

*ON COMPLETENESS
WITH RESPECT TO A CARATHÉODORY-LIKE METRIC*

BY

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1. Introduction. In [3], completeness with respect to the Carathéodory metric is discussed. However, in the case of the unit disc, for example, where the Carathéodory metric is known explicitly, we obtain the same metric if

$$(*) \quad d_1(x, y) = \sup_f \frac{1}{2} \log \frac{1 + [f(x), f(y)]}{1 - [f(x), f(y)]},$$

where f is an analytic function which can be approximated uniformly by rational functions whose poles lie off the closed unit disc and which map to the unit disc; here,

$$[a, b] = \frac{|a - b|}{|1 - \bar{b}a|}.$$

In this paper* we consider a metric on X defined in terms of the functions which are uniformly approximable by rational functions whose poles lie off the closure of X . We determine necessary and sufficient conditions for completeness of this metric and we investigate how completeness with respect to the Carathéodory metric is related to completeness with respect to the new metric.

2. Definitions and preliminaries. Let X be a bounded region in the plane with $\text{int } \bar{X} = X$. Let $R(\bar{X})$ denote the Banach space of analytic functions which can be approximated uniformly by rational functions whose poles lie off \bar{X} . For $f \in R(\bar{X})$, $\|f\| \leq 1$, where $\|f\| = \text{Max } |f(x)|$ with $x \in \bar{X}$, we define d_1 by (*). It is easily shown that d_1 is a metric which yields the same relative topology on X as the ordinary metric.

Endowed with the usual pointwise operations, $R(\bar{X})$ is a Banach algebra with unit. If M is the maximal ideal space of non-trivial complex homomorphisms of $R(\bar{X})$, then $R(\bar{X})$ is isometrically isomorphic to a closed subalgebra of $C(M)$, the complex-valued continuous functions of M .

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In this case, it is easy to see that the point evaluations are all the homomorphisms. Hence, M can be identified with \bar{X} .

Gleason pointed out (see [1], p. 128-129) that, for $\varphi_1, \varphi_2 \in M$, $\varphi_1 \sim \varphi_2$ if and only if $\|\varphi_1 - \varphi_2\| < 2$ is an equivalence relation on M , where

$$\|\varphi\| = \sup_f |\varphi(f)|, \quad f \in R(\bar{X}), \|\varphi\| \leq 1.$$

Each equivalence class is called a *Gleason part*. If for $\lambda \in \bar{X}$ we set

$$X_\lambda = \{\varphi \in M, \varphi(x) = \lambda\},$$

then X_λ is called a *fibre*. In this case the fibre contains exactly one element. If there exists $f \in R(\bar{X})$ such that $\varphi(f) = 1$ for all $\varphi \in X_\lambda$ while $|\varphi(f)| < 1$, $\varphi \in X_\lambda$, then X_λ is called a *peak fibre* (point). We denote by X^* the union of all the fibres corresponding to interior points of X . By $\varphi_x \leftrightarrow x$, X^* can be identified with X .

We write

$$\varrho(\varphi_1, \varphi_2) = \sup_f \left\{ \frac{|\varphi_1(f)|}{\|\varphi_1\|} \leq 1, \varphi_2(f) = 0, f \in R(\bar{X}) \right\}.$$

ϱ is a metric on M with $\varrho(\varphi_1, \varphi_2) < 1$ if and only if $\|\varphi_1 - \varphi_2\| < 2$ (see [1], p. 128-129). We will need the following result:

THEOREM 1. *Suppose that $\{x_n\}$ is a sequence of points in X and let $\{\varphi_{x_n}\}$ be the associated homomorphisms. Then $\{x_n\}$ is Cauchy with respect to d_1 if and only if $\{\varphi_{x_n}\}$ is Cauchy with respect to ϱ .*

Proof. Suppose that $\{\varphi_{x_n}\}$ is Cauchy with respect to ϱ . Let $\varepsilon > 0$ be arbitrary and let

$$\delta = \frac{e^{2\varepsilon} - 1}{e^{2\varepsilon} + 1}.$$

Since $\{\varphi_{x_n}\}$ is Cauchy, there exists N_0 such that $n, m \geq N_0$ imply $\varrho(\varphi_{x_n}, \varphi_{x_m}) < \delta$. Hence for any $n, m \geq N_0$ we have

$$\sup_f |f(x_n)| < \delta \quad \text{for all } f, \|f\| \leq 1, f(x_m) = 0, \text{ and } f \in R(\bar{X}).$$

Let $f \in R(\bar{X})$; then $[f(x_n), f(x_m)] \in R(\bar{X})$ and

$$\frac{1 + [f(x_n), f(x_m)]}{1 - [f(x_n), f(x_m)]} < \frac{1 + \delta}{1 - \delta}.$$

Therefore,

$$\frac{1}{2} \log \frac{1 + [f(x_n), f(x_m)]}{1 - [f(x_n), f(x_m)]} < \frac{1}{2} \log \frac{1 + \delta}{1 - \delta} < \varepsilon.$$

This implies $d_1(x_n, x_m) \leq \varepsilon$ for $n, m \geq N_0$. Hence $\{x_n\}$ is Cauchy.

Conversely, suppose that $\{x_n\}$ is Cauchy with respect to d_1 . Let $\varepsilon > 0$ be arbitrary. Then for $n, m \geq N_1$ we have

$$d(x_n, x_m) < \frac{1}{2} \log \frac{1 + \varepsilon}{1 - \varepsilon}.$$

Hence $[f(x_n), f(x_m)] < \varepsilon$. Therefore

$$\sup_f |\varphi_{x_n}(f)| \leq \varepsilon, \quad f \in R(\bar{X}), \|f\| \leq 1, \text{ and } f(x_m) = 0,$$

i.e., $\varrho(\varphi_{x_n}, \varphi_{x_m}) \leq \varepsilon$.

The following theorem is easy to prove.

THEOREM 2. *Let $\{x_n\}$ be a sequence in X and assume that $\lambda \in \bar{X}$ is such that $x_n \rightarrow \lambda$. Then $\|\varphi_{x_n} - \varphi_\lambda\| \rightarrow 0$ if and only if $\varrho(\varphi_{x_n}, \varphi_\lambda) \rightarrow 0$.*

3. Main result. We will show that the main result of [3] is valid with a slight technical addition. We assume that X is a bounded region in C with $\text{int } \bar{X} = X$ and

$$\lim_{n \rightarrow \infty} \frac{m(\Delta_n(\lambda) \cap \partial X)}{m(\Delta_n)} = \alpha, \quad \alpha < 1,$$

where m denotes two-dimensional Lebesgue measure, ∂X is the boundary of X , and

$$\Delta_n(\lambda) = \left\{ y \in C \mid |y - \lambda| < \frac{1}{n} \right\},$$

λ being a non-peak point.

THEOREM 3. *Let X be as above. The following propositions are equivalent:*

- (a) X is complete with respect to d_1 .
- (b) X^* is an entire Gleason part.
- (c) X_λ is a peak point for all boundary points λ .
- (d) The d_1 -closed and bounded sets are compact.
- (e) X^* is closed with respect to $\|\cdot\|$ as a subset of the maximal ideal space.

We begin by showing (b) \Rightarrow (a). It is not difficult to show that X^* is always contained in a Gleason part (see [1], p. 130). Let $\{x_n\}$ be a Cauchy sequence in X with respect to d_1 . Then, by Theorem 1, $\{\varphi_{x_n}\}$ is Cauchy with respect to ϱ and hence with respect to $\|\cdot\|$. However, the complex homomorphisms are complete with respect to this norm, i.e., there exists $\varphi \in M$ such that $\|\varphi_{x_n} - \varphi\| \rightarrow 0$. However, X^* is the entire Gleason part by assumption. Therefore, $\varphi \in X^*$, i.e., there exists $x_0 \in X$ such that $\varphi = \varphi_{x_0}$. Therefore, $\|\varphi_{x_n} - \varphi_{x_0}\| \rightarrow 0$. This implies $|x_n - x_0| \rightarrow 0$. Since d_1 and the ordinary metric yield the same topology, $d_1(x_n, x_0) \rightarrow 0$ and d_1 is complete.

We next show (c) \Rightarrow (b). Suppose that $\varphi \notin X^*$. Then $\varphi(z) = \lambda$, where λ is a boundary point. By (c), there exists $f \in R(\bar{X})$, $\|f\| \leq 1$, $f(\lambda) = 1$ while $|f(x)| < 1$ for all $x \in \bar{X}$, $x \neq \lambda$. Form $g = [f, f(x_0)]$, $x_0 \in X$. Now

$$\varrho(\varphi_\lambda, \varphi_{x_0}) = 1 \quad \text{and} \quad \varphi_\lambda \not\sim \varphi_{x_0}.$$

In order to show the difficult part of the theorem we need a few definitions used in rational approximation.

Let A be a compact subset of the plane, and μ a complex Borel measure on A . We write

$$\tilde{\mu}(x) = \int_A \frac{d|\mu|(z)}{|z-x|}.$$

Using Fubini's theorem, we can show that $\tilde{\mu}(x)$ is locally an L^1 -function. Hence $\tilde{\mu}(x) < \infty$ a.e. We say that a complex measure ν represents a homomorphism φ if

$$\varphi(f) = \int_{\bar{X}} f d\nu \quad \text{for all } f \in R(\bar{X}).$$

The following theorems are due to Browder (see [1], p. 176).

THEOREM A. *Let $x \in \bar{X}$ and let μ be a measure which represents φ_x . Assume that*

$$\varepsilon > 0 \quad \text{and} \quad \delta = \frac{\varepsilon}{\|\mu\| + 1 + \varepsilon}.$$

Then $\|\varphi_x - \varphi_y\| < \varepsilon$ whenever $|y - x| \tilde{\mu}(y) < \delta$.

THEOREM B. *Let μ be a compactly supported measure in C . Let $x \in C$ and let Δ_n be the disc centred at x of radius $1/n$. Then*

$$|\mu\{x\}| = \lim_{n \rightarrow \infty} \frac{n^2}{\pi} \int_{\Delta_n} |x - z| \tilde{\mu} dm,$$

where m denotes two-dimensional Lebesgue measure (see [1], p. 157).

We are finally ready to show (a) \Rightarrow (c). Suppose that λ is a boundary point which is not a peak point. Then there exists a representing measure μ with $\mu(\{\lambda\}) = 0$ (see [5], p. 5). Let

$$P_\varepsilon = \{y \in \bar{X} \mid \|\varphi_\lambda - \varphi_y\| < \varepsilon\} \quad \text{and} \quad \Delta_n(\lambda) = \left\{ y \in C \mid |y - \lambda| < \frac{1}{n} \right\}.$$

Let

$$\delta = \frac{\varepsilon}{1 + \|\mu\| + \varepsilon}, \quad \varepsilon > 0.$$

By Theorem A,

$$P_\varepsilon \supset \{y \in C \mid |y - \lambda| \tilde{\mu}(y) < \delta\},$$

$$P_\varepsilon \cap \Delta_n \supset \{y \in \Delta_n \mid |y - \lambda| \tilde{\mu}(y) < \delta\}.$$

Therefore

$$m(P_\varepsilon \cap \Delta_n) \geq m(\{y \in \Delta_n \mid |y - \lambda| \tilde{\mu}(y) < \delta\}) \geq m(\Delta_n) - \frac{1}{\delta} \int_{\Delta_n} |\lambda - y| \tilde{\mu} dm(y).$$

Hence

$$\frac{m(P_\varepsilon \cap \Delta_n)}{m(\Delta_n)} \geq 1 - \frac{n^2}{\pi \delta} \int_{\Delta_n} |\lambda - y| \tilde{\mu} dm(y).$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{m(P_\varepsilon \cap \Delta_n)}{m(\Delta_n)} = 1.$$

Hence

$$\frac{m(P_\varepsilon \cap \partial X \cap \Delta_n)}{m(\Delta_n)} + \frac{m(P_\varepsilon \cap X \cap \Delta_n)}{m(\Delta_n)} \rightarrow 1.$$

By assumption,

$$\lim_{n \rightarrow \infty} \frac{m(P_\varepsilon \cap \partial X \cap \Delta_n)}{m(\Delta_n)} = a < 1.$$

Therefore, there exists a subsequence $\{n_k\}$ such that

$$\frac{m(P_\varepsilon \cap X \cap \Delta_{n_k})}{m(\Delta_{n_k})} \rightarrow \beta, \quad \beta > 0.$$

By relabelling, we call the subsequence $\{n\}$, i.e.

$$\frac{m(P_\varepsilon \cap X \cap \Delta_n)}{m(\Delta_n)} \rightarrow \beta > 0.$$

Consider $\varepsilon_l = 1/l$. Therefore, for each $l > 0$ there exists

$$x_{j(l)} \in \Delta_{j(l)} \cap X \cap P_{1/l}, \quad \text{i.e.,} \quad \|\varphi_\lambda - \varphi_{x_{j(l)}}\| < \frac{1}{l}.$$

Hence $\varrho(\varphi_{x_{j(l)}}, \varphi_\lambda) \rightarrow 0$. Therefore $\{\varphi_{x_{j(l)}}\}$ is Cauchy with respect to ϱ . Consequently, $\{x_{j(l)}\}$ is Cauchy with respect to d_1 . If there exists $x_0 \in X$ with $d(x_{j(l)}, x_0) \rightarrow 0$, then $|x_{j(l)} - x_0| \rightarrow 0$. However, $|x_{j(l)} - \lambda| \rightarrow 0$, and hence $x_0 = \lambda$, a contradiction. Therefore, X is not complete with respect to d_1 .

We have shown that (a), (b), and (c) are equivalent. Clearly, (d) \Rightarrow (a). Let A be a d_1 -closed and bounded set of X . Then A is relatively closed. We need only to show that A is compactly contained in X to conclude that A is compact since X is bounded.

Suppose that A is not compactly contained in X . Then there exists a sequence $\{x_n\} \subset A$ with $x_n \rightarrow \lambda$, where λ is a boundary point. By assumption, there exists K such that $d(x_n, x_m) \leq K$ for all n and m . Since X is

complete, λ is a peak point. Therefore, we can find $f \in R(\bar{X})$ with $f(\lambda) = 1$, $\|f\| \leq 1$, and $|f(x)| < 1$ for all $x \in \bar{X}$, $x \neq \lambda$. By composing with a Möbius transformation, we may assume that $f(x_1) = 0$, $\|f\| \leq 1$, $|f(x)| < 1$, $x \in X$, and $|f(\lambda)| = 1$. Since f is continuous at λ , $|f(x_n)| \rightarrow 1$. We have

$$\log \frac{1 + |f(x_n)|}{1 - |f(x_n)|} \leq 2d(x_1, x_n) \leq 2K.$$

Let $n \rightarrow \infty$ and we have a contradiction. Hence (a) \Rightarrow (d).

Finally, suppose that X is complete and $\|\varphi_{x_n} - \varphi\| \rightarrow 0$. Since (a)-(d) are equivalent, X^* is an entire Gleason part. Therefore, $\varphi_{x_n} \sim \varphi$ since $\|\varphi_{x_n} - \varphi\| < 2$ for n large enough. We conclude that $\varphi \in X^*$, i.e., $\varphi = \varphi_{x_0}$ for some x_0 .

Conversely, if X^* is closed and there is a boundary point λ which is not a peak point, then, as before, there exists $\{x_n\} \subset X$ with $x_n \rightarrow \lambda$ and $\|\varphi_{x_n} - \varphi_\lambda\| \rightarrow 0$. Since X^* is closed, $\varphi_\lambda = \varphi_{x_0}$, $x_0 \in X$. Hence, $\varphi_\lambda(z) = \varphi_{x_0}(z)$ or $\lambda = x_0$, a contradiction.

4. Examples. Recall the Carathéodory metric:

$$d(x, y) = \sup_f \frac{1}{2} \log \frac{1 + [f(x), f(y)]}{1 - [f(x), f(y)]}, \quad \|f\| \leq 1, f \in H^\infty(X).$$

Clearly, every region X complete with respect to d_1 is complete with respect to the Carathéodory metric. However, we will give an example of a bounded region X with $\text{int } \bar{X} = X$, and $m(\partial X) = 0$ such that it is complete with respect to d but not d_1 .

There is a criterion to determine when a boundary point λ is a peak point. This is due to Melnikov.

THEOREM 4 (see [5]). λ is a peak point for $R(\bar{X})$ if and only if

$$\sum_{n=1}^{\infty} 2^n \gamma(E_n(\lambda) - \bar{X}) = \infty,$$

where γ denotes the analytic capacity and

$$E_n(\lambda) = \{z \mid 2^{-n-1} < |z - \lambda| < 2^{-n}\}.$$

From this we easily see that every X with the boundary points linearly accessible from the exterior are peak points. In [4] it is shown that if $\text{int } \bar{X} = X$, then the peak points are dense in the boundary.

Consider the following regions:

$$Y = \Delta(0; 1), \quad A = \bigcup_{n=1}^{\infty} \bar{\Delta}(x_n; r_n),$$

where

$$\begin{aligned} \Delta(x_n; r_n) &= \{x \mid |x - x_n| \leq r_n\}, \quad 1 > x_1 > x_2 > \dots > x_n \rightarrow 0, \\ x_1 + r_1 &< 1, \quad x_{n+1} + r_{n+1} < x_n - r_n. \end{aligned}$$

For $X = Y - (A \cup \{0\})$, $\text{int } \bar{X} = X$ and X is complete with respect to the Carathéodory metric if and only if

$$(**) \quad \sum_{n=1}^{\infty} \frac{r_n}{x_n} = \infty$$

(see [4]). It can also be shown that $\{0\}$ is a peak point for $R(\bar{X})$ if and only if $(**)$ holds. Therefore, for these regions at least the Carathéodory metric is complete if and only if it is complete with respect to d_1 .

We now construct a region with the properties stated at the beginning of this section, that is, complete with respect to d but not d_1 . In [3], X is complete with respect to d if and only if X_λ is a peak fibre for every boundary point λ . We know (see [2]) that X_λ is a peak fibre if and only if

$$\sum_{n=1}^{\infty} 2^n \gamma(E_n(\lambda) - X) = \infty.$$

Using this we see that $\Delta(0, 1) - [0, \frac{3}{4}]$ is complete with respect to the Carathéodory metric. However, this region does not satisfy $\text{int } \bar{X} = X$. Consider

$$Y = \Delta(0; 1), \quad E_n = \{z \mid 2^{-n-1} \leq |z| \leq 2^{-n}\}.$$

From E_n delete closed discs $\Delta_{n,k}$ satisfying the following:

- (1) $\Delta_{n,k} \cap \Delta_{n,j} = \emptyset, k \neq j$.
- (2) $\sum_{k=1}^{\infty} r_{n,k} < 10^{-n}$, where $r_{n,k}$ is the radius of $\Delta_{n,k}$.
- (3) The closed discs accumulate exactly to $I \cap E_n$, where $I = [0, \frac{3}{4}]$.

An accumulation as in (3) is possible, for we only have to take the centres of these discs to have rational coordinates converging exactly to $I \cap E_n$. Let

$$X = Y - \bigcup_{n,k} \Delta_{n,k} - I,$$

where X satisfies $\text{int } \bar{X} = X$ and $m(\partial X) = 0$. Clearly, every boundary point is a peak fibre of $H^\infty(X)$. Hence, it is complete with respect to the Carathéodory metric.

To show that X is not complete with respect to d_1 , we prove that $\{0\}$ is not a peak point for $R(\bar{X})$. First,

$$\gamma(\bar{\bar{X}} \cap E_n) \leq \gamma\left(\bigcup_{n,k} \Delta_{n,k}\right),$$

where $\bar{\bar{X}}$ denotes the complement of \bar{X} .

We need the following theorem (see [5]):

THEOREM 5. *Let K be a compact set and let σ be a rectifiable contour which has the winding number 1 around each point of K . Then*

$$\gamma(K) \leq \frac{1}{2\pi} \text{length}(\sigma).$$

To estimate $\gamma(\bigcup_k \Delta_{n,k})$ we set

$$\gamma(\bigcup_j \Delta_{n,j}) = \sup_{K \subset \bigcup_j \Delta_{n,j}} \gamma(K),$$

where K is compact. Since finitely many of the $\Delta_{n,k}$ cover K , we have

$$\gamma(K) \leq \frac{1}{2\pi} \sum_{j=1}^{\infty} 2\pi r_{n,j} < \frac{1}{10^n}.$$

Therefore

$$\gamma(\bigcup_j \Delta_{n,j}) \leq \frac{1}{10^n} \quad \text{and} \quad \gamma(\tilde{X} \cap E_n) \leq \frac{1}{10^n}.$$

Consequently,

$$\sum_{n=1}^{\infty} 2^n \gamma(\tilde{X} \cap E_n) \leq \frac{1}{4}$$

and $\{0\}$ is not a peak point of $R(\bar{X})$. Therefore, d_1 is not complete.

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