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ON THIN-TALL SCATTERED SPACES

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Throughout this article*, the usual set-theoretic conventions will be used; undefined terms may be found in [2] or [4]. The well-known Cantor-Bendixson process is as follows. If X is a topological space, define $I_\beta(X)$ to be the set of all isolated points of the subspace $X-\{I_\alpha(X): \alpha<\beta\}$. The *Cantor-Bendixson derivative* is the set $X-\bigcup\{I_\alpha(X): \alpha<\beta\}$, where β is the least ordinal such that $I_\beta(X)=\emptyset$.

A space X is called *scattered* (or *dispersed*) if every closed subspace of X has an isolated point. It is easy to see that X is scattered iff the Cantor-Bendixson derivative of X is the empty set.

It is natural to define an ordinal function, the *Cantor-Bendixson height*, on the class of scattered spaces, as follows:

$$\mathrm{ht}(X)=\sup\{\alpha+1: I_\alpha(X)\neq\emptyset\}.$$

We can also define a cardinal function, the *Cantor-Bendixon width*, on the class of scattered spaces, as follows:

$$\mathrm{wd}(X)=\sup\{|I_\alpha(X)|: \alpha<\mathrm{ht}(X)\}.$$

The following problem was first posed by R. Telgársky in 1968. Although never appearing in print, it has nevertheless been widely known and was of interest to a number of mathematicians. We reformulate it in terms of the definitions above:

Does there exist a locally compact, T_2, scattered space X with $\mathrm{wd}(X)=\omega$ and $\mathrm{ht}(X)=\omega_1$?

The problem has recently seen four positive solutions. One is by Ostaszewski [8] using $\diamondsuit$, which is countably compact and perfectly normal. Another, by Juhász et al. [5], uses CH and is perfectly normal. The third, by Rajagopalan, oral communication, uses CH and is countably compact.

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Furthermore, Rajagopalan [9] has given a positive solution using no extra set-theoretic axioms. However, these constructions of Rajagopalan are fairly long and complicated. We propose a short construction, without using extra set-theoretic hypotheses, which gives a positive solution to Telgársky's problem.

The construction resembles those in [8] and [5]. In fact, it is basically the same, without the use of and fruits of extra set-theoretic axioms.

THEOREM 1. *There exists a locally compact, T_2, scattered space X such that* $\mathrm{wd}(X) = \omega$ *and* $\mathrm{ht}(X) = \omega_1$.

Construction. For this construction we need the following definition: if α is an ordinal, we let $\hat{\alpha}$ be the maximum limit ordinal not greater than α. The basic underlying set will be $X = \{x_\alpha \colon \alpha < \omega_1\}$, where $x_\alpha \neq x_\beta$ for $\alpha < \beta < \omega_1$. We construct a topology τ on X by recursively constructing topologies τ_β on $X_\beta = \{x_\alpha \colon \alpha < \beta\}$, such that

(i) $\langle X_\beta, \tau_\beta\rangle$ is locally compact and T_2;

(ii) for all $\alpha < \beta$, $P(X_\alpha) \cap \tau_\beta = \tau_\alpha$;

(iii) if $\alpha < \beta$ and U is an open neighbourhood of x_α in $\langle X_\beta, \tau_\beta\rangle$, then $\{\gamma < \hat{\alpha} \colon x_\gamma \in U\}$ is cofinal in $\hat{\alpha}$;

(iv) there exists a set B, cofinal in $\hat{\beta}$, such that $\{x_\alpha \colon \alpha \in B\}$ is a closed discrete subset of $\langle X_\beta, \tau_\beta\rangle$.

If β is a limit ordinal, including $\beta = \omega_1$, define τ_β to be the topology on X_β generated by $\bigcup\{\tau_\alpha \colon \alpha < \beta\}$.

Suppose that $\langle X_\beta, \tau_\beta\rangle$ has already been defined. By (iv) we have a set B, cofinal in $\hat{\beta}$, such that $\{x_\alpha \colon \alpha \in B\}$ is closed and discrete in $\langle X_\beta, \tau_\beta\rangle$. Let us decompose B into two disjoint sets $\{\alpha_n \colon n < \omega\}$ and B', both cofinal in $\hat{\beta}$. For each n, let U_n be a compact open neighbourhood of x_{α_n} such that $\{U_n \colon n < \omega\}$ is discrete and

$$\bigcup\{U_n \colon n < \omega\} \cap B' = \emptyset.$$

Write

$$V_m = \{x_\beta\} \cup \bigcup\{U_n \colon n \geqslant m\}$$

and let $\tau_{\beta+1}$ be the topology on $X_{\beta+1}$ generated by $\tau_\beta \cup \{V_m \colon m < \omega\}$.

Clearly, if we put $\tau = \tau_{\omega_1}$, then $\langle X, \tau\rangle$ is locally compact and T_2. It is also straightforward to show that for every $\alpha < \omega_1$ we have

$$I_\alpha(X) = \{x_\beta \colon \omega\cdot\alpha \leqslant \beta < \omega\cdot(\alpha+1)\}.$$

Thus X is scattered, and $\mathrm{wd}(X) = \omega$ and $\mathrm{ht}(X) = \omega_1$.

COROLLARY. *There exists a compact, T_2, scattered space Y such that* $\mathrm{wd}(Y) = \omega$ *and* $\mathrm{ht}(Y) = \omega_1 + 1$.

For the construction, let Y be the one-point compactification of the space X of Theorem 1.

In order to obtain spaces which answer Telgársky's problem and have also additional properties, we can do the same basic construction and add some additional inductive hypothesis. We illustrate this with two examples.

THEOREM 2. *There exists a space X which is locally compact, submetrizable and scattered so that* $\mathrm{ht}(X) = \omega_1$, $\mathrm{wd}(X) = \omega$ *and* $s(X) = \omega_1$. (*Thus the width of X is strictly less than the spread s of X.*)

Construction. The construction is similar to that for Theorem 1 with the exception that we take $X = \{x_\alpha: \alpha < \omega_1\}$ to be a subspace of the Sorgenfrey square S, described in [10], p. 103. We also require that X has uncountable spread and that, for each limit ordinal γ, the subset $\{x_\alpha: \gamma \leqslant \alpha < \gamma+\omega\}$ is dense in S. We use the inductive hypotheses (i)-(iv) of Theorem 1 and add the following:

(v) τ_β is finer than the subspace Sorgenfrey topology on X_β.

The initial step and the limit ordinal step are the same as in Theorem 1. However, at successor steps, we first choose a countable descending local base $\{B_n: n < \omega\}$ for x_β in the Sorgenfrey topology. We then pick $\{x_{\alpha_n}: n < \omega\}$ having the properties as in Theorem 1 and such that, for each $n < \omega$, $x_{\alpha_n} \in B_n$. We also choose U_n to have the additional property that $U_n \subset B_n$. Thus the topology $\tau_{\beta+1}$ will refine the subspace Sorgenfrey topology on $X_{\beta+1}$.

Remark 1. Assuming Martin's axiom and $2^\omega > \omega_1$, we infer immediately from theorems in [1] or [6] that the space X of Theorem 2 is perfectly normal.

THEOREM 3. *Assume* $2^\omega = \omega_1$. *Then there exists a locally compact, T_2, scattered, countably compact space X such that* $\mathrm{wd}(X) = \omega$ *and* $\mathrm{ht}(X) = \omega_1$.

Construction. Again we perform a slight modification of the recursive construction of Theorem 1. This time we let $\{E_\alpha: \omega \leqslant \alpha < \omega_1\}$ be a (not necessarily one-to-one) enumeration of all countably infinite subsets of X such that $E_\alpha \subset X_\alpha$ for each $\alpha < \omega_1$. We again use the inductive hypotheses (i)-(iv) of Theorem 1, but now we add the following:

(v′) for all $\alpha < \beta$, E_α has a limit point in $\langle X_\beta, \tau_\beta\rangle$.

The initial and limit ordinal steps remain as in Theorem 1. At the $(\beta+1)$-st successor step we check if E_β has a limit point in $\langle X_\beta, \tau_\beta\rangle$. If it does, we proceed exactly as in Theorem 1. If it does not, then we choose our sequence $\{x_{\alpha_n}: n < \omega\}$ to have infinite intersection with E_β and all other properties. Thus E_β will have a limit point in $\langle X_{\beta+1}, \tau_{\beta+1}\rangle$.

Remark 2. If we assume Martin's axiom plus $2^\omega > \omega_1$, then the space constructed for Theorem 3 does not exist. This is an immediate consequence of the following theorem of Hechler [3]:

If Martin's axiom is assumed and X is a countably compact, separable, regular, T_2 space of cardinality less than 2^ω, then X is compact.

Since ht is an ordinal-valued function, it may be possible to construct locally compact, T_2, scattered spaces such that $\mathrm{wd}(X) = \omega$ and $\mathrm{ht}(X) > \omega_1$. It follows from a result in [7], p. 288-298, that the cardinality of a locally compact, T_2, separable, scattered space is not greater than 2^ω. Thus the height of such a space must be less than $(2^\omega)^+$. Hence, without extra set-theoretic assumptions, we cannot hope to prove that it can be not less than ω_2. This shows that our next result is, in a sense, best possible.

THEOREM 4. *For every $a < \omega_2$ there is a locally compact, T_2, scattered space X such that* $\mathrm{ht}(X) = a$ *and* $\mathrm{wd}(X) = \omega$.

Before we can prove this, however, we do some preparations. For any ordinal a let us denote by $\mathcal{S}_a$ the class of all locally compact, σ-compact, T_2, scattered spaces X such that $\mathrm{ht}(X) = a$ and $|I_\beta(X)| = \omega$ for every $\beta < a$. If $X \in \mathcal{S}_a$, we fix $\{x_n^{(\beta)}: n \in \omega\}$, a one-to-one enumeration of $I_\beta(X)$ for every $\beta < a$. Now, if $X \in \mathcal{S}_{a+1}$, then $I^a(X)$ is clearly a closed discrete subspace of X. Since X is also σ-compact (and, therefore, collectionwise normal), we can separate the points $x_n^{(a)}$ by a discrete collection $\{U_n: n \in \omega\}$ of compact open neighbourhoods, which we shall also keep fixed from now on.

Now, let $X \in \mathcal{S}_{a+1}$ and $Y \in \mathcal{S}_\beta$ be such that $X \cap Y = \emptyset$ and define the space $X \otimes Y$ as follows. The underlying set of $X \otimes Y$ is $X \cup Y \setminus I_0(Y)$; a basic neighbourhood of a point p in X is a neighbourhood of p in X, while if $p \in Y \setminus I_0(Y)$, then a basic neighbourhood of p is of the form V^*, where V is any neighbourhood of p in Y and

$$V^* = [V \setminus I_0(Y)] \cup \bigcup \{U_n: y_n^{(0)} \in V\}.$$

It is easy to check that this indeed defines a topology, which is T_2, since clearly $V \cap W = \emptyset$ implies $V^* \cap W^* = \emptyset$. It is also easy to see that V^* is also compact if V is. Hence $X \otimes Y$ is locally compact and, moreover, X is an open subspace of $X \otimes Y$ while $[Y \setminus I_0(Y)] \cup I_a(X)$ as a subspace of $X \otimes Y$ is homeomorphic to Y. Since X and Y are σ-compact, so is $X \otimes Y$. Finally, it can be seen that $X \otimes Y$ is scattered with $\mathrm{ht}(X \otimes Y) = a + \beta$, and

$$I_\gamma(X \otimes Y) = \begin{cases} I_\gamma(X) & \text{if } \gamma \leqslant a, \\ I_\delta(Y) & \text{if } \gamma = a + \delta \text{ for } 1 \leqslant \delta < \beta. \end{cases}$$

In short, we have $X \otimes Y \in \mathcal{S}_{a+\beta}$.

Next, assume that $X \in \mathcal{S}_a$, where $\mathrm{cf}(a) = \omega$. Then we can fix a sequence $\langle a_n: n \in \omega \rangle$ of ordinals which converges to a in a strictly increasing way. Let $T = \{t_n: n \in \omega\}$ be a set of distinct elements not occurring in X. We define the space $H(X)$ on the underlying set $X \cup T$ as follows. Pick $p_h \in I_{a_h}(X)$ for each $h \in \omega$. Then $\{p_h: h \in \omega\}$ will be closed and discrete in X. Hence we can pick a discrete collection $\{W_h: h \in \omega\}$ of compact open neighbourhoods of the points p_h in X. We also fix a decomposition $\{a_n: n \in \omega\}$ of ω into the disjoint infinite subsets a_n. Now, any neighbourhood of a point $p \in X$ in X will serve as a basic neighbourhood of p in $H(X)$, while the sets

$$V_m(t_n) = \{t_n\} \cup \bigcup \{W_k: k \in a_n \text{ and } k \geqslant m\}$$

will be the basic neighbourhoods of t_n in $H(X)$. Quite similarly as above we can see that $H(X)$ is locally compact, σ-compact, T_2, and scattered. Moreover, X is an open subspace of $H(X)$, $\operatorname{ht}(H(X)) = \alpha+1$, $I_\gamma(H(X)) = I_\gamma(X)$ for $\gamma < \alpha$, and $I_\alpha(H(X)) = T$. In particular, $H(X) \in \mathscr{S}_{\alpha+1}$.

Proof of Theorem 4. We can actually prove more, namely that $\mathscr{S}_{\alpha+1} \neq \emptyset$ for every $\alpha < \omega_2$. We shall do this by induction on α. Thus assume that $\mathscr{S}_{\beta+1} \neq \emptyset$ for every $\beta < \alpha$. The cases in which $\operatorname{cf}(\alpha) \leqslant \omega$ are very easy, hence we omit them. Thus we assume that $\operatorname{cf}(\alpha) = \omega_1$. Then we have

$$\alpha = \sum_{\gamma < \omega_1} \beta_\gamma$$

(ordinal addition!) with $0 < \beta_\gamma < \alpha$. Let us put

$$\alpha_\nu = \sum_{\mu < \nu} \beta_\mu \quad \text{for any } \nu < \omega_1.$$

By the inductive hypothesis there exists a $Y_\nu \in \mathscr{S}_{\beta_\nu+1}$ for every $\nu < \omega_1$.

Now we define, by a subinduction on $\nu < \omega_1$, spaces $X_\nu \in \mathscr{S}_{\alpha_\nu+1}$ such that the following inductive hypothesis is satisfied:

(i′) If $\mu < \nu$, then X_μ is an open subspace of X_ν such that $I_\gamma(X_\nu) = I_\gamma(X_\mu)$ for every $\gamma \leqslant \alpha_\mu$.

Put $X_0 = Y_0$, and then suppose that $\nu > 0$ and that X_μ has already been defined for every $\mu < \nu$ in such a way that (i) holds. If $\nu = \mu+1$, then we can put $X_\nu = X_\mu \otimes Y_\nu$. If ν is limit, i.e. $\operatorname{cf}(\nu) = \omega$, then let Z be the direct union of the spaces $\{X_\mu: \mu < \nu\}$. Then it is clear from (i) that $Z \in \mathscr{S}_{\alpha_\nu}$, and since $\operatorname{cf}(\alpha_\nu) = \operatorname{cf}(\nu) = \omega$, we can put $X_\nu = H(Z)$. Having completed this subinduction, by (i) we can take $X = \bigcup \{X_\nu: \nu < \omega_1\}$, again with the direct union topology. Then X is locally compact, T_2, and scattered with $\operatorname{ht}(X) = \alpha$ and $\operatorname{wd}(X) = \omega$. Let $\hat{X}$ be the one-point compactification of X; clearly, the topological sum of ω disjoint copies of $\hat{X}$ is a member of $\mathscr{S}_{\alpha+1}$. This completes the induction and the proof.

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