM$, with ${}^*\pi \circ \nu = \mathrm{id}_M$, at the point $(0, 0)$ of M is quite arbitrary — it is not determined by the values of ν in any neighbourhood of $(0, 0)$.

On the other hand, Theorem 3.2 states that a regular differential form ω on M determines the mapping $\tilde{\omega}\colon M \to {}^*TM$ which is, on a neighbourhood of $(0, 0)$, of the shape $\beta_1 dx^1 + \beta_2 dx^2$ for some $\beta_1, \beta_2 \in \mathscr{F}(M)$. Consequently, we have

$$\beta_1((0, 0)) = \lim_{t\to 0} \tilde{\omega}((t, 0))\left(\frac{\partial}{\partial x^1}\bigg|_{(t,0)}\right)$$

and

$$\beta_2((0, 0)) = \lim_{t\to 0} \tilde{\omega}((0, t))\left(\frac{\partial}{\partial x^2}\bigg|_{(0,t)}\right).$$

Thus $\tilde{\omega}((0, 0))$ is uniquely determined by other values of $\tilde{\omega}$.

For example, the mapping $\nu\colon \mathrm{set}M \to \mathrm{set}{}^*TM$ defined by

$$\nu_p = \begin{cases} dx^1_p & \text{for } p \neq (0, 0), \\ dx^2_p & \text{for } p = (0, 0) \end{cases}$$

satisfies the condition ${}^*\pi \circ \nu = \mathrm{id}_{\mathrm{set}M}$, determines a smooth mapping $\nu\colon M \to {}^*TM$, and cannot be obtained from any regular differential form on M.

Remark 3.2. For a given smooth mapping $\nu\colon M \to {}^*TM$ with ${}^*\pi \circ \nu = \mathrm{id}_M$ (even such as in the example above) and a smooth vector field X on M, the evaluation $\langle \nu, X\rangle\colon M \to \boldsymbol{R}$, $p \mapsto \nu_p(X_p)$, is smooth since it is equal to the composition $\tilde{X} \circ \nu$.

4. More about the differential structure of ${}^*TM$. Consider an arbitrary but fixed $\xi \in \mathscr{F}({}^*TM)$. Take any $p \in \mathrm{set}M$.

Each pair (ω, ν) of covectors in ${}^*T_pM$ determines a curve

$$\boldsymbol{R} \ni t \mapsto \omega + t\nu \in \mathrm{set}{}^*TM. \tag{13}$$

If $\omega = d\alpha_p$, $\nu = d\beta_p$ $(\alpha, \beta \in \mathscr{F}(M))$, then the curve admits a lift

$$\boldsymbol{R} \ni t \mapsto (1, t; p) \in \boldsymbol{R}^{\{\alpha,\beta\}} \times \operatorname{set} M \hookrightarrow \boldsymbol{R}^{[\mathscr{F}(M)]} \times \operatorname{set} M$$

(adding an appropriate constant, if necessary, we may assume that $\alpha \neq \beta$). Consequently, the curve (13) is smooth and we put

$$d\xi(\nu; \omega) := \frac{d}{dt}\bigg|_0 \xi(\omega + t\nu), \tag{14}$$

which is the value of $d\xi$ on the vector tangent to the curve at ω.

LEMMA 4.1. *The function* $d\xi(\cdot; \omega)$: ${}^*T_pM \to \boldsymbol{R}$ *is linear for any* $\omega \in {}^*T_pM$.

Proof. Fix an arbitrary ω in ${}^*T_pM$. The condition

$$d\xi(t\nu; \omega) = td\xi(\nu; \omega)$$

for $t \in \boldsymbol{R}$, $\nu \in {}^*T_pM$, is obvious. It is sufficient to prove only the additivity. Let $\nu^i = d\beta_p^i$ $(\beta^i \in \mathscr{F}(M))$, $i = 1, 2$, be any two covectors in ${}^*T_pM$. As before, $\omega = d\alpha_p$ for some $\alpha \in \mathscr{F}(M)$ and we may assume that α, β^1, β^2 are different from each other. We have

$$d\xi(\nu^1 + \nu^2; \omega) = \frac{d}{dt}\bigg|_0 \xi(\omega + t\nu^1 + t\nu^2) = \frac{d}{dt}\bigg|_0 \big(\xi \circ \Phi|(\boldsymbol{R}^{\{\alpha,\beta^1,\beta^2\}} \times \{p\})\big)(1, t, t, p)$$

$$= \frac{d}{dt}\bigg|_0 \big(\xi \circ \Phi|(\boldsymbol{R}^{\{\alpha,\beta^1,\beta^2\}} \times \{p\})\big)(1, t, 0, p) + \frac{d}{dt}\bigg|_0 \big(\xi \circ \Phi|(\boldsymbol{R}^{\{\alpha,\beta^1,\beta^2\}} \times \{p\})\big)(1, 0, t, p)$$

$$= \frac{d}{dt}\bigg|_0 \xi(\omega + t\nu^1) + \frac{d}{dt}\bigg|_0 \xi(\omega + t\nu^2) = d\xi(\nu^1; \omega) + d\xi(\nu^2; \omega).$$

The lemma implies that $d\xi(\cdot; \omega) \circ i_p$ is a vector.

Definition 4.1. A *tangent mapping* of $\xi \in \mathscr{F}({}^*TM)$,

$$T\xi\colon \operatorname{set}{}^*TM \to \operatorname{set} TM, \qquad \pi \circ T\xi = {}^*\pi,$$

is defined by

$$T\xi(\omega) = d\xi(\cdot; \omega) \circ i_{*\pi(\omega)}.$$

THEOREM 4.1. *The tangent mapping* $T\xi$ *defines a smooth mapping* $T\xi$: ${}^*TM \to TM$ *for any* $\xi \in \mathscr{F}({}^*TM)$.

Proof. Choose an arbitrary but fixed $\xi \in \mathscr{F}({}^*TM)$ and take any $\alpha \in \mathscr{F}(M)$.

(i) $(\alpha \circ \pi) \circ T\xi = \alpha \circ {}^*\pi$ and is smooth.

(ii) Let B be any finite subset of $\mathscr{F}(M)$. Attaching α, if necessary, we may assume that $\alpha \in B$. We show the smoothness of $d\alpha \circ T\xi \circ \Phi|(\boldsymbol{R}^B \times \operatorname{set} M)$.

We have

$$\big(da\circ T\xi\circ\Phi|(\boldsymbol{R}^B\times\operatorname{set}M)\big)(c,p) = \frac{d}{dt}\bigg|_0 \xi\Big(\sum_{\beta\in B} c_\beta d\beta_p + t da_p\Big)$$

$$= \frac{d}{dt}\bigg|_0 \big(\xi\circ\Phi|(\boldsymbol{R}^B\times\operatorname{set}M)\big)\big((c_\beta+t\delta_{\beta\alpha})_{\beta\in B}, p\big) = \big(\xi\circ\Phi|(\boldsymbol{R}^B\times\operatorname{set}M)\big)_{|\alpha}(c,p),$$

where $\delta_{\beta\alpha}$ is the Kronecker symbol. Thus the function

$$da\circ T\xi\circ\Phi|(\boldsymbol{R}^B\times\operatorname{set}M) = \big(\xi\circ\Phi|(\boldsymbol{R}^B\times\operatorname{set}M)\big)_{|\alpha}$$

is smooth and $da\circ T\xi\in\mathscr{F}({}^*TM)$.

According to (i) and (ii) we conclude that $T\xi$ is smooth.

COROLLARY 4.1. *For any point p of M, a vector $v\in T_pM$ gives rise to a smooth function $v\circ i_p^{-1}$ on $\big({}^*T_pM, \mathscr{F}({}^*TM)_{*T_pM}\big)$ if and only if there is a smooth vector field X on M such that $X_p = v$.*

Proof. The "if" part is obvious by Theorem 3.1.

The "only if" part. Fix an arbitrary point p of M and assume that $v\circ i_p^{-1}\in\mathscr{F}({}^*TM)_{*T_pM}$ for some $v\in T_pM$. Then there exists a smooth function $\xi\in\mathscr{F}({}^*TM)$ that coincides with $v\circ i_p^{-1}$ on a neighbourhood V of $o_p\in{}^*T_pM$, $o\colon M\to{}^*TM$ being the zero-mapping. We claim that $v = T\xi(o_p)$.

For $\nu\in{}^*T_pM$, the curve $\boldsymbol{R}\ni t\mapsto t\nu\in{}^*T_pM$ is smooth and, therefore, its values are in V for $|t|$ sufficiently small. Thus we get

$$T\xi(o_p)(i_p^{-1}\nu) = d\xi(\nu; o_p) = \frac{d}{dt}\bigg|_0 \xi(t\nu) = \frac{d}{dt}\bigg|_0 (v\circ i_p^{-1})(t\nu) = v(i_p^{-1}\nu)$$

for $\nu\in{}^*T_pM$. Consequently, the vector v extends to the smooth vector field $T\xi\circ o$.

THEOREM 4.2. *For any point p of M, if $\dim{}^*T_pM<\infty$ (equivalently, $\dim T_pM<\infty$), then the differential structure $\mathscr{F}({}^*TM)$ coincides on ${}^*T_pM$ with the differential structure generated by all smooth vector fields on M. In other words, $\mathscr{F}({}^*TM)_{*T_pM}$ is generated by the set*

$$\{\tilde{X}|{}^*T_pM\colon X \text{ is a smooth vector field on } M\}.$$

Proof. Assume that $\dim T_pM = n<\infty$ for some $p\in\operatorname{set}M$. Let m denote the dimension of the subspace of all vectors in T_pM which extend to smooth vector fields. Take vector fields $X_1,\dots,X_m$ on M so that the vectors X_{hp}, $h\leqslant m$, are independent. Now, choose functions $a^1,\dots,a^n$ in $\mathscr{F}(M)$ such that $(da_p^i)_{i\leqslant n}$ is a basis in ${}^*T_pM$ and $da_p^i(X_{hp}) = \delta_h^i$ for $i\leqslant n$, $h\leqslant m$.

We denote the mapping

$$\boldsymbol{R}^n \ni c \mapsto \sum_{i\leqslant n} c_i da_p^i \in {}^*T_pM$$

by Ψ. It corresponds (in some sense) to $\Phi|\boldsymbol{R}^{\{a_1,\ldots,a_n\}} \times \{p\}$ and, therefore, is smooth.

Let us consider an arbitrary function $\beta \in \mathscr{F}({}^*TM)_{*T_pM}$. We have $\beta\circ\Psi \in C^\infty(\boldsymbol{R}^n)$. Moreover, for any $\omega \in {}^*T_pM$ there is a function $\xi \in \mathscr{F}({}^*TM)$ which coincides with β on a neighbourhood of ω in ${}^*T_pM$. All the values of $T\xi$ on ${}^*T_pM$ belong to the linear subspace of T_pM spanned by X_{hp}, $h = 1, \ldots, m$ (cf. Theorem 4.1). Thus the covectors da_p^i, $m+1 \leqslant i \leqslant n$, vanish on all those subspaces. Consequently, since the formula

$$da_p^i \circ T\xi \circ \Psi = (\xi\circ\Psi)_{|i} = (\beta\circ\Psi)_{|i} \quad \text{for } i \leqslant n$$

holds in some neighbourhood V of $\Psi^{-1}(\omega) \in \boldsymbol{R}^n$ (see the proof of Theorem 4.1), we get

$$(\beta\circ\Psi)_{|m+1} = \ldots = (\beta\circ\Psi)_{|n} = 0 \tag{15}$$

on V. Since ω is arbitrary, equalities (15) hold everywhere on $\boldsymbol{R}^n$ and, therefore,

$$(\beta\circ\Psi)(c) = (\beta\circ\Psi)(c_1, \ldots, c_m, 0, \ldots, 0) \quad \text{for all } c \in \boldsymbol{R}^n.$$

Thus, we have

$$\beta(\omega) = (\beta\circ\Psi)\big(\tilde{X}_1(\omega), \ldots, \tilde{X}_m(\omega), 0, \ldots, 0\big) \quad \text{for } \omega \in {}^*T_pM.$$

This completes the proof.

Remark 4.1. The set of all $\tilde{X}$, with X being a smooth vector field on M, and of all $a\circ\pi^*$, $a \in \mathscr{F}(M)$, generates a differential structure $\mathscr{F}(T^*M)$ on

$$\operatorname{set} T^*M = \bigcup_p T_p^*M$$

(cf. [1]). As $\operatorname{set}{}^*TM \subset \operatorname{set} T^*M$, one can ask whether (or when) ${}^*TM$ is a differential subspace of T^*M. Do these two differential structures coincide in the case of $\mathscr{D}_0$-spaces (cf. [3])? (**P 1261**) The author does not know the answers.

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