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MEAN VALUE AND HARNACK INEQUALITIES FOR DEGENERATE PARABOLIC EQUATIONS

BY

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1. Introduction. In this paper we study the behavior of solutions of degenerate parabolic equations of the form

$$v(x)u_t(x,t) = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x,t)D_{x_j}u(x,t)) \tag{1.1}$$

where the coefficients are measurable functions, and the coefficient matrix $A = (a_{ij})$ is symmetric and satisfies

$$w_1(x)|\xi|^2 \le \langle A\xi, \xi\rangle \le w_2(x)|\xi|^2, \tag{1.2}$$

for $\xi \in \mathbf{R}^n$ and $(x,t) \in \Omega \times (a,b)$, Ω a domain in $\mathbf{R}^n$. Such equations arise for example by making a quasiconformal change of variable $y = \varphi(x)$ in the heat equation $D_t = \Delta_y$; in this case, $v = |\det(\varphi')|$ and w_1 and w_2 are constant multiples of $v^{(n-2)/n}$. They also appear in the problem of heat conduction in a non-homogeneous and non-isotropic medium; in this case $v(x)$ represents the product of the density times the specific heat at the point x, and a_{ij} is the thermal conductivity.

We prove mean value inequalities for solutions of (1.1), which in particular imply the local boundedness of solutions, and we derive a Harnack-type inequality. The proof of our results depends on the iterative techniques introduced by Moser in [11] and [12] to study the case when v, w_1 and w_2 are positive constants. The main tool used in this technique is a certain class of Sobolev interpolation inequalities. In our case, these inequalities have been proved by us in [8].

Let $u = u(x,t)$ be a locally integrable function in $Q = \Omega \times (a,b)$. We say u is a *solution* of the equation (1.1) in Q if $u \in L^2_{v+w_2}(Q)$, i.e.,

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$\iint_Q |u(x,t)|^2(v(x)+w_2(x))\,dx\,dt < \infty$, and u satisfies

$$\iint_Q |u_t(x,t)|^2 v(x)\,dx\,dt < \infty, \tag{1.3}$$

$$\iint_Q |\nabla u(x,t)|^2 w_2(x)\,dx\,dt < \infty \tag{1.4}$$

and

$$\iint_Q \{u_t\varphi v + \langle A\nabla u, \nabla\varphi\rangle\}\,dx\,dt = 0 \tag{1.5}$$

for any $\varphi \in C_0^1(Q)$. Here the derivatives are understood in the sense of distributions, and $\nabla = \nabla_x$ throughout the paper. If in (1.5) we replace "$=$" by "$\leq$" or "$\geq$" and assume that the resulting condition holds for all $\varphi \geq 0$, $\varphi \in C_0^1(Q)$, then we say u is a *subsolution* or *supersolution*, respectively.

We say that a non-negative and locally integrable function w is a *doubling weight* (i.e., $w \in D$) if there exists a constant $c > 0$ such that

$$w(2B) \leq cw(B) \tag{1.6}$$

for every ball B in $\mathbf{R}^n$, where $2B$ is the ball with the same center as B and twice the radius, and $w(B) = \int_B w\,dx$. We use the notation $B_\alpha(x_0)$ to denote the ball in $\mathbf{R}^n$ with center x_0 and radius α.

Given $1 < p < \infty$, we say $w \in A_p$ if there exists a constant $c > 0$ such that for all balls B in $\mathbf{R}^n$

$$\left[|B|^{-1}\int_B w(x)\,dx\right]\left[|B|^{-1}\int_B w(x)^{-1/(p-1)}\,dx\right]^{p-1} \leq c. \tag{1.7}$$

Also, if v is a weight, $w \in A_p(v)$ means an analogous inequality holds with dx and $|B|$ replaced by $v(x)\,dx$ and $v(B)$, respectively. We also use the notation $A_\infty(v) = \bigcup_{p>1} A_p(v)$. It is well known (see [6]) that $w/v \in A_\infty(v)$ if and only if there exist constants $\delta > 0$ and $c > 0$ such that

$$w(E)/w(B) \leq c[v(E)/v(B)]^\delta \tag{1.8}$$

for all balls B in $\mathbf{R}^n$ and every measurable set $E \subset B$.

We will use the notation $w\otimes 1$ to denote the measure $w(x)\,dx\,dt$ on $\mathbf{R}^{n+1}$.

Given $q \geq 2$, we say that *the Poincaré inequality holds for* w_1, w_2 if there exists a constant $c > 0$ such that for any ball B and every $u \in \mathrm{Lip}(\bar{B})$

we have

$$(1.9)\quad \left[w_2(B)^{-1}\int_B |u-\mathrm{av}_{\mu,B}u|^q w_2\,dx\right]^{1/q} \le c|B|^{1/n}\left[w_1(B)^{-1}\int_B |\nabla u|^2 w_1\,dx\right]^{1/2},$$

where $\mathrm{av}_{\mu,B}u = \int_B u\,d\mu/\mu(B)$. Here $\mathrm{Lip}(\bar{B})$ denotes the class of functions which are Lipschitz continuous in $\bar{B}$. Similarly, if $q \ge 2$, we say that *Sobolev's inequality holds for* w_1, w_2 if for every $u \in \mathrm{Lip}(\bar{B})$ with compact support in B

$$(1.10)\quad \left[w_2(B)^{-1}\int_B |u|^q w_2\,dx\right]^{1/q} \le c|B|^{1/n}\left[w_1(B)^{-1}\int_B |\nabla u|^2 w_1\,dx\right]^{1/2},$$

with c independent of u and B.

By the results of [2], a sufficient condition for the validity of (1.9) or (1.10), with $\mu = 1$ or $\mu = w_2$, assuming that $w_1 \in A_2$ and $q > 2$ is the following:

$$(1.11)\qquad [|\widetilde{B}|/|B|]^{1/n}[w_2(\widetilde{B})/w_2(B)]^{1/q} \le c[w_1(\widetilde{B})/w_1(B)]^{1/2}$$

for all balls $\widetilde{B}$, B satisfying $\widetilde{B} \subset 2B$, with c independent of the balls. We also know from [2] and [8] that if (1.9) holds and w_1, w_2 and μ are doubling weights, then (1.11) holds.

We can now state our main results.

THEOREM A (Harnack's Inequality). *Suppose that*

(i) $v, w_1, w_2 \in A_2$;
(ii) *the Poincaré inequality holds for* w_1, w_2 *with* $\mu = 1$ *and some* $q > 2$;
(iii) *the Poincaré inequality holds for* w_1, v *with any* μ *and some* $q > 2$;
(iv) $w_2/v \in A_\infty(v)$.

If u is a non-negative solution of (1.1) *in the cylinder* $R = B_\alpha(x_0) \times (t_0 - \beta, t_0 + \beta)$, *then*

$$(1.12)\qquad \operatorname*{ess\,sup}_{R^-} u \le c_1 e^{c_2[\alpha^{-2}\beta\Lambda(B_\alpha(x_0))+\alpha^2\beta^{-1}\lambda(B_\alpha(x_0))^{-1}]} \operatorname*{ess\,inf}_{R^+} u,$$

where

$$R^- = B_{\alpha/2}(x_0) \times (t_0 - \tfrac{3}{4}\beta, t_0 - \tfrac{1}{4}\beta), \qquad R^+ = B_{\alpha/2}(x_0) \times (t_0 + \tfrac{1}{4}\beta, t_0 + \beta),$$

$$\Lambda(B) = \frac{w_2(B)}{v(B)} \quad \text{and} \quad \lambda(B) = \frac{w_1(B)}{v(B)} \quad \text{for a ball } B.$$

Here the constants c_1, c_2 *depend only on the constants which arise in* (i)–(iv).

Several comments about this result are in order. Given the dimension α of the cylinder R, the choice of β leads to different constants in the exponent in (1.12) as well as to different cylinders. In particular, the value of β that minimizes the exponential in (1.12) is given by

$$\beta = \alpha^2 \frac{v(B_\alpha(x_0))}{[w_1(B_\alpha(x_0))w_2(B_\alpha(x_0))]^{1/2}} .$$

In this case (1.12) becomes

$$\operatorname*{ess\,sup}_{R^-} u \le c_1 e^{c_2[w_2(B_\alpha(x_0))/w_1(B_\alpha(x_0))]^{1/2}} \operatorname*{ess\,inf}_{R^+} u, \tag{1.13}$$

and this inequality is sharp as the example on p. 1114 of [3] shows.

In the case when $\beta = \alpha^2$ in (1.12) (i.e., for the usual parabolic cylinders), we obtain

$$\operatorname*{ess\,sup}_{R^-} u \le c_1 e^{c_2[\Lambda(B_\alpha(x_0))+\lambda(B_\alpha(x_0))^{-1}]} \operatorname*{ess\,inf}_{R^+} u.$$

If the weight functions are positive constants, this reduces to the inequality proved by Moser in [11]. In case $\beta = \alpha^2\lambda(B_\alpha(x_0))^{-1}$ or $\beta = \alpha^2\Lambda(B_\alpha(x_0))^{-1}$, the term in the exponent in (1.12) becomes $w_2(B_\alpha(x_0))/w_1(B_\alpha(x_0))$.

We also mention that for any given choice of the cylinder R the exponent in (1.12) is optimal. This can be seen as in [11] by considering in $\mathbf{R}^2$ the equation

$$u_t = \lambda u_{xx} + \Lambda u_{yy}, \tag{1.14}$$

where $0 < \lambda < \Lambda$, in the cylinder $B_\alpha(0,0) \times (-\beta,\beta) = R$. The function

$$u(x,y,t) = e^{at+bx}\cos cy$$

is a non-negative solution of (1.14) in R provided $|c\alpha| \le \pi/2$ and $\lambda b^2 - \Lambda c^2 = a$. By picking $c = 1/\alpha$ and $b = \alpha/4\lambda\beta$, we have

$$\begin{aligned}\log\left[\frac{u(\alpha/2,0,-\beta/2)}{u(0,0,\beta/2)}\right] &= \Lambda\alpha^{-2}\beta + \frac{1}{16}\alpha^2\beta^{-1}\frac{1}{\lambda}\\ &\ge \frac{1}{16}\left(\Lambda\alpha^{-2}\beta + \alpha^2\beta^{-1}\frac{1}{\lambda}\right).\end{aligned}$$

The hypothesis (i) in the statement of the Harnack inequality is used to pass to the limit in the interpolation inequalities, and also sometimes in their hypotheses. However, if one is only interested in classical solutions of (1.1), then the Harnack inequality is true with weaker assumptions than (i); see the discussion at the beginning of Section 3. The proof of our result is done in several steps and none of the steps requires all the hypotheses (i)–(iv); for example, (iv) is used only in Section 4 to prove distribution function estimates of the logarithm of u.

We mention that a Harnack inequality for solutions of (1.1) when $v = 1$ and $w_1 \approx w_2 \in A_{1+2/n}$ has been proved in [4], and when $v \approx w_1 \approx w_2 \in A_2$ in [5]. These are special cases of our result. To see this, first note that when $w_1 \approx w_2 \in A_2$, condition (ii) is satisfied since (1.11) holds for some $q > 2$. Also, in case $v = 1$ and $w_1 \in A_{1+2/n}$, the pair of weights $(1, w_1)$ satisfies (1.11) for some $q > 2$. This is because in this case (1.11) is equivalent to assuming that w_1 satisfies a doubling condition of some order $d < 1+2/n$ (i.e., $w_1(B_{tr}) \le ct^{nd}w_1(B_r)$ for some $d < 1+2/n$, for every $t \ge 1$, $r > 0$, with c independent of t and r), and this is implied by $w_1 \in A_{1+2/n}$. Therefore, condition (iii) holds and (iv) is obviously true. Note also that when $w_1 \in A_{1+2/n}$, we have

$$\alpha^{n+2}/w_1(B_\alpha) \approx \Big[\int_{B_\alpha} w_1^{-n/2}\, dx \Big]^{2/n},$$

and then (1.13) is the result of [4]. In case $v \approx w_1 \approx w_2 \in A_2$, the result in [5] follows immediately from Theorem A.

In proving Harnack's inequality we will derive a number of mean value inequalities for solutions (as well as for subsolutions and supersolutions). As an example, we have the following result in which we use the notation

$$⨌_R f(x,t)m(x,t)\, dx\, dt = \iint_R f(x,t)m(x,t)\, dx\, dt \Big/ \iint_R m(x,t)\, dx\, dt.$$

THEOREM B. *Assume that hypotheses* (i)–(iii) *of Theorem A hold. Let* $0 < p < \infty$, $\alpha, \beta > 0$, $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$, *and let* $B_\alpha(x_0) = B$, $B_{\alpha'}(x_0) = B'$ *and*

$$R = R_{\alpha,\beta} = B \times (t_0 - \beta, t_0 + \beta), \qquad R'^{+} = B' \times (t_0 - \beta', t_0 + \beta).$$

If u is a solution of (1.1) *in R, then u is bounded in R'^{+} and*

$$\operatorname*{ess\,sup}_{R'^{+}} |u|^p \le D(\alpha^2\beta^{-1}\lambda(B)^{-1} + 1)^{1/(h-1)}$$
$$\times (\alpha^{-2}\beta\Lambda(B) + 1)^{h/(h-1)} ⨌_R |u|^p(\alpha^{-2}\beta w_2 + v)\, dx\, dt,$$

where

$$D \le \begin{cases} C^{1/(h-1)} & \text{if } p \ge 2, \\ c^{\log(3/p)} C^c & \text{if } 0 < p < 2, \end{cases}$$
$$C = c\frac{\alpha^2\beta}{(\alpha - \alpha')^2(\beta - \beta')} .$$

Here $h > 1$ and $c > 0$ are constants which are independent of $u, p, \alpha, \alpha', \beta, \beta'$.

Our results also imply estimates in the elliptic case, i.e., when $u(x,t)$ is independent of t, so that $\partial u/\partial t = 0$. A solution of an elliptic equation $\nabla \cdot A\nabla = 0$ in a ball $B_\alpha(x_0)$ is a solution of the corresponding parabolic equation in $B_\alpha(x_0) \times (t_0 - \beta, t_0 + \beta)$ for any t_0, β and for any $v(x)$. To obtain a Harnack inequality like that in [3], namely, (1.13) above with R^- and R^+ replaced by $B_{\alpha/2}(x_0)$, we choose β as mentioned just before (1.13), and we choose $v = w_2$. With this choice of v, hypotheses (ii) and (iii) of Theorem A are the same, and hypothesis (iv) is automatically true. Thus, the hypotheses are like those in [3], except that it is not assumed in [3] that $w_2 \in A_2$ or that $\nabla u \in L^2_{w_2}$ if u is a solution. (See, however, the remark on p. 1130 of [3].) We have made these stronger hypotheses in order to simplify some of the technical difficulties in [3]; for example, as shown in Lemma (2.2) below, the stronger assumptions make it simple to approximate general functions by smooth ones, thereby allowing us to avoid defining solutions in terms of Cauchy sequences as in [3].

The organization of the paper is as follows. In Section 2 we derive Caccioppoli-type estimates for solutions, subsolutions and supersolutions. In Section 3 we prove mean value inequalities, including Theorem B, and in Section 4 estimates for the distribution function of the logarithm of a positive solution are derived. In Section 5 we include a proof of an L^∞-estimate of Bombieri type needed in our case. In Section 6 we combine the results of the previous sections to obtain Harnack's inequality. The general outline of the proof follows Moser's method in [11] and [12], but there are complications due to the presence of the weights. To make the paper understandable, we have given the details of most of the proofs.

Finally, in Section 7, which is separate from the rest of the paper, some remarks are made about the elliptic case [3]. These include a comment about the details of one of the proofs in [3] and the outline of an argument which corrects an error in one of the proofs in [3].

2. Caccioppoli estimates. We first show that if u is a solution of (1.1) in $Q = \Omega \times (a,b)$, then the formula (cf. (1.5))

$$\iint_Q \{u_t\varphi v + \langle A\nabla u, \nabla\varphi\rangle\}\, dx\, dt = 0 \tag{2.1}$$

holds for a class of test functions φ which is larger than $C_0^1(Q)$. We recall that by definition u is a solution if (2.1) holds for all $\varphi \in C_0^1(Q)$, and $u \in L^2_{v+w_2}(Q)$, $u_t \in L^2_v(Q)$ and $\nabla u \in L^2_{w_2}(Q)$.

LEMMA (2.2). *Let u be a solution of* (1.1) *in Q, and assume that $w_2 \in A_2$ and that any one of the following is valid:*

(a) *Sobolev's inequality* (1.10) *holds for w_1, v with $q = 2$;*

(b) *Sobolev's inequality* (1.10) *holds for* w_2, v *with* $q = 2$;
(c) $v \in A_2$.

Then (2.1) *is valid for any* $\varphi(x,t) \in L^1(Q)$ *which has compact support in* Q *and which satisfies*

$$\iint_Q |\nabla\varphi(x,t)|^2 w_2(x)\,dx\,dt < \infty.$$

Proof. We will give the proof first assuming that (a) holds. Let $\psi(x,t) \in C_0^\infty(\mathbf{R}^{n+1})$, $\iint \psi\,dx\,dt = 1$, and define $\psi_\varepsilon(x,t) = \varepsilon^{-n-1}\psi(x/\varepsilon, t/\varepsilon)$, $\varepsilon > 0$. If φ satisfies the conditions of the lemma, then for small $\varepsilon > 0$ the function φ_ε defined by $\varphi_\varepsilon(x,t) = (\varphi * \psi_\varepsilon)(x,t)$ belongs to $C_0^\infty(Q)$. Thus, (2.1) holds for φ_ε for small $\varepsilon > 0$, and we want to show that it holds for φ by passing to the limit. Since $\nabla\varphi \in L^2_{w_2}(Q)$ by hypothesis and since $1/w_2 \in L^1_{\text{loc}}(\mathbf{R}^{n+1})$ (in fact, $1/w_2 \in L^1_{\text{loc}}(\mathbf{R}^n)$ since $w_2 \in A_2$), we have $\nabla\varphi \in L^1(Q)$, and therefore, $\nabla\varphi_\varepsilon = (\nabla\varphi) * \psi_\varepsilon \to \nabla\varphi$ a.e. in Q as $\varepsilon \to 0$. Also,

$$|\nabla\varphi_\varepsilon(x,t)| \le cM(\chi_Q|\nabla\varphi|)(x,t),$$

where M denotes the Hardy–Littlewood maximal operator in $\mathbf{R}^{n+1}$. Note that $w_2 \otimes 1 \in A_2(\mathbf{R}^{n+1})$ since $w_2 \in A_2(\mathbf{R}^n)$. Therefore, by [10], M is bounded on $L^2_{w_2}(\mathbf{R}^{n+1})$, and it follows that

$$\iint M(\chi_Q|\nabla\varphi|)(x,t)^2 w_2(x)\,dx\,dt \le c \iint_Q |\nabla\varphi(x,t)|^2 w_2(x)\,dx\,dt.$$

By dominated convergence, we see that $\nabla\varphi_\varepsilon \to \nabla\varphi$ in $L^2_{w_2}(Q)$ as $\varepsilon \to 0$. Consequently, using (1.2) and the fact that $\nabla u \in L^2_{w_2}(Q)$ (since u is a solution), we obtain

$$\iint_Q \langle A\nabla u, \nabla\varphi_\varepsilon\rangle\,dx\,dt \to \iint_Q \langle A\nabla u, \nabla\varphi\rangle\,dx\,dt$$

as $\varepsilon \to 0$. It remains only to show that

$$\iint_Q u_t\varphi_\varepsilon v\,dx\,dt \to \iint_Q u_t\varphi v\,dx\,dt. \tag{2.3}$$

By Fatou's lemma and Sobolev's inequality (1.10) for w_1, v and $q = 2$, we obtain

$$\begin{aligned}\iint_Q |\varphi|^2 v\,dx\,dt &\le \liminf_{\varepsilon\to 0} \iint_Q |\varphi_\varepsilon|^2 v\,dx\,dt \\ &\le c\liminf_{\varepsilon\to 0} \iint_Q |\nabla\varphi_\varepsilon|^2 w_1\,dx\,dt,\end{aligned}$$

with $c = c_{Q,v,w_1}$. Since $w_1 \le w_2$, it then follows from the earlier estimates that

$$\iint\limits_Q |\varphi|^2 v\,dx\,dt \le c \iint\limits_Q |\nabla\varphi|^2 w_2\,dx\,dt.$$

Since a similar inequality holds for $\varphi_\varepsilon - \varphi$ and since $\nabla\varphi_\varepsilon \to \nabla\varphi$ in $L^2_{w_2}(Q)$, it follows that $\varphi_\varepsilon \to \varphi$ in $L^2_v(Q)$. However, $u_t \in L^2_v(Q)$ by hypothesis, since u is a solution, and we then obtain (2.3) by Schwarz's inequality.

Instead of assuming that Sobolev's inequality with $q = 2$ is valid for w_1, v, we may assume it instead for the pair w_2, v. This is clear from the proof. Alternatively, we can replace the Sobolev assumptions by the assumption that $v \in A_2$, since then by using the maximal function we immediately find that $\varphi_\varepsilon \to \varphi$ in $L^2_v(Q)$.

Remarks. (i) We note that assumption (a) of Lemma (2.2) is fulfilled if $w_1 \in A_2$ and assumption (iii) of Theorem A is true. In fact, under assumption (iii), we have (1.11) with w_2 replaced by v; by the results of [2], since $w_1 \in A_2$, this implies that Sobolev's inequality holds for w_1, v with the same q, and therefore, also with $q = 2$ by Hölder's inequality.

(ii) There are analogues of Lemma (2.2) in case u is just a subsolution or a supersolution. In fact, if we replace (2.1) by the appropriate inequality (namely, "≤ 0" for subsolutions, and "≥ 0" for supersolutions, assuming in either case that $\varphi \ge 0$), then the conclusion remains valid with the additional hypothesis that $\varphi \ge 0$. The only change needed in the proof is to choose $\psi \ge 0$ since this implies that $\varphi_\varepsilon \ge 0$ if $\varphi \ge 0$.

In the next lemma, we consider Caccioppoli estimates for u_+^p, $u_+ = \max\{u, 0\}$, in case $2 \le p < \infty$ and u is a subsolution of (1.1). This of course includes the powers u^p, $2 \le p < \infty$, of a non-negative solution u.

LEMMA (2.4). *Let $2 \le p < \infty$ and u be a subsolution of* (1.1) *in $Q = \Omega \times (a,b)$ which satisfies $u_+ \in L^p_{v+w_2}(Q)$. Let $w_2 \in A_2$ and assume that either* (a), (b) *or* (c) *of Lemma* (2.2) *holds. Then for every smooth function $\eta(x,t)$ which is zero in a neighborhood of $\partial\Omega \times (a,b)$ and for a.e. τ_1, τ_2 satisfying $a < \tau_1 < \tau_2 < b$, we have*

$$\int\limits_\Omega (\eta^2 u_+^p)\,\Big|_{t=\tau_1}^{\tau_2} v\,dx + 2\,\frac{p-1}{p} \int\limits_\Omega \int\limits_{\tau_1}^{\tau_2} |\nabla(u_+^{p/2})|^2 \eta^2 w_1\,dx\,dt$$

$$\le \int\limits_\Omega \int\limits_{\tau_1}^{\tau_2} |(\eta^2)_t| u_+^p v\,dx\,dt + 2\,\frac{p}{p-1} \int\limits_\Omega \int\limits_{\tau_1}^{\tau_2} |\nabla\eta|^2 u_+^p w_2\,dx\,dt.$$

Before giving the proof, we note that the conclusion of the lemma has no content if the right side is $+\infty$. For this reason we have assumed that $u_+ \in L^p_{v+w_2}(Q)$; the lemma would still have meaning if we only assumed that $u_+ \in L^p_{v+w_2}(Q)$ on every compact subset of Q. Note also that the first term on the left side of the conclusion, which is the only one of the four terms which may not be non-negative, is finite for a.e. $\tau_1, \tau_2 \in (a,b)$ by Fubini's theorem if $u_+ \in L^p_v(Q)$.

The hypothesis that (a), (b) or (c) of Lemma (2.2) be valid is used in order to apply remark (ii) following Lemma (2.2).

Proof. Given $\beta \geq 1$ and $\ell > 0$, define $g(s)$ and $h(s)$ for $s \in (-\infty, \infty)$ by

$$g(s) = \begin{cases} s^\beta & \text{if } 0 \leq s \leq \ell, \\ \ell^{\beta-1}s & \text{if } s > \ell, \\ 0 & \text{if } s < 0, \end{cases}$$

$$h(s) = \begin{cases} \dfrac{1}{\beta+1} s^{\beta+1} & \text{if } 0 \leq s \leq \ell, \\ \dfrac{1}{2}\ell^{\beta-1}s^2 - \dfrac{\beta-1}{2(\beta+1)}\ell^{\beta+1} & \text{if } s > \ell, \\ 0 & \text{if } s < 0. \end{cases}$$

Note that $h(s) \geq 0$ and that $h'(s) = g(s)$ for all s.

Let η satisfy the hypothesis of the lemma, and let $\chi(t, \tau_1, \tau_2)$ be the characteristic function of (τ_1, τ_2), $a < \tau_1 < \tau_2 < b$. We will show that if u is a subsolution in Q then the function φ defined by

$$\varphi(x,t) = \eta^2 g(u)\chi(t, \tau_1, \tau_2)$$

satisfies the conditions of Lemma (2.2). Clearly, φ has compact support in Q, and φ is integrable on Q since u is. To see that $\nabla\varphi \in L^2_{w_2}(Q)$, first note that (cf. Lemma 4 of [1])

$$\nabla\varphi = \begin{cases} \eta^2 g'(u)\nabla u + 2\eta\nabla\eta g(u), & \tau_1 < t < \tau_2, \\ 0, & t \notin (\tau_1, \tau_2). \end{cases}$$

Since g' is bounded (recall that $\beta \geq 1$) and $\nabla u \in L^2_{w_2}(Q)$ by hypothesis,

$$\iint_Q |\eta^2 g'(u)\nabla u|^2 w_2 \, dx \, dt < \infty.$$

Also,

$$\iint_Q \eta^2 |\nabla\eta|^2 g(u)^2 w_2 \, dx \, dt$$

$$\leq c_\eta \iint_Q \ell^{2(\beta-1)} u^2 \chi_{(u>\ell)} w_2 \, dx \, dt + c_\eta \iint_Q u^{2\beta}\chi_{(0\leq u\leq \ell)} w_2 \, dx \, dt.$$

The first integral on the right is finite since $u \in L^2_{w_2}(Q)$, and the second is clearly finite. This shows that $\nabla\varphi \in L^2_{w_2}(Q)$.

By remark (ii) following Lemma (2.2), noting that $\varphi \geq 0$, we obtain

$$\iint_Q \{u_t \varphi v + \langle A\nabla u, \nabla\varphi\rangle\}\, dx\, dt \leq 0$$

for the function φ defined above. Since $h' = g$ is a Lipschitz function, Lemma 5 of [1] implies that $\eta^2 h(u)$ is differentiable and

$$(\eta^2 h(u))_t = (\eta^2)_t h(u) + \eta^2 g(u) u_t.$$

Therefore,

$$\begin{aligned} u_t\varphi &= \eta^2 g(u) u_t \chi(t, \tau_1, \tau_2) \\ &= (\eta^2 h(u))_t \chi(t, \tau_1, \tau_2) - (\eta^2)_t h(u)\chi(t, \tau_1, \tau_2). \end{aligned}$$

Hence, using Lemma 6 of [1] and the fact that $u \in L^2_v(Q)$, so that $h(u) \in L^1_v(Q)$, we obtain

$$\int_\Omega (\eta^2 h(u))\,|_{t=\tau_1}^{\tau_2} v(x)\, dx + \iint_Q \langle A\nabla u, \nabla\varphi\rangle\, dx\, dt$$
$$\leq \int_\Omega \int_{\tau_1}^{\tau_2} (\eta^2)_t h(u) v(x)\, dx\, dt.$$

By the formula for $\nabla\varphi$,

$$\text{(2.5)} \qquad \begin{aligned} &\int_\Omega (\eta^2 h(u))\,|_{t=\tau_1}^{\tau_2} v(x)\, dx + \int_\Omega \int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla u\rangle \eta^2 g'(u)\, dx\, dt \\ &\leq \int_\Omega \int_{\tau_1}^{\tau_2} (\eta^2)_t h(u) v(x)\, dx\, dt - \int_\Omega \int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla\eta\rangle 2\eta g(u)\, dx\, dt, \end{aligned}$$

where we have used the observations made earlier that both $\eta^2 g'(u)\nabla u$ and $\eta\nabla\eta g(u)$ belong to $L^2_{w_2}(Q)$ to insure that the second and fourth integrals are finite.

Now define $\theta(s)$ for $-\infty < s < \infty$ by

$$\theta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \ell, \\ \beta(\ell/s)^{\beta-1} & \text{if } s > \ell, \\ 0 & \text{if } s < 0. \end{cases}$$

Note that $\theta(s) \to \chi_{[0,\infty)}(s)$ as $\ell \to \infty$, that $0 \leq \theta(s) \leq \beta$, and that

$$g^2(u) = \frac{1}{\beta} u_+^{\beta+1} g'(u)\theta(u).$$

Thus,

$$
\begin{aligned}
&-\int_{\Omega}\int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla\eta\rangle 2\eta g(u)\,dx\,dt \\
&\quad = -2\int_{\Omega}\int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla\eta\rangle \eta \frac{1}{\beta^{1/2}} u_+^{(\beta+1)/2} g'(u)^{1/2}\theta(u)^{1/2}\,dx\,dt \\
&\quad \le \varepsilon \int_{\Omega}\int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla u\rangle \eta^2 g'(u)\,dx\,dt \\
&\qquad + \frac{1}{\varepsilon\beta}\int_{\Omega}\int_{\tau_1}^{\tau_2} \langle A\nabla\eta, \nabla\eta\rangle u_+^{\beta+1}\theta(u)\,dx\,dt,
\end{aligned}
$$

where we have used the inequality

$$
|\langle Ax, y\rangle| \le \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2} \le \frac{\varepsilon}{2}\langle Ax, x\rangle + \frac{1}{2\varepsilon}\langle Ay, y\rangle, \qquad \varepsilon > 0.
$$

Combining this with (2.5) and using (1.2) we obtain for $0 < \varepsilon < 1$

$$
\begin{aligned}
(2.6)\qquad &\int_{\Omega} (\eta^2 h(u))\big|_{t=\tau_1}^{\tau_2} v(x)\,dx + (1-\varepsilon)\int_{\Omega}\int_{\tau_1}^{\tau_2} |\nabla u|^2 g'(u)\eta^2 w_1\,dx\,dt \\
&\le \int_{\Omega}\int_{\tau_1}^{\tau_2} |(\eta^2)_t| h(u) v(x)\,dx\,dt + \frac{1}{\varepsilon\beta}\int_{\Omega}\int_{\tau_1}^{\tau_2} |\nabla\eta|^2 u_+^{\beta+1}\theta(u) w_2(x)\,dx\,dt.
\end{aligned}
$$

Now let $\ell \to \infty$ and note that $g'(u) \nearrow \beta u_+^{\beta-1}\chi_{(u>0)}$, $h(u) \nearrow \frac{1}{\beta+1} u_+^{\beta+1}$ and $\theta(u) \to \chi_{(u>0)}$, $0 \le \theta(u) \le \beta$. Given p with $2 \le p < \infty$, select $\beta \ge 1$ such that $\beta + 1 = p$. By monotone convergence, the second and third expressions above converge respectively to

$$
(1-\varepsilon)(p-1)\int_{\Omega}\int_{\tau_1}^{\tau_2} |\nabla(u_+)|^2 u_+^{p-2}\eta^2 w_1(x)\,dx\,dt
$$

and

$$
\frac{1}{p}\int_{\Omega}\int_{\tau_1}^{\tau_2} |(\eta^2)_t| u_+^p v(x)\,dx\,dt,
$$

the latter being finite because of the hypothesis that $u_+ \in L_v^p(Q)$, although this hypothesis is not necessary here. Since $u_+ \in L_{w_2}^p(Q)$, the fourth expression in (2.6) tends by dominated convergence to

$$
\frac{1}{\varepsilon(p-1)}\int_{\Omega}\int_{\tau_1}^{\tau_2} |\nabla\eta|^2 u_+^p w_2(x)\,dx\,dt.
$$

Again, the hypothesis that $u_+ \in L^p_{w_2}(Q)$ is not needed since the lemma is trivially true if the last expression is $+\infty$. Finally, consider the first term in (2.6). If

$$\int_Q (\eta^2 u_+^p)\,|_{t=\tau} v(x)\,dx < \infty$$

for both $\tau = \tau_1$ and $\tau = \tau_2$ (which is the case for a.e. $\tau_1, \tau_2 \in (a,b)$ by Fubini's theorem when $u_+ \in L^p_v(Q)$), we easily obtain from (2.6) by passing to the limit

$$\frac{1}{p}\int_\Omega (\eta^2 u_+^p)\,|_{t=\tau_1}^{\tau_2} v(x)\,dx + (1-\varepsilon)(p-1)\int_\Omega \int_{\tau_1}^{\tau_2} |\nabla(u_+)|^2 u_+^{p-2}\eta^2 w_1\,dx\,dt$$
$$\leq \frac{1}{p}\int_\Omega \int_{\tau_1}^{\tau_2} |(\eta^2)_t| u_+^p v(x)\,dx\,dt + \frac{1}{\varepsilon(p-1)}\int_\Omega \int_{\tau_1}^{\tau_2} |\nabla\eta|^2 u_+^p w_2(x)\,dx\,dt.$$

Since $\nabla(u_+^{p/2}) = (p/2)u_+^{(p-2)/2}\nabla(u_+)$, the second expression equals

$$\frac{4}{p^2}(1-\varepsilon)(p-1)\int_\Omega \int_{\tau_1}^{\tau_2} |\nabla(u_+^{p/2})|^2 \eta^2 w_1(x)\,dx\,dt,$$

and Lemma (2.4) follows immediately by choosing $\varepsilon = \frac{1}{2}$. Note that $u_+^{p/2} \in L^1(Q)$ since $u_+^{p/2} \in L^2_{w_2}(Q)$ and $w_2 \in A_2$.

We will eventually prove that if u is a solution in Q then u is locally bounded in Q. We will show in the next lemma that if u is a solution in Q which is (locally) bounded above and also below away from zero, then there is an analogue of Lemma (2.4) for all $p \neq 0, 1$, although this analogue is slightly different depending on the sign of $(p-1)/p$.

LEMMA (2.7). *Let $-\infty < p < \infty$, $p \neq 0, 1$, and suppose when $p < 1$ that u is a supersolution and when $p > 1$ that u is a subsolution of* (1.1) *in $Q = \Omega \times (a,b)$ which satisfies $0 < m \leq u(x,t) \leq M < \infty$ in Q. Let $w_2 \in A_2$ and assume that* (a), (b) *or* (c) *of Lemma* (2.2) *is valid. Then for every smooth function $\eta(x,t)$ which is zero in a neighborhood of $\partial\Omega \times (a,b)$, and for a.e. τ_1, τ_2 with $a < \tau_1 < \tau_2 < b$, we have*

$$\operatorname{sign}\left(\frac{p-1}{p}\right)\int_\Omega (\eta^2 u^p)\,|_{t=\tau_1}^{\tau_2} v\,dx + 2\left|\frac{p-1}{p}\right| \int_\Omega \int_{\tau_1}^{\tau_2} |\nabla(u^{p/2})|^2 \eta^2 w_1\,dx\,dt$$
$$\leq \int_\Omega \int_{\tau_1}^{\tau_2} |(\eta^2)_t| u^p v\,dx\,dt + 2\left|\frac{p}{p-1}\right| \int_\Omega \int_{\tau_1}^{\tau_2} |\nabla\eta|^2 u^p w_2\,dx\,dt.$$

Proof. We will consider the case when $p < 1$, $p \neq 0$, and u is a supersolution; the argument for $p > 1$ and subsolutions is similar. For

$a < \tau_1 < \tau_2 < b$, and η as in the hypothesis, define

$$\varphi(x,t) = \eta^2 u^{p-1} \chi(t, \tau_1, \tau_2).$$

Since $p - 1 < 0$, it follows easily from the facts that $u(x,t) \geq m > 0$ in Q and $\nabla u \in L^2_{w_2}(Q)$ that φ satisfies the conditions of Lemma (2.2). Hence, by remark (ii) to Lemma (2.2), since u is a supersolution and $\varphi \geq 0$,

$$\iint_Q \{u_t \varphi v + \langle A\nabla u, \nabla\varphi\rangle\}\, dx\, dt \geq 0.$$

Thus,

$$\int_\Omega \int_{\tau_1}^{\tau_2} u_t \eta^2 u^{p-1} v\, dx\, dt + (p-1) \int_\Omega \int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla u\rangle u^{p-2}\eta^2\, dx\, dt$$
$$\geq -2 \int_\Omega \int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla\eta\rangle u^{p-1}\eta\, dx\, dt,$$

so that (cf. (2.5))

$$\frac{1}{p}\int_\Omega (\eta^2 u^p)\,|_{t=\tau_1}^{\tau_2} v\, dx + (p-1)\int_\Omega \int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla u\rangle u^{p-2}\eta^2\, dx\, dt$$
$$\geq \frac{1}{p}\int_\Omega \int_{\tau_1}^{\tau_2} (\eta^2)_t u^p v\, dx\, dt - 2\int_\Omega \int_{\tau_1}^{\tau_2} \langle A\nabla u, \nabla\eta\rangle u^{p-1}\eta\, dx\, dt.$$

Note that $u^{(p-2)/2}\nabla u = (2/p)\nabla(u^{p/2})$, and that $p\,\mathrm{sign}((p-1)/p) < 0$ since $p < 1,\ p \neq 0$. Hence, multiplying through by $p\,\mathrm{sign}((p-1)/p)$, we obtain

$$\mathrm{sign}\left(\frac{p-1}{p}\right) \int_\Omega (\eta^2 u^p)\,|_{t=\tau_1}^{\tau_2} v\, dx$$
$$+ 4\left|\frac{p-1}{p}\right| \int_\Omega \int_{\tau_1}^{\tau_2} \langle A\nabla(u^{p/2}), \nabla(u^{p/2})\rangle \eta^2\, dx\, dt$$
$$\leq \int_\Omega \int_{\tau_1}^{\tau_2} |(\eta^2)_t| u^p v\, dx\, dt + 4\int_\Omega \int_{\tau_1}^{\tau_2} |\langle A\nabla(u^{p/2}), \nabla\eta\rangle| u^{p/2}|\eta|\, dx\, dt.$$

If we use the inequality

$$|\langle A\nabla(u^{p/2}), \nabla\eta\rangle| u^{p/2}|\eta| \leq \frac{\varepsilon}{2}\langle A\nabla(u^{p/2}), \nabla(u^{p/2})\rangle\eta^2 + \frac{1}{2\varepsilon}\langle A\nabla\eta, \nabla\eta\rangle u^p$$

with $\varepsilon = 4|(p-1)/p|$ to estimate the last integral on the right, and then combine terms, we get

$$\operatorname{sign}\left(\frac{p-1}{p}\right)\int_{\Omega}(\eta^2u^p)\big|_{t=\tau_1}^{\tau_2}v\,dx$$

$$+2\left|\frac{p-1}{p}\right|\int_{\Omega}\int_{\tau_1}^{\tau_2}\langle A\nabla(u^{p/2}),\nabla(u^{p/2})\rangle\eta^2\,dx\,dt$$

$$\leq\int_{\Omega}\int_{\tau_1}^{\tau_2}|(\eta^2)_t|u^pv\,dx\,dt+2\left|\frac{p}{p-1}\right|\int_{\Omega}\int_{\tau_1}^{\tau_2}\langle A\nabla\eta,\nabla\eta\rangle u^p\,dx\,dt.$$

Lemma (2.7) now follows immediately by applying the degeneracy condition (1.2) to the second and fourth integrals.

Given $(x_0,t_0)\in\mathbf{R}^{n+1}$, we will use the notation

$$B_r=B_r(x_0)=\{x:|x-x_0|<r\},$$
$$R_{r,s}=R(r,x_0;s,t_0)=B_r(x_0)\times(t_0-s,t_0+s).$$

For fixed r,s we will consider ρ, σ satisfying

$$r/2<\rho<r,\qquad s/2<\sigma<s,$$

and for simplicity we will write

$$\begin{aligned}&B=B_r,\quad R=R_{r,s},\\&B'=B_\rho,\quad R'=R_{\rho,\sigma}=B'\times(t_0-\sigma,t_0+\sigma),\\&R'^{+}=B'\times(t_0-\sigma,t_0+s),\\&R'^{-}=B'\times(t_0-s,t_0+\sigma).\end{aligned}\tag{2.8}$$

LEMMA (2.9). *Let $2\leq p<\infty$ and u be a subsolution of* (1.1) *in $R=R_{r,s}$. Let $w_2\in A_2$ and assume that* (a), (b) *or* (c) *of Lemma* (2.2) *holds. Let ρ and σ satisfy $r/2<\rho<r$, $s/2<\sigma<s$ and let B' and R'^{+} be as above. Then*

$$\operatorname*{ess\,sup}_{\tau\in(t_0-\sigma,t_0+s)}\int_{B'}u_+(x,\tau)^pv\,dx+\frac{p-1}{p}\iint_{R'^{+}}|\nabla(u_+^{p/2})|^2w_1\,dx\,dt$$
$$\leq c\iint_R u_+^p\left[\frac{p}{p-1}\frac{w_2}{(r-\rho)^2}+\frac{v}{s-\sigma}\right]dx\,dt$$

with c independent of all the parameters.

Proof. We may assume that $u_+\in L^2_{w_2+v}(R)$ or there is nothing to prove. The lemma will follow by applying Lemma (2.4) with $\Omega=B_r=B$ and $(a,b)=(t_0-s,t_0+s)$. In fact, pick $\eta(x,t)$ to be zero in a neighborhood of $\{\partial B\times(t_0-s,t_0+s)\}\cup\{B\times(t=t_0-s)\}$, i.e., $\eta=0$ near ∂R except on the top, $\eta=1$ in R'^{+}, $|\nabla\eta|\leq c/(r-\rho)$, $|\eta_t|\leq c/(s-\sigma)$. Apply the conclusion of Lemma (2.4) with this η and with τ_1 chosen so close to $t_0 - s$ that $\eta(x, \tau_1) = 0$ for all $x \in B$. Then the first term on the left in Lemma (2.4) equals

$$\int_B \eta(x,\tau_2)^2 u_+(x,\tau_2)^p v(x)dx, \tag{2.10}$$

which is non-negative and which, when $\tau_2 \in (t_0 - \sigma, t_0 + s)$, is at least as large as

$$\int_{B'} u_+(x,\tau_2)^p v(x)dx. \tag{2.11}$$

Since the right side of the conclusion of Lemma (2.4) is bounded by the expression on the right in Lemma (2.9), we see (by dropping the second term on the left in Lemma (2.4)) that this expression is also a bound for (2.11) for a.e. $\tau_2 \in (t_0 - \sigma, t_0 + s)$. This proves part of the conclusion of Lemma (2.9), i.e., the part involving the ess sup of (2.11). For the other part, we use Lemma (2.4) with the same function η and the same τ_1 as above, drop the first term on the left in Lemma (2.4) (i.e., drop (2.10), which is non-negative) and let $\tau_2 \nearrow t_0 + s$ in the second term. The desired estimate then follows since $\eta = 1$ on R'^+.

In a similar way, we obtain the following corollary of Lemma (2.7).

LEMMA (2.12). *Let* $-\infty < p < \infty$, $p \neq 0, 1$, *let* u *satisfy* $0 < m \leq u(x,t) \leq M < \infty$ *in* $R = R_{r,s}$, $w_2 \in A_2$, *and suppose that* (a), (b) *or* (c) *of Lemma* (2.2) *holds. Let* ρ *and* σ *satisfy* $r/2 < \rho < r$, $s/2 < \sigma < s$, *and let* B', R'^+ *and* R'^- *be defined by* (2.8). *Then if* $p > 1$ *and* u *is a subsolution of* (1.1) *in* R, *or if* $p < 0$ *and* u *is a supersolution in* R,

$$\operatorname*{ess\,sup}_{\tau \in (t_0-\sigma, t_0+s)} \int_{B'} u(x,\tau)^p v\,dx + \frac{p-1}{p} \iint_{R'^+} |\nabla(u^{p/2})|^2 w_1\,dx\,dt$$
$$\leq c \iint_R u^p \left[\frac{p}{p-1}\frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma}\right] dx\,dt.$$

Moreover, if $0 < p < 1$ *and* u *is a supersolution in* R, *then*

$$\operatorname*{ess\,sup}_{\tau \in (t_0-s, t_0+\sigma)} \int_{B'} u(x,\tau)^p v\,dx + \left|\frac{p-1}{p}\right| \iint_{R'^-} |\nabla(u^{p/2})|^2 w_1\,dx\,dt$$
$$\leq c \iint_R u^p \left[\left|\frac{p}{p-1}\right| \frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma}\right] dx\,dt.$$

In any case, the constant c *is independent of all parameters.*

Proof. In case $p < 0$ or $p > 1$, we have $\operatorname{sign}((p-1)/p) > 0$, and the proof is essentially the same as that of Lemma (2.9), using Lemma (2.7) instead of Lemma (2.4). If $0 < p < 1$ then $\operatorname{sign}((p-1)/p) < 0$, and the proof is again like that of Lemma (2.9) except that we choose η to vanish on ∂R except on the top, with $\eta = 1$ on R'^{-}.

3. Mean value inequalities. In this section, we will use the following Sobolev interpolation inequality: there exists $h > 1$ so that

$$(3.1)\qquad \frac{1}{w(\widetilde{B})}\int_{\widetilde{B}} |F(x)|^{2h} w(x)\,dx \le c\left[\frac{1}{v(\widetilde{B})}\int_{\widetilde{B}} F(x)^2 v(x)\,dx\right]^{h-1}$$
$$\times\left[\frac{|\widetilde{B}|^{2/n}}{w_1(\widetilde{B})}\int_{\widetilde{B}} |\nabla F(x)|^2 w_1(x)\,dx + \frac{1}{v(\widetilde{B})}\int_{\widetilde{B}} F(x)^2 v(x)\,dx\right],$$

where $\widetilde{B}$ is a ball in $\mathbf{R}^n$ and c is independent of $\widetilde{B}$ and F. We will need this inequality with w taken to be either w_2 or v. Such inequalities are studied in [8] in case $F \in \operatorname{Lip}(\widetilde{B})$, but the results hold in general if $v, w_1 \in A_2$. In fact, if $v, w_1 \in A_2$, then for any F with $F \in L^2_v(\widetilde{B})$ and $\nabla F \in L^2_{w_1}(\widetilde{B})$, a limiting argument like that in the proof of Lemma (2.2) shows that (3.1) holds; for the left side we only need to use Fatou's lemma. On the other hand, if $F \notin L^2_v(\widetilde{B})$ or if $\nabla F \notin L^2_{w_1}(\widetilde{B})$, then (3.1) is trivially true. It is shown in [8] that (3.1) is valid with $w = w_2$ and some $h > 1$ for all $F \in \operatorname{Lip}(\widetilde{B})$ if *either* of the following is true:

(i) $w_1, v \in A_2$, $w_2 \in D$ (i.e., (1.6) holds for w_2) and Poincaré's inequality (1.9) holds for w_1, w_2 with some $q > 2$ and $\mu \equiv 1$;

(i)′ $v/w_2 \in A_2(w_2)$, $w_1 \in D$, $w_2 \in D$ and Poincaré's inequality (1.9) holds for w_1, w_2 with some $q > 2$ and $\mu = w_2$.

Moreover, (3.1) is valid with $w = v$ and some $h > 1$ for all $F \in \operatorname{Lip}(\widetilde{B})$ if

(ii) $w_1, v \in D$ and Poincaré's inequality (1.9) holds for w_1, v with some $q > 2$ and $\mu = v$.

Since we will also use the results from §2, we need the weights to satisfy the conditions there too. Thus, we will assume throughout this section that

(3.2)
(a) $w_1, w_2, v \in A_2$,
(b) Poincaré's inequality holds for both of the pairs w_1, w_2 and w_1, v with some $q > 2$ and $\mu \equiv 1$.

These assumptions are made for simplicity and can be varied somewhat.

In the following lemma, we use the notation described in (2.8), and we write

$$⨌_R f(x,t)m(x,t)\,dx\,dt = \iint_R f(x,t)m(x,t)\,dx\,dt \Big/ \iint_R m(x,t)\,dx\,dt.$$

Given $R = R_{r,s}$ and $R' = R_{\rho,\sigma}$ with $r/2 < \rho < r$ and $s/2 < \sigma < s$, let

$$C = c\frac{r^2 s}{(r-\rho)^2(s-\sigma)}, \tag{3.3}$$

where c is a positive constant which may be different at different occurrences, but which only depends on the weights and on h, h being the index for which (3.1) holds for both $w = w_2$ and $w = v$. We also write $\lambda(B) = w_1(B)/v(B)$ and $\Lambda(B) = w_2(B)/v(B)$ for a ball $B \subset \mathbf{R}^n$.

LEMMA (3.4). *Assume that* (3.2) *holds, let* $0 < p < \infty$, $r, s > 0$, $r/2 < \rho < r$, $s/2 < \sigma < s$, *and let* R, R'^{+} *be defined as in* (2.8). *If* u *is a subsolution of* (1.1) *in* R, *then* u_+ *is bounded on* R'^{+} *and*

$$\operatorname*{ess\,sup}_{R'^{+}} u_+^p \le D\left[\frac{r^2}{s}\frac{1}{\lambda(B)} + 1\right]^{1/(h-1)} \left[\frac{s}{r^2}\Lambda(B) + 1\right]^{h/(h-1)} \times ⨌_R u_+^p \left[\frac{s}{r^2}w_2 + v\right] dx\,dt,$$

where

$$D \le \begin{cases} C^{h/(h-1)} & \text{if } p \ge 2, \\ c^{\log(3/p)} C^c & \text{if } 0 < p < 2, \end{cases}$$

C being given by (3.3).

Proof. We will prove the case $p \ge 2$ by an iteration argument, and then deduce the case $0 < p < 2$ by using a method due to Hardy and Littlewood.

By applying the interpolation inequality (3.1) with $w = w_2$, $F(x) = u_+(x,\tau)^{p/2}$ and $\widetilde{B} = B_\rho = B'$, and combining this inequality with the part of the estimate in Lemma (2.9) involving the ess sup term, we obtain

$$\begin{aligned} &\frac{1}{w_2(B')}\int_{B'} u_+(x,\tau)^{ph} w_2\,dx \\ &\le c\left[\frac{1}{v(B')}\iint_R u_+^p\left(\frac{p}{p-1}\frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma}\right)dx\,dt\right]^{h-1} \\ &\quad\times \left[\frac{\rho^2}{w_1(B')}\int_{B'} |\nabla(u_+(x,\tau)^{p/2})|^2 w_1\,dx \right. \\ &\qquad\left. + \frac{1}{v(B')}\iint_R u_+^p\left(\frac{p}{p-1}\frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma}\right)dx\,dt\right] \end{aligned}$$

for a.e. $\tau \in (t_0 - \sigma, t_0 + s)$. Now integrate both sides with respect to τ over $(t_0 - \sigma, t_0 + s)$, recall that $R'^{+} = B' \times (t_0 - \sigma, t_0 + s)$, and use the part of the estimate in Lemma (2.9) involving the gradient term to bound the right side of the resulting expression. This gives

$$\frac{1}{w_2(B')} \iint_{R'^{+}} u_+(x,\tau)^{ph} w_2 \, dx\, dt \le \frac{c}{v(B')^{h-1}} \left[\frac{p}{p-1} \frac{\rho^2}{w_1(B')} + \frac{s+\sigma}{v(B')} \right]$$
$$\times \left[\iint_R u_+^p \left(\frac{p}{p-1} \frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma} \right) dx\, dt \right]^h .$$

Thus, since $r/2 < \rho < r$ and $s/2 < \sigma < s$, we obtain from the doubling property of the weights

$$(3.5) \qquad \frac{1}{w_2(B')} \iint_{R'^{+}} u_+^{ph} w_2 \, dx\, dt$$
$$\le \frac{c}{v(B)^h} \left(\frac{p}{p-1} \frac{r^2}{\lambda(B)} + s \right) \left[\iint_R u_+^p \left(\frac{p}{p-1} \frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma} \right) dx\, dt \right]^h .$$

A similar inequality holds with w_2 replaced by v on the left, and if we add the two inequalities, we obtain

$$(3.6) \qquad \iint_{R'^{+}} u_+^{ph} \left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)} \right) dx\, dt$$
$$\le \frac{c}{v(B)^h} \left(\frac{p}{p-1} \frac{r^2}{\lambda(B)} + s \right) \left[\iint_R u_+^p \left(\frac{p}{p-1} \frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma} \right) dx\, dt \right]^h .$$

Note that

$$\frac{p}{p-1} \frac{w_2}{(r-\rho)^2} + \frac{v}{s-\sigma} \le \frac{r^2}{(r-\rho)^2 (s-\sigma)} \left(\frac{p}{p-1} \frac{s}{r^2} w_2 + v \right),$$
$$\iint_{R'^{+}} \left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)} \right) dx\, dt \approx s,$$

$$\iint_R \left(\frac{p}{p-1} \frac{s}{r^2} w_2 + v \right) dx\, dt \approx s \left(\frac{p}{p-1} \frac{s}{r^2} w_2(B) + v(B) \right)$$
$$= s v(B) \left(\frac{p}{p-1} \frac{s}{r^2} \Lambda(B) + 1 \right).$$

Thus, we obtain from (3.6) by raising both sides of (3.6) to the power $1/h$ that

$$\left[\iint_{R'^+} u_+^{ph}\left(\frac{w_2}{w_2(B)}+\frac{v}{v(B)}\right)dx\,dt\right]^{1/h} \le C\left[\frac{p}{p-1}\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h} \times\left[\frac{p}{p-1}\frac{s}{r^2}\Lambda(B)+1\right]\iint_R u_+^p\left(\frac{p}{p-1}\frac{s}{r^2}w_2+v\right)dx\,dt, \tag{3.7}$$

where C is defined by (3.3).

Note that the inequality

$$\frac{a+b}{c+d}\le\frac{a}{c}+\frac{b}{d}\qquad\text{if } a,b\ge 0,\ c,d>0, \tag{3.8}$$

gives

$$\frac{sr^{-2}w_2(x)+v(x)}{sr^{-2}w_2(B)+v(B)}\le\frac{w_2(x)}{w_2(B)}+\frac{v(x)}{v(B)}\,.$$

Using this inequality on the left side of (3.7) and noting that $p\ge 2$ implies $1<p/(p-1)\le 2$, we obtain from (3.7) for $p\ge 2$,

$$\left[\iint_{R'^+} u_+^{ph}\left(\frac{s}{r^2}w_2+v\right)dx\,dt\right]^{1/h} \le C\left[\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h}\left[\frac{s}{r^2}\Lambda(B)+1\right]\iint_R u_+^p\left(\frac{s}{r^2}w_2+v\right)dx\,dt. \tag{3.9}$$

We now wish to iterate (3.9). Fix r, s, ρ, σ with $r/2<\rho<r$ and $s/2<\sigma<s$. For $k=1,2,\dots$, define sequences $\{r_k\}_{k=1}^\infty$ and $\{s_k\}_{k=1}^\infty$, by $r_1=r$, $r_k-r_{k+1}=(r-\rho)/2^k$ for $k\ge 1$, and $s_1=s$, $s_k-s_{k+1}=(s-\sigma)/2^k$ for $k\ge 1$. Also, define

$$R_k=B_{r_k}\times(t_0-s_k,t_0+s),\qquad k\ge 1,$$

so that $R_1=R$ and $\bigcap_{k=1}^\infty R_k\supset R'^+$. Since $\frac{1}{2}sr^{-2}\le s_kr_k^{-2}\le 4sr^{-2}$, we have by (3.9) with p replaced by ph^{k-1}, $p\ge 2$,

$$\left[\iint_{R_{k+1}} u_+^{ph^k}\left(\frac{s}{r^2}w_2+v\right)dx\,dt\right]^{1/h^k} \le\left\{C_k\left[\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h}\left[\frac{s}{r^2}\Lambda(B)+1\right]\right\}^{1/h^{k-1}} \times\left[\iint_{R_k} u_+^{ph^{k-1}}\left(\frac{s}{r^2}w_2+v\right)dx\,dt\right]^{1/h^{k-1}} \tag{3.10}$$

where

$$C_k = c\frac{r_k^2 s_k}{(r_k - r_{k+1})^2(s_k - s_{k+1})} \le 2^{3k}C.$$

In particular,

$$\text{(3.11)} \qquad \operatorname*{ess\,sup}_{R'+} u_+^p \le \prod_{k=1}^{\infty}\left\{2^{3k}C\left[\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h}\left[\frac{s}{r^2}\Lambda(B)+1\right]\right\}^{1/h^{k-1}} \times \iint_R u_+^p\left(\frac{s}{r^2}w_2+v\right)dx\,dt.$$

Clearly,

$$\sum_{k=1}^{\infty}\frac{1}{h^{k-1}} = \frac{h}{h-1} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{k}{h^{k-1}} = \left(\frac{h}{h-1}\right)^2.$$

Thus (3.11) implies that

$$\operatorname*{ess\,sup}_{R'+} u_+^p \le C^{h/(h-1)}\left[\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/(h-1)} \times \left[\frac{s}{r^2}\Lambda(B)+1\right]^{h/(h-1)} \iint_R u_+^p\left(\frac{s}{r^2}w_2+v\right)dx\,dt.$$

This completes the proof of the estimates in Lemma (3.4) when $p \ge 2$. Since u is a subsolution, the integral on the right is finite by definition when $p = 2$, and it follows that u_+ is bounded on R'^+.

For $0 < p \le 2$ and $\frac{1}{2} < \alpha, \beta \le 1$, define

$$I_p(\alpha,\beta)^p = \left[\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/(h-1)}\left[\frac{s}{r^2}\Lambda(B)+1\right]^{h/(h-1)} \times \iint_{R^+_{\alpha r,\beta s}} u_+^p\left(\frac{s}{r^2}w_2+v\right)dx\,dt$$

where

$$R^+_{\alpha r,\beta s} = B_{\alpha r} \times (t_0 - \beta s, t_0 + s).$$

Note that if $\alpha = \beta = 1$, then $R^+_{r,s}$ just means $R_{r,s}$. Also define

$$I_\infty(\alpha,\beta) = \operatorname*{ess\,sup}_{R^+_{\alpha r,\beta s}} u_+.$$

From the inequality already derived for $p = 2$, using the doubling property of the weights, we have for $\frac{1}{2} < \alpha' < \alpha \le 1$ and $\frac{1}{2} < \beta' < \beta \le 1$,

$$\text{(3.12)} \qquad I_\infty(\alpha',\beta')^2 \le c\left\{\frac{1}{(\alpha-\alpha')^2(\beta-\beta')}\right\}^{h/(h-1)} I_2(\alpha,\beta)^2.$$

If $0 < p < 2$,

$$I_2(\alpha,\beta)^2 \le I_\infty(\alpha,\beta)^{2-p} I_p(\alpha,\beta)^p$$

and, therefore, since $I_p(\alpha,\beta)^p \le c I_p(1,1)^p$ by doubling, and since we may assume that $I_p(1,1) = 1$,

$$I_\infty(\alpha',\beta')^2 \le c\left\{\frac{1}{(\alpha-\alpha')^2(\beta-\beta')}\right\}^{h/(h-1)} I_\infty(\alpha,\beta)^{2-p}.$$

Letting $\theta = (2-p)/2$, taking logarithms and observing that the expression above in curly brackets exceeds 8, we obtain

$$\log I_\infty(\alpha',\beta')^2 \le c \log \frac{1}{(\alpha-\alpha')^2(\beta-\beta')} + \theta \log I_\infty(\alpha',\beta'). \tag{3.13}$$

Now fix ρ, σ with $r/2 < \rho < r$, $s/2 < \sigma < s$, and write $\rho = \alpha_0 r$, $\sigma = \beta_0 s$ for appropriate α_0, β_0 with $\frac{1}{2} < \alpha_0, \beta_0 < 1$. Consider sequences $\{\alpha_j\}_{j=0}^\infty$, $\{\beta_j\}_{j=0}^\infty$ with

$$\alpha_j = \alpha_0 + (1-\alpha_0)\Big(\sum_{k=1}^{j} k^{-2}/c_0\Big), \qquad j \ge 1,\ c_0 > \sum_{k=1}^\infty k^{-2},$$

and β_j defined similarly. Choose α', α and β', β to be successive entries in $\{\alpha_j\}$ and $\{\beta_j\}$, respectively, and note that

$$(\alpha_{j+1}-\alpha_j)^2 = (1-\alpha_0)^2[(j+1)^{-2}/c_0]^2, \ (\beta_{j+1}-\beta_j) \ge (1-\beta_0)[(j+1)^{-2}/c_0]^2.$$

Also, $(1-\alpha_0)^2(1-\beta_0) = (r-\rho)^2(s-\sigma)/r^2 s$. By iteration (with a different constant c in C),

$$\log I_\infty(\alpha_0,\beta_0) \le c \sum_{j=0}^{k} \theta^j \log[(j+1)C] + \theta^{k+1} \log I_\infty(\alpha_{k+1},\beta_{k+1}).$$

The second term on the right tends to 0 as $k \to \infty$ since $\theta^{k+1} \to 0$ (note $0 < \theta < 1$) and since $c_0 > \sum_{k=1}^\infty k^{-2}$ and u_+ is bounded on each R'^+. But it can be shown that there is a constant c independent of θ such that

$$\sum_{j=0}^\infty \theta^j \{\log(j+1) + \log C\} \le \frac{c}{1-\theta}\left\{\log \frac{c}{1-\theta} + \log C\right\}.$$

Therefore, since $1-\theta = p/2$,

$$\log I_\infty(\alpha_0,\beta_0) \le \frac{c}{p}\left\{\log \frac{c}{p} + \log C\right\},$$

$$I_\infty(\alpha_0,\beta_0)^p \le c^{\log(3/p)} C^c = c^{\log(3/p)} C^c I_p(1,1)^p.$$

This completes the proof of Lemma (3.4) in all cases.

If u is a solution, then by applying Lemma (3.4) to both u and $-u$, we immediately obtain Theorem B of the introduction, with α, β, α', β' there taken to be r, s, ρ, σ respectively.

We will now derive a mean value inequality for u^p when $-\infty < p < \infty$ and u is a non-negative solution. We begin by noting the following analogues of (3.7), which are proved in exactly the same way as (3.7) except that Lemma (2.12) is used instead of Lemma (2.9). We assume that (3.2) holds, that $0 < m \le u(x,t) \le M < \infty$ in $R = R_{r,s}$, and that $r/2 < \rho < r$, $s/2 < \sigma < s$. Then if $p > 1$ and u is a subsolution in R, or if $p < 0$ and u is a supersolution in R,

$$(3.14)\quad \left[\iint_{R'^+} u^{ph}\left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)}\right) dx\, dt\right]^{1/h} \le$$

$$C\left[\frac{p}{p-1}\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h}\left[\frac{p}{p-1}\frac{s}{r^2}\Lambda(B)+1\right]\iint_R u^p\left(\frac{p}{p-1}\frac{s}{r^2}w_2+v\right) dx\, dt,$$

where C is defined by (3.3). Note that $p/(p-1) > 0$ in either case. Moreover, if $0 < p < 1$ and u is a supersolution in R, then

$$(3.15)\quad \left[\iint_{R'^-} u^{ph}\left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)}\right) dx\, dt\right]^{1/h}$$

$$\le C\left[\frac{p}{|p-1|}\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h}\left[\frac{p}{|p-1|}\frac{s}{r^2}\Lambda(B)+1\right]$$

$$\times \iint_R u^p\left(\frac{p}{|p-1|}\frac{s}{r^2}w_2+v\right) dx\, dt.$$

We also note that (3.14) and (3.15) remain true if the integrals on the right are replaced by the larger integral

$$(3.16)\quad \iint_R u^p\left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)}\right) dx\, dt;$$

the fact that (3.16) is larger follows from (3.8).

LEMMA (3.17). *Assume that* (3.2) *holds*, $r, s > 0$, $r/2 < \rho < r$, $s/2 < \sigma < s$, *and let* R, R', R'^+ *be defined as in* (2.8). *If* u *is a non-negative solution of* (1.1) *in* R *then for* $p > 0$,

$$(3.18)\quad \operatorname*{ess\,sup}_{R'} u^p \le C^c\left[p\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/(h-1)}\left[p\frac{s}{r^2}\Lambda(B)+1\right]^{h/(h-1)}$$

$$\times \iint_R u^p\left[\frac{w_2}{w_2(B)} + \frac{v}{v(B)}\right] dx\, dt,$$

and for $p < 0$,

$$\operatorname*{ess\,sup}_{R'^+} u^p \le C^{h/(h-1)} \left[|p| \frac{r^2}{s} \frac{1}{\lambda(B)} + 1 \right]^{1/(h-1)} \times \left[|p| \frac{s}{r^2} \Lambda(B) + 1 \right]^{h/(h-1)} \iint_R u^p \left[\frac{w_2}{w_2(B)} + \frac{v}{v(B)} \right] dx\, dt, \tag{3.19}$$

where C is defined by (3.3).

Proof. Suppose first that $p > 0$, and let u be a non-negative solution in R. We may assume that $u(x,t) \ge m > 0$ in R by considering the functions $u(x,t) + m$ for $m > 0$, and then letting $m \to 0$. Moreover, since u is bounded on R'^+ by Theorem B, we may assume by replacing R by a slightly smaller set of this type that $u(x,t) \le M < \infty$ in R. Then, by combining (3.14) and (3.15), and by noting that R' is a subset of both R'^+ and R'^-, we have for $p > 0,\ p \ne 1$,

$$\left[\iint_{R'} u^{ph} \left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)} \right) dx\, dt \right]^{1/h} \le C \left[\frac{p}{|p-1|} \frac{r^2}{s} \frac{1}{\lambda(B)} + 1 \right]^{1/h} \left[\frac{p}{|p-1|} \frac{s}{r^2} \Lambda(B) + 1 \right] \times \iint_R u^p \left(\frac{p}{|p-1|} \frac{s}{r^2} w_2 + v \right) dx\, dt. \tag{3.20}$$

Recall that the integral on the right side is bounded by (3.16). Now fix r, s, ρ, σ with $r/2 < \rho < r$, $s/2 < \sigma < s$, and define $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ as in the proof of Lemma (3.4), and let $R_k = R_{r_k, s_k}$ for $k \ge 1$. Then $R_1 = R$ and $R' \subset \bigcap R_k$. For fixed $p > 0$ satisfying $ph^{k-1} - 1 \ne 0$ for all $k = 1, 2, \ldots$, we successively apply first inequality (3.20) and then, for $k \ge 2$, the inequalities

$$\left[\iint_{R_{k+1}} u^{ph^k} \left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)} \right) dx\, dt \right]^{1/h^k} \le \left\{ C_k \left[\frac{ph^{k-1}}{|ph^{k-1} - 1|} \frac{r_k^2}{s_k} \frac{1}{\lambda(B_{r_k})} + 1 \right]^{1/h} \left[\frac{ph^{k-1}}{|ph^{k-1} - 1|} \frac{s_k}{r_k^2} \Lambda(B_{r_k}) + 1 \right] \right\}^{1/h^{k-1}} \times \left[\iint_{R_k} u^{ph^{k-1}} \left(\frac{w_2}{w_2(B)} + \frac{v}{v(B)} \right) dx\, dt \right]^{1/h^{k-1}},$$

where $C_k \le 2^{3k} C$ as in the proof of Lemma (3.4). We initially choose p to be one of the values $\frac{1}{2} h^j (h+1),\ j = 0, \pm 1, \pm 2, \ldots$ Then it is easy to see that $|p - 1| \ge (h-1)/2h > 0$. Since ph^{k-1} has the same form as p, it follows that $|ph^{k-1} - 1| \ge (h-1)/2h$ for all $k \ge 1$, and from this and the fact that

$\lambda(B_{r_k}) \approx \lambda(B)$ due to the doubling property of the weights, we obtain

$$\frac{ph^{k-1}}{|ph^{k-1}-1|}\frac{r_k^2}{s_k}\frac{1}{\lambda(B_{r_k})}+1 \le c_1\left[ph^{k-1}\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]$$

with c_1 depending only on h and the doubling constants of the weights. A similar estimate holds for the factor which involves Λ. Hence, by incorporating c_1 into the constant c in (3.3), we obtain by iteration

$$\text{(3.21)} \quad \operatorname*{ess\,sup}_{R'} u^p$$
$$\le \prod_{k=1}^{\infty}\left\{2^{3k}C\left[ph^{k-1}\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h}\left[ph^{k-1}\frac{s}{r^2}\Lambda(B)+1\right]\right\}^{1/h^{k-1}}$$
$$\times \iint_R u^p\left(\frac{p}{|p-1|}\frac{s}{r^2}w_2+v\right)dx\,dt.$$

As in the proof of Lemma (3.4),

$$\prod_{k=1}^{\infty}(2^{3k}C)^{1/h^{k-1}} = cC^{h/(h-1)},$$

$$\prod_{k=1}^{\infty}\left(ph^{k-1}\frac{s}{r^2}\Lambda(B)+1\right)^{1/h^{k-1}} \le \prod_{k=1}^{\infty}\left[h^{k-1}\left(p\frac{s}{r^2}\Lambda(B)+1\right)\right]^{1/h^{k-1}}$$
$$= c\left(p\frac{s}{r^2}\Lambda(B)+1\right)^{h/(h-1)},$$

$$\prod_{k=1}^{\infty}\left(ph^{k-1}\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right)^{1/h^{k-1}} \le c\left(p\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right)^{h/(h-1)}.$$

Hence,

$$\text{(3.22)} \quad \operatorname*{ess\,sup}_{R'} u^p \le C^{h/(h-1)}\left[p\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/(h-1)}\left[p\frac{s}{r^2}\Lambda(B)+1\right]^{h/(h-1)}$$
$$\times \iint_R u^p\left[\frac{p}{|p-1|}\frac{s}{r^2}w_2+v\right]dx\,dt.$$

This inequality is valid for $p=\frac{1}{2}h^j(h+1)$, $j=0,\pm1,\pm2,\dots$ in case u is a non-negative solution in R, and since the integral on the right is bounded by (3.16), (3.22) implies (3.18) for this particular sequence of p values.

To prove (3.18) for a value of $p>0$ which is not in the sequence above, we again use the technique of Hardy and Littlewood, with a few modifications. Choose j so that $\frac{1}{2}h^j(h+1)<p<\frac{1}{2}h^{j+1}(h+1)$. Let $\bar{p}=\frac{1}{2}h^{j+1}(h+1)$, so that $\bar{p}/h < p < \bar{p}$. For $\frac{1}{2} < \alpha, \beta \leq 1$, define

$$I_p(\alpha,\beta)^p = \left[p\frac{r^2}{s}\frac{1}{\lambda(B)} + 1\right]^{1/(h-1)} \left[p\frac{s}{r^2}\Lambda(B) + 1\right]^{h/(h-1)} \times \iint_{R_{\alpha r, \beta s}} u^p \left[\frac{w_2}{w_2(B)} + \frac{v}{v(B)}\right] dx\, dt$$

and

$$I_\infty(\alpha,\beta) = \operatorname*{ess\,sup}_{R_{\alpha r,\beta s}} u.$$

Using the inequality already derived for $p = \bar{p}$, we have for $\frac{1}{2} < \alpha' < \alpha \leq 1$ and $\frac{1}{2} < \beta' < \beta \leq 1$ (cf. (3.12))

$$I_\infty(\alpha',\beta')^{\bar{p}} \leq c\left\{\frac{1}{(\alpha-\alpha')^2(\beta-\beta')}\right\}^{h/(h-1)} I_{\bar{p}}(\alpha,\beta)^{\bar{p}}.$$

Since

$$\bar{p}\frac{r^2}{s}\frac{1}{\lambda(B)} + 1 \leq \left[p\frac{r^2}{s}\frac{1}{\lambda(B)} + 1\right]\frac{\bar{p}}{p} \leq \left[p\frac{r^2}{s}\frac{1}{\lambda(B)} + 1\right]h,$$

and a similar inequality holds for $p(s/r^2)\Lambda(B) + 1$,

$$\begin{aligned} I_{\bar{p}}(\alpha',\beta')^{\bar{p}} &\leq h^{(h+1)/(h-1)} I_\infty(\alpha,\beta)^{\bar{p}-p} I_p(\alpha,\beta)^p \\ &\leq c_1 I_\infty(\alpha,\beta)^{\bar{p}-p} I_p(1,1)^p \end{aligned}$$

with c_1 independent of p, $\bar{p}$. Assuming as we may that $I_p(1,1) = 1$, we obtain

$$I_\infty(\alpha',\beta')^{\bar{p}} \leq c\left\{\frac{1}{(\alpha-\alpha')^2(\beta-\beta')}\right\}^{h/(h-1)} I_\infty(\alpha,\beta)^{\bar{p}-p},$$

or taking logarithms and letting $\theta = (\bar{p} - p)/\bar{p}$,

$$\log I_\infty(\alpha',\beta') \leq \frac{c}{\bar{p}}\log\frac{1}{(\alpha-\alpha')^2(\beta-\beta')} + \theta \log I_\infty(\alpha,\beta)$$

(cf. (3.13)).

Now fix ρ, σ with $r/2 < \rho < r$, $s/2 < \sigma < s$ and define $\{\alpha_j\}_{j=0}^\infty$, $\{\beta_j\}_{j=0}^\infty$ as in the proof of Lemma (3.4). As before,

$$\begin{aligned} \log I_\infty(\alpha_0,\beta_0) &\leq \frac{c}{\bar{p}}\sum_{j=0}^\infty \theta^j\{\log(j+1) + \log C\}, \\ &\leq \frac{c}{\bar{p}}\frac{1}{1-\theta}\left\{\log\frac{c}{1-\theta} + \log C\right\}. \end{aligned}$$

Since $1 - \theta = p/\bar{p} \geq 1/h$,

$$\log I_\infty(\alpha_0,\beta_0) \leq \frac{c}{p}\{\log ch + \log C\} \leq \frac{c}{p}\log C.$$

Hence $I_\infty(\alpha_0, \beta_0)^p \le C^c = C^c I_p(1,1)^p$. This completes the proof of Lemma (3.17) for $p > 0$.

The proof of Lemma (3.17) for $p < 0$ is similar but simpler since we do not have to avoid $p = 1$ in the iteration. For $p < 0$, assuming as we may that u is bounded above as well as below away from zero, we use (3.14) together with the observation that the integral on the right in (3.14) is bounded by (3.16). Arguing as in the case $p > 0$, except that now $|ph^{k-1} - 1| > 1$ at each stage, we obtain the following analogue of (3.21):

$$\begin{aligned}\operatorname*{ess\,sup}_{R'+} u^p &\le \prod_{k=1}^{\infty}\left\{2^{3k}C\left[|p|h^{k-1}\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/h}\left[|p|h^{k-1}\frac{s}{r^2}\Lambda(B)+1\right]\right\}^{1/h^{k-1}} \\ &\qquad\times \iint_R u^p\left(\frac{|p|}{|p-1|}\frac{s}{r^2}w_2+v\right)dx\,dt.\end{aligned}$$

Thus, by estimating the infinite product as usual, we get

$$\begin{aligned}\operatorname*{ess\,sup}_{R'+} u^p &\le C^{h/(h-1)}\left[|p|\frac{r^2}{s}\frac{1}{\lambda(B)}+1\right]^{1/(h-1)}\left[|p|\frac{s}{r^2}\Lambda(B)+1\right]^{h/(h-1)} \\ &\qquad\times \iint_R u^p\left(\frac{|p|}{|p-1|}\frac{s}{r^2}w_2+v\right)dx\,dt.\end{aligned}\tag{3.23}$$

This analogue of (3.22) is valid for all $p < 0$ and implies (3.19) by using (3.8). Note that (3.23) is actually a stronger conclusion than is stated in (3.19). This completes the proof of Lemma (3.17) in all cases.

4. The logarithm estimate.

We begin with

LEMMA (4.1). *Let v and w_1 be weights such that there exists $s > 1$ with*

$$\left[\frac{|I|}{|B|}\right]^{2/n}\left[\frac{1}{|I|}\int_I\left(\frac{v}{v(B)}\right)^s dx\right]^{1/s}\left[\frac{1}{|I|}\int_I\left(\frac{w_1}{w_1(B)}\right)^{-s} dx\right]^{1/s} \le c \tag{4.2}$$

for all balls I, B, $I \subset 2B$, where c is a constant independent of the balls. Let $B = B_r(x_0)$ and $\varphi \in C_0(B)$, $0 \le \varphi \le 1$, and suppose that the level sets of φ (i.e., $\{x : \varphi(x) \ge \lambda\}$) are convex. Then

$$\int_B |u(x) - A_B|^2\varphi(x)v(x)\,dx \le c\frac{|B|^2}{\varphi(B)^2}\frac{v(B)|B|^{2/n}}{w_1(B)}\int_B |\nabla u|^2\varphi w_1\,dx, \tag{4.3}$$

for all $u \in C^1(\mathbf{R}^n)$, $A_B = \varphi(B)^{-1} \int_B u(x)\varphi(x)\,dx$.

P r o o f. By inequality (2.7) of [4], for $x \in B$,

$$|u(x) - A_B|\sqrt{\varphi(x)} \leq c\frac{|B|}{\varphi(B)} \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\sqrt{\varphi(y)}\,dy.$$

Hence,

$$\int_B |u(x) - A_B|^2 \varphi(x) v(x)\,dx \leq c\frac{|B|^2}{\varphi(B)^2} \|I_1(\chi_B |\nabla u| \sqrt{\varphi})\|^2_{L^2_v},$$

where I_1 is the fractional integral of order 1. Then by (4.2) and the results of [14], the lemma follows.

R e m a r k s.

(4.4) Note that (4.3) implies an analogous inequality with

$$A_B = \frac{1}{(\varphi v)(B)} \int_B u\varphi v\,dx.$$

(4.5) In case $w_1 \in A_2$, condition (4.2) becomes

$$\left[\frac{|I|}{|B|}\right]^{2/n} \left[\frac{1}{|I|} \int_I \left(\frac{v}{v(B)}\right)^s dx\right]^{1/s} \leq \frac{1}{|I|}\frac{w_1(I)}{w_1(B)}\,. \tag{4.6}$$

If we assume that Sobolev or Poincaré inequalities hold for w_1, v with $q = 2$, then

$$\left[\frac{|I|}{|B|}\right]^{2/n} \frac{v(I)}{v(B)} \leq c\frac{w_1(I)}{w_1(B)}\,, \tag{4.7}$$

$I \subset 2B$ (see [2]). Note that (4.7) implies (4.6) for some $s > 1$ if $v \in A_\infty$. Therefore, if we assume $v \in A_\infty$ and $w_1 \in A_2$, then the validity of Sobolev or Poincaré inequalities implies the conclusion of Lemma (4.1).

(4.8) In case $1/v \in L^1_{\text{loc}}(\Omega)$ and $w_2 \in A_2$, we claim that inequality (4.3) holds if $u \in L^2_v(\Omega)$ and $\nabla u \in L^2_{w_2}(\Omega)$. This follows because then $u \in L^1_{\text{loc}}(\Omega)$, and if ψ_ε is a smooth approximation of the identity, we have for $2B \subset \Omega$

$$u_\varepsilon(x) = (\chi_{2B} u) * \psi_\varepsilon'(x) \to u(x) \quad \text{a.e. in } B,$$
$$u_\varepsilon \to u \quad \text{in } L^1(B),$$
$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } B.$$

Then using Fatou's lemma and (4.3), we have

$$\int_B |u(x) - A_B|^2 \varphi v\,dx \le \varliminf_{\varepsilon\to 0} \int_B |u_\varepsilon(x) - A_B(\varepsilon)|^2 \varphi v\,dx$$

$$\le c\frac{|B|^2}{\varphi(B)^2}\frac{v(B)|B|^{2/n}}{w_1(B)} \varliminf_{\varepsilon\to 0} \int_B |\nabla u_\varepsilon(x)|^2 w_1 \varphi\,dx.$$

But $|\nabla u_\varepsilon(x)| = |\nabla(\chi_{2B}u) * \psi_\varepsilon(x)| \le M(\nabla(\chi_{2B}u))(x)$, where M is the Hardy–Littlewood maximal operator. Since $w_2 \in A_2$, $M(\nabla(\chi_{2B}u)) \in L^2_{w_2}(2B)$, and the claim follows from the Lebesgue dominated convergence theorem and the fact that $w_1 \le w_2$.

LEMMA (4.9). *Suppose $v \in D$, $v^{-1} \in L^1_{\mathrm{loc}}(\Omega)$, $w_2 \in A_2$, (4.2) holds, and $w_2/v \in A_\infty(v)$. Let B_R be a ball of radius R, $t_0 \in (a,b)$ and*

$$\tilde{w}_2(x) = w_2(x)/w_2(B_R), \qquad \tilde{v}(x) = v(x)/v(B_R).$$

If u is a solution of (1.1) in $B_{3R/2} \times (a,b)$ which is bounded below by a positive constant, then there are constants c_1, M_2, δ and V such that if for $s > 0$ we define

$$E^+ = \{(x,t) \in B_R \times (t_0,b) : \log u < -s - M_2(b-t_0) - V\},$$
$$E^- = \{(x,t) \in B_R \times (a,t_0) : \log u > s - M_2(a-t_0) - V\},$$

then

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E^+) \le c_1 \left\{\frac{1}{s}\frac{v(B_R)}{w_1(B_R)}\frac{R^2}{b-t_0}\right\}^\delta (b-t_0), \tag{4.10}$$

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E^-) \le c_1 \left\{\frac{1}{s}\frac{v(B_R)}{w_1(B_R)}\frac{R^2}{t_0-a}\right\}^\delta (t_0-a). \tag{4.11}$$

Here c_1 and δ depend only on the constants in the conditions on v and w_2, $M_2 \approx w_2(B_R)/R^2 v(B_R)$ with the constants in this equivalence depending only on the constants in the conditions on v and w_2, and V is a constant which depends on u.

Proof. Let u be a solution in $B^* \times (a,b)$, where $B^* = B_{3R/2}$, and assume that $u \ge \varepsilon > 0$. Let $a < \tau_1 < \tau_2 < b$ and $\eta(x,t) = \psi(x)\chi(t,\tau_1,\tau_2)$ where $\psi \ge 0$ is a smooth function having compact support in B^*. Set $\vartheta = -\log u$ and let $\Phi = \eta^2(1/u)$. Since u^{-1} is bounded, Φ is an admissible test function by Lemma (2.2) (a), using the fact from [14] that (4.2) implies Sobolev's inequality (1.10) for w_1, v with $q = 2$. Then by (2.1)

$$\iint \eta^2 \vartheta_t v\,dx\,dt + \iint \langle A\nabla\vartheta, \eta^2\nabla\vartheta\rangle\,dx\,dt = -\iint \langle A\nabla\vartheta, \nabla\eta^2\rangle\,dx\,dt,$$

and consequently,

$$\int_{B^*} \vartheta \,|_{t=\tau_1}^{\tau_2} \psi^2 v\, dx + \int_{\tau_1}^{\tau_2} \int_{B^*} \psi^2 \langle A\nabla\vartheta, \nabla\vartheta\rangle \, dx\, dt = -2 \int_{\tau_1}^{\tau_2} \int_{B^*} \psi \langle A\nabla\vartheta, \nabla\psi\rangle \, dx\, dt. \tag{4.12}$$

By Schwarz's inequality,

$$\begin{aligned} &\Big| \int_{\tau_1}^{\tau_2} \int_{B^*} \psi \langle A\nabla\vartheta, \nabla\psi\rangle \, dx\, dt \Big| \\ &\le \Big[\int_{\tau_1}^{\tau_2} \int_{B^*} \psi^2 \langle A\nabla\vartheta, \nabla\vartheta\rangle \, dx\, dt \Big]^{1/2} \Big[\int_{\tau_1}^{\tau_2} \int_{B^*} \langle A\nabla\psi, \nabla\psi\rangle \, dx\, dt \Big]^{1/2} \\ &\le \frac{1}{4} \int_{\tau_1}^{\tau_2} \int_{B^*} \psi^2 \langle A\nabla\vartheta, \nabla\vartheta\rangle \, dx\, dt + \int_{\tau_1}^{\tau_2} \int_{B^*} \langle A\nabla\psi, \nabla\psi\rangle \, dx\, dt, \end{aligned}$$

where we have used the inequality $(ab)^{1/2} \le a/4 + b$. Therefore,

$$\int_{B^*} \vartheta \,|_{t=\tau_1}^{\tau_2} \psi^2 v\, dx + \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{B^*} \psi^2 |\nabla\vartheta|^2 w_1 \, dx\, dt \le 2 \int_{\tau_1}^{\tau_2} \int_{B^*} |\nabla\psi|^2 w_2 \, dx\, dt. \tag{4.13}$$

Now let

$$\psi(x) = \begin{cases} 1 & \text{in } B_R, \\ 0 & \text{in } (B^*)^c, \end{cases}$$

ψ smooth, $0 \le \psi \le 1$, $|\nabla\psi| \le c/R$ and let the level sets of ψ be convex. Then by (4.13), Lemma (4.1) and Remark (4.8) with $\varphi = \psi^2$, we obtain

$$\int_{B^*} \vartheta \,|_{t=\tau_1}^{\tau_2} \varphi v\, dx + c_1 \frac{w_1(B^*)\varphi(B_{2R})^2}{v(B^*)|B^*|^{2+2/n}} \int_{\tau_1}^{\tau_2} \int_{B^*} |\vartheta - V(t)|^2 \varphi v \, dx\, dt \le \frac{c}{R^2} w_2(B^*)(\tau_2 - \tau_1),$$

where

$$V(t) = \frac{1}{(\varphi v)(B^*)} \int_{B^*} \vartheta \varphi v \, dx.$$

Note that (4.12) implies that V can be written as an indefinite integral with respect to t, so that V is absolutely continuous in (a,b). Therefore,

$$\frac{dV}{dt}+M_1\frac{1}{(\varphi v)(B^*)}\int_{B^*}|\vartheta(x,t)-V(t)|^2\varphi v\,dx\leq M_2,$$

where

$$M_1=c_1\frac{w_1(B^*)\varphi(B^*)^2}{v(B^*)|B^*|^{2+2/n}}\,,\qquad M_2=C\frac{w_2(B^*)}{R^2(\varphi v)(B^*)}\,.$$

Now for fixed $t_0\in(a,b)$ we set

$$w(x,t)=\vartheta(x,t)-M_2(t-t_0)-V(t_0),\qquad W(t)=V(t)-M_2(t-t_0)-V(t_0).$$

Then we have

$$\frac{dW}{dt}+M_1\frac{1}{(\varphi v)(B^*)}\int_{B^*}|w(x,t)-W(t)|^2\varphi v\,dx\leq 0. \tag{4.14}$$

Given $s>0$ and $t\in(a,b)$, define

$$Q_s(t)=\{\,x\in B^*:w(x,t)>s\,\}.$$

By (4.14), $W(t)$ is non-increasing, and since $W(t_0)=0$ for $x\in Q_s(t)$ and $t\geq t_0$, we have $w(x,t)-W(t)\geq s-W(t)\geq s-W(t_0)=s>0$. Then by (4.14)

$$\frac{1}{[s-W(t)]^2}\frac{dW}{dt}+M_1\frac{(\varphi v)(Q_s(t))}{(\varphi v)(B^*)}\leq 0$$

for $t\geq t_0$, or equivalently,

$$\frac{1}{[s-W(t)]^2}\frac{d}{dt}(s-W)\geq M_1\frac{(\varphi v)(Q_s(t))}{(\varphi v)(B^*)}\,.$$

Integration of this inequality over $[t_0,b)$ leads to

$$\frac{1}{s}\geq M_1\frac{1}{(\varphi v)(B^*)}\int_{t_0}^{b}(\varphi v)(Q_s(t))\,dt.$$

Since v and w_1 are doubling and $\varphi\equiv 1$ in B_R, we obtain from this and the definition of M_1 that

$$\iint_{\{(x,t):x\in B_R,\,t_0<t<b,\,w(x,t)>s\}}v(x)\,dx\,dt\leq c\frac{1}{s}\frac{v(B_R)^2}{w_1(B_R)}R^2.$$

If for $E\subset\mathbf{R}^{n+1}$ we set $(v\otimes 1)(E)=\int_E v(x)\,dx\,dt$, then

$$\begin{aligned}(v\otimes 1)\{\,(x,t):x\in B_R,\ t_0<t<b,\ \log u<-s-M_2(b-t_0)-V(t_0)\,\}\\ \leq c\frac{1}{s}\frac{v(B_R)^2}{w_1(B_R)}R^2=c\frac{1}{s}(v\otimes 1)(B_R\times(t_0,b))\frac{v(B_R)}{w_1(B_R)}\frac{R^2}{(b-t_0)}\,.\end{aligned}\tag{4.15}$$

Defining

$$\widetilde{Q}_s(t)=\{\,x\in B^*:w(x,t)<-s\,\},\qquad s>0,$$

and observing that for $x \in \tilde{Q}_s(t)$ and $t < t_0$ we have $|w(x,t) - W(t)| \geq s + W(t)$, by an analogous argument we get

$$(4.16) \quad (v \otimes 1)\{(x,t) : x \in B_R,\ a < t < t_0,\ \log u > s - M_2(a-t_0) - V(t_0)\}$$
$$\leq c\frac{1}{s}\frac{v(B_R)^2}{w_1(B_R)}R^2 = c\frac{1}{s}(v \otimes 1)(B_R \times (t_0, a))\frac{v(B_R)}{w_1(B_R)}\frac{R^2}{t_0 - a}.$$

Now assume that $w_2/v \in A_\infty(v)$ and set

$$\tilde{w}_2(x) = w_2(x)/w_2(B_R), \qquad \tilde{v}(x) = v(x)/v(B_R).$$

Note that inequalities (4.15) and (4.16) can be written as

$$(4.17) \qquad (\tilde{v} \otimes 1)(E^+) \leq c\frac{1}{s}\frac{v(B_R)}{w_1(B_R)}R^2,$$

$$(4.18) \qquad (\tilde{v} \otimes 1)(E^-) \leq c\frac{1}{s}\frac{v(B_R)}{w_1(B_R)}R^2.$$

If $E \subset B_R \times (\alpha, \beta)$, then define $E_t = \{x \in B_R : (x,t) \in E\}$. Since $w_2/v \in A_\infty(v)$, there exists δ, $0 < \delta < 1$, such that $\tilde{w}_2(E_t) \leq c\tilde{v}(E_t)^\delta$ for all $t \in (\alpha, \beta)$. Then

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E) = \int_\alpha^\beta (\tilde{v} + \tilde{w}_2)(E_t)\,dt = \int_\alpha^\beta \tilde{v}(E_t)\,dt + \int_\alpha^\beta \tilde{w}_2(E_t)\,dt$$
$$\leq (\tilde{v} \otimes 1)(E) + c\int_\alpha^\beta \tilde{v}(E_t)^\delta\,dt \qquad \text{since } w_2/v \in A_\infty(v)$$
$$\leq (\tilde{v} \otimes 1)(E) + c(\beta - \alpha)^{1-\delta}\Big[\int_\alpha^\beta \tilde{v}(E_t)\,dt\Big]^\delta$$
$$= (\tilde{v} \otimes 1)(E) + c(\beta - \alpha)^{1-\delta}(\tilde{v} \otimes 1)(E)^\delta.$$

Note that $(\tilde{v} \otimes 1)(E) \leq (\tilde{v} \otimes 1)(B_R \times (\alpha, \beta)) = \beta - \alpha$. Thus, since $0 < \delta < 1$, $(\tilde{v} \otimes 1)(E) \leq (\beta - \alpha)^{1-\delta}(\tilde{v} \otimes 1)(E)^\delta$, and we obtain

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E) \leq c(\beta - \alpha)^{1-\delta}(\tilde{v} \otimes 1)(E)^\delta.$$

Combining this with (4.17) and (4.18) completes the proof of Lemma (4.9).

5. Bombieri's Lemma

LEMMA (5.1). *Let $R(\rho)$ be a one-parameter family of rectangles in $\mathbf{R}^{n+1}$, $R(\sigma) \subseteq R(\rho)$, $\frac{1}{2} \leq \sigma \leq \rho \leq 1$ and let ν be a doubling measure in $\mathbf{R}^{n+1}$. Let A, μ, M, m, θ and δ be positive constants such that $M \geq 1/\mu$, and suppose that f is a positive measurable function defined in a neighborhood of $R(1)$*

satisfying

$$\operatorname*{ess\,sup}_{R(\sigma)} f^p \le \frac{A}{(\rho-\sigma)^m} \iint_{R(\rho)} f^p \nu(x,t)\,dx\,dt \tag{5.2}$$

for all σ, ρ, p, $\frac{1}{2} \le \theta \le \sigma < \rho < 1$, $0 < p < M$, *and*

$$\nu\{(x,t) \in R(1) : \log f > s\} \le (\mu/s)^\delta \nu(R(1)) \qquad \textit{for all } s > 0. \tag{5.3}$$

Then there is a constant $\gamma = \gamma(A, m, \delta) > 0$ *such that*

$$\log(\operatorname*{ess\,sup}_{R(\theta)} f) \le \frac{\gamma}{(1-\theta)^{2m}}\mu.$$

Proof. For $\theta \le \rho \le 1$, set

$$\varphi(\rho) = \operatorname*{ess\,sup}_{R(\rho)} \log f.$$

The function φ is non-decreasing and we may assume $\varphi(\theta) > \mu$. For $\theta \le \rho < 1$ we write

$$\iint_{R(\rho)} f^p \nu\,dx\,dt = \iint_{R(\rho)\cap\{\log f > \varphi(\rho)/2\}} + \iint_{R(\rho)\cap\{\log f \le \varphi(\rho)/2\}} = \mathrm{I} + \mathrm{II}.$$

Then

$$\mathrm{II} \le e^{p\varphi(\rho)/2}\nu(R(\rho)).$$

Also since $f \le e^{\varphi(\rho)}$ a.e. on $R(\rho)$,

$$\begin{aligned} \mathrm{I} &\le e^{p\varphi(\rho)}\nu\{(x,t) \in R(1) : \log f > \varphi(\rho)/2\} \\ &\le \mu^\delta [2/\varphi(\rho)]^\delta \nu(R(1)) e^{p\varphi(\rho)} \qquad \text{by (5.3)}. \end{aligned}$$

Since $\nu(R(1)) \le c_1 \nu(R(\rho))$ for $\theta \le \rho \le 1$ and some $c_1 \ge 1$,

$$⨌_{R(\rho)} f^p \nu\,dx\,dt \le e^{p\varphi(\rho)/2} + c_1[2\mu/\varphi(\rho)]^\delta e^{p\varphi(\rho)}.$$

We claim that we may assume there is a value of p, $0 < p < M$, such that

$$e^{p\varphi(\rho)/2} = L e^{p\varphi(\rho)}, \qquad L = c_1[2\mu/\varphi(\rho)]^\delta. \tag{5.4}$$

In fact, if we define

$$p = \frac{2}{\varphi(\rho)} \log \frac{1}{L}, \tag{5.5}$$

then (5.4) holds and we will show by using $M \ge 1/\mu$ that there is a constant N depending only on δ and c_1 such that $0 < p < M$ if $\varphi(\rho) > N\mu$. To see this, first note that if $\varphi(\rho) > c_1^{1/\delta} 2\mu$ then $L < 1$, so $p > 0$. Next note that

$$
\begin{aligned}
p &= \frac{2}{\varphi(\rho)}\log\frac{1}{L} = \frac{2}{\varphi(\rho)}\log\left[\frac{1}{c_1}\left[\frac{\varphi(\rho)}{2\mu}\right]^{\delta}\right] \\
&\le \frac{2}{\varphi(\rho)}\log\left[\frac{\varphi(\rho)}{2\mu}\right]^{\delta} \qquad \text{since } c_1 \ge 1 \\
&< 1/\mu \le M,
\end{aligned}
$$

where to obtain the penultimate inequality we assume that $\varphi(\rho)/2\mu > c_2$ for c_2 chosen so large (depending only on δ) that $\log(x^{\delta}) < x$ when $x > c_2$. Thus, by defining $N = \max\{2c_1^{1/\delta}, 2c_2\}$, we see that $0 < p < M$ if $\varphi(\rho) > N\mu$. If, on the other hand, $\varphi(\rho) \le N\mu$ for N as above, then the lemma is immediately true.

Therefore, for the value of p given by (5.5) we have

$$
\operatorname*{ess\,sup}_{R(\sigma)} f^p \le \frac{2A}{(\rho-\sigma)^m} e^{p\varphi(\rho)/2}.
$$

Then taking logarithms and using (5.5), we obtain

$$
\varphi(\sigma) \le \frac{\varphi(\rho)}{2}\left\{1 + \frac{\log(2A(\rho-\sigma)^{-m})}{\log(1/L)}\right\}, \qquad \sigma < \rho.
$$

Given $0 < \varepsilon < 1$, the expression above in curly brackets is either $\le 2\varepsilon$ or $> 2\varepsilon$. In the first case, $\varphi(\sigma) \le \varepsilon\varphi(\rho)$, and in the second case,

$$
\left\{\frac{\varphi(\rho)^{\delta}}{c_1(2\mu)^{\delta}}\right\}^{2\varepsilon-1} \le \frac{2A}{(\rho-\sigma)^m}.
$$

Therefore, if $\frac{1}{2} < \varepsilon < 1$, we have in any case

$$
\varphi(\sigma) \le \varepsilon\varphi(\rho) + \frac{(2A)^{1/\delta(2\varepsilon-1)}\mu c_1^{1/\delta}}{(\rho-\sigma)^{m/\delta(2\varepsilon-1)}},
$$

for $\theta \le \sigma < \rho < 1$. Now the result follows by the iteration argument on p. 733 of [11].

6. Proof of Harnack's inequality. Let u be a non-negative solution of (1.1) in the cylinder $R_{\alpha,\beta}(x_0,t_0) = B_\alpha(x_0)\times(t_0-\beta, t_0+\beta)$. We consider the transformation $T(x,t) = (\alpha x + x_0, \beta t + t_0)$ and define $\bar{u}(x,t) = u(T(x,t))$. Then $\bar{u}$ is a solution in $T^{-1}(R_{\alpha,\beta}(x_0,t_0))$ of the equation

$$
\bar{v}\bar{u}_t = \operatorname{div}(\bar{A}(x,t)\nabla\bar{u}),
$$

where $\bar{A}(x,t) = (\bar{a}_{ij}(x,t))$ with $\bar{a}_{ij}(x,t) = \alpha^{-2}\beta a_{ij}(\alpha x + x_0, \beta t + t_0)$, and we have

$$
\bar{w}_1(x)|\xi|^2 \le \langle \bar{A}\xi, \xi\rangle \le \bar{w}_2(x)|\xi|^2,
$$

$\bar{v}(x) = v(\alpha x + x_0)$, $\bar{w}_i(x) = \alpha^{-2}\beta w_i(\alpha x + x_0)$. Note that

$$T^{-1}(R_{\alpha,\beta}(x_0, t_0)) = \{(x,t) : |x| < 1,\ |t| < 1\}.$$

If we take $\frac{1}{2} \le \rho < r < 1$ and

$$|p| < \left[\bar{\Lambda}(B_1(0)) + \frac{1}{\bar{\lambda}(B_1(0))}\right]^{-1} \tag{6.1}$$

with $\bar{\Lambda}(B) = \bar{w}_2(B)/\bar{v}(B)$, $\bar{\lambda}(B) = \bar{w}_1(B)/\bar{v}(B)$, then the mean value inequalities in Lemma (3.17) applied to $\bar{u}$ give:

(a) for $p > 0$ satisfying (6.1) and $\frac{1}{2} \le \rho < r < 1$,

$$\operatorname*{ess\,sup}_{B_{(\rho+1)/3}(0)\times(-1/2-\rho/2,-1/2+\rho/2)} \bar{u}^p \le c\frac{1}{(r-\rho)^m}$$
$$\times \iint_{B_{(r+1)/3}(0)\times(-1/2-r/2,-1/2+r/2)} \bar{u}^p\left[\frac{\bar{w}_2(x)}{\bar{w}_2(B_1(0))} + \frac{\bar{v}(x)}{\bar{v}(B_1(0))}\right]dx\,dt$$

for some $m > 0$, and

(b) for $p < 0$ satisfying (6.1) and $\frac{1}{2} \le \rho < r < 1$,

$$\operatorname*{ess\,sup}_{B_{(\rho+1)/3}(0)\times((1-\rho)/2,1)} \bar{u}^p$$
$$\le c\frac{1}{(r-\rho)^m} \iint_{B_{(r+1)/3}(0)\times((1-r)/2,1)} \bar{u}^p\left[\frac{\bar{w}_2(x)}{\bar{w}_2(B_1(0))} + \frac{\bar{v}(x)}{\bar{v}(B_1(0))}\right]dx\,dt.$$

By Theorem B, $\bar{u}$ is bounded above on the cylinders which appear in (a) and (b), and by adding $\varepsilon > 0$, we may assume by letting $\varepsilon \to 0$ at the end of the proof that $\bar{u}$ is bounded below there by a positive constant.

It is easy to check that condition (4.2) holds for $\bar{v}$ and $\bar{w}_1$ assuming that it holds for v, w_1. Then by Lemma (4.9) we have

$$\left[\left(\frac{\bar{v}}{\bar{v}(B_1(0))} + \frac{\bar{w}_2}{\bar{w}_2(B_1(0))}\right)\otimes 1\right](E^{\pm})$$
$$\le c\left\{\frac{1}{s}\frac{\bar{v}(B_1(0))}{\bar{w}_1(B_1(0))}\right\}^{\delta} \le c\left\{\frac{1}{s}\left[\bar{\Lambda}(B_1(0)) + \frac{1}{\bar{\lambda}(B_1(0))}\right]\right\}^{\delta},$$

where $E^{\pm}$ are defined as in Lemma (4.9) with $R = 2/3$, $a = -1$, $b = 1$, $t_0 = 0$, $u = \bar{u}$ and $M_2 \approx \bar{\Lambda}(B_1(0))$. The constant $V = V(0)$ in Lemma (4.9) depends on $\bar{u}$ but is the same for both E^+ and E^-.

By (a) we can apply Bombieri's Lemma to the family of rectangles $R^-(\rho) = B_{(\rho+1)/3}(0) \times (-1/2 - \rho/2, -1/2 + \rho/2)$ with $\mu = \bar{\Lambda}(B_1(0)) + \bar{\lambda}(B_1(0))^{-1}$, $M = 1/\mu$ and $f = e^{-M_2+V(0)}\bar{u}$, where $V(0)$ and M_2 are as above. We obtain

$$\operatorname*{ess\,sup}_{R^-(1/2)} f \le Ce^{c[\bar{\Lambda}(B_1(0))+1/\bar{\lambda}(B_1(0))]},$$

which implies

$$\operatorname*{ess\,sup}_{R^-(1/2)} \bar{u} \le Ce^{c[\bar{\Lambda}(B_1(0))+1/\bar{\lambda}(B_1(0))]}e^{-V(0)}. \tag{6.2}$$

Also, by (b), we can apply Bombieri's Lemma to the family of rectangles $R^+(\rho) = B_{(\rho+1)/3}(0) \times ((1-\rho)/2, 1)$ and $f = e^{-M_2-V(0)}\bar{u}^{-1}$, with μ, M, M_2 and $V(0)$ as before, and we obtain

$$\operatorname*{ess\,sup}_{R^+(1/2)} f \le Ce^{c[\bar{\Lambda}(B_1(0))+1/\bar{\lambda}(B_1(0))]},$$

which implies

$$e^{-V(0)} \le Ce^{c[\bar{\Lambda}(B_1(0))+1/\bar{\lambda}(B_1(0))]} \operatorname*{ess\,inf}_{R^+(1/2)} \bar{u}. \tag{6.3}$$

Then (6.2) and (6.3) imply

$$\operatorname*{ess\,sup}_{R^-(1/2)} \bar{u} \le Ce^{c[\bar{\Lambda}(B_1(0))+1/\bar{\lambda}(B_1(0))]} \operatorname*{ess\,inf}_{R^+(1/2)} \bar{u}.$$

Now observe by changing variables that

$$\begin{aligned}\bar{\Lambda}(B_1(0)) + \frac{1}{\bar{\lambda}(B_1(0))} &= \frac{\alpha^{-n-2}\beta w_2(B_\alpha(x_0))}{\alpha^{-n}v(B_\alpha(x_0))} + \frac{\alpha^{-n}v(B_\alpha(x_0))}{\alpha^{-n-2}\beta w_1(B_\alpha(x_0))} \\ &= \alpha^{-2}\beta\Lambda(B_\alpha(x_0)) + \alpha^2\beta^{-1}\lambda(B_\alpha(x_0))^{-1},\end{aligned}$$

and that

$$\operatorname*{ess\,sup}_{R^-(1/2)} \bar{u} = \operatorname*{ess\,sup}_{R^-} u, \qquad \operatorname*{ess\,sup}_{R^+(1/2)} \bar{u} = \operatorname*{ess\,inf}_{R^+} u.$$

This proves Harnack's inequality.

7. Comments about the elliptic case. This section is separate from the rest of the paper and is intended to clarify two proofs in [3]. The references and numbering pertain to [3], and the notation is that of [3].

First, the argument which is needed near the bottom of p. 1123 of [3] in order to piece together the estimates is like the argument given above in the proof of Lemma (3.17) for $p > 0$.

Second, there is an error on p. 1121, line 6, of [3] in the way that the factor $p^{1/p}$ is treated. Although this affects the proof of Lemma (3.1) of [3], the statement of Lemma (3.1) is correct. In order to correct the proof, we follow the proof as given except that on p. 1121, line 6, we replace μ by μp. This proves an inequality like that stated in Lemma (3.1) but with μ replaced by μp. In particular, it shows that $\tilde{u}$ is bounded on each ball αB, $0 < \alpha < 1$. Thus, by replacing B by a slightly smaller ball, we may assume that $\tilde{u}$ is bounded on B. Now we repeat the proof with a few modifications, the goal being to prove the inequality on p. 1120, line 5 from the bottom, with the factor β which appears in the constant $c\beta s/(t-s)$ replaced by $\beta/(2\beta-1)$. Since $\beta/(2\beta-1) \le 1$, this will lead to (3.10) with $c\mu rs/(t-s)$ replaced by just $c\mu s/(t-s)$, and Lemma (3.1) then follows by iteration. The necessary modifications involve the way in which we bound the expressions

$$\frac{1}{H'_M(u)} \int_0^u H'_M(t)^2\,dt \tag{7.1}$$

in (3.7), and $\mathrm{av}_{tB} H_M(u)$ on p. 1120, line 6. Instead of estimating these in terms of $uH'_M(u)$ as indicated, we use the exact expressions for H'_M and H_M, considering the domains of integration $\{u \le M\}$ and $\{u > M\}$ separately. Now recall that $\tilde{u}$ is bounded and choose M to be large compared to this bound. Then, after letting $k_j \to \infty$ (recall that $u_{k_j} \to \tilde{u}$ in L^2_v), all integrals extended over $\{tB : \tilde{u} \ge M\}$ are zero because of the choice of M, and the desired estimate follows. Some further details are given below.

By definition of H_M, (7.1) above equals $\beta u^\beta/(2\beta-1)$ if $u \le M$ and equals

$$\beta M^{\beta-1}\left[u - \frac{2(\beta-1)}{2\beta-1}M\right] \quad \text{if } u > M.$$

Thus, the first term on the right side of (3.8) of [3] becomes

$$4\int_{u\le M} |\nabla\eta|^2 v\left[\frac{\beta}{2\beta-1}u^\beta\right]^2 + 4\int_{u>M} |\nabla\eta|^2 v\beta^2 M^{2(\beta-1)}\left[u - \frac{2(\beta-1)}{2\beta-1}M\right]^2.$$

Moreover,

$$\begin{aligned}\mathrm{av}_{tB} H_M(u) &\le \left[\frac{1}{v(tB)}\int_{tB} H_M(u)^2 v\right]^{1/2} \\ &\le \left[\frac{1}{v(tB)}\int_{\substack{tB\\ u\le M}} (u^\beta)^2 v + \frac{1}{v(tB)}\int_{\substack{tB\\ u>M}} [M^\beta + \beta M^{\beta-1}(u-M)]^2 v\right]^{1/2}.\end{aligned}$$

Therefore, the first term on the right of (3.9) is

$$\begin{aligned} c\frac{s}{t-s}\mu\Bigg\{&\frac{1}{v(B)}\int_{\substack{tB\\ u\le M}} \left[\frac{\beta}{2\beta-1}u^\beta\right]^2 v \\ &+ \frac{1}{v(B)}\int_{\substack{tB\\ u>M}} \beta^2 M^{2(\beta-1)}\left[u - \frac{2(\beta-1)}{2\beta-1}M\right]^2 v\Bigg\}^{1/2}\end{aligned}$$

$$+\left\{\frac{1}{v(B)}\int\limits_{\substack{tB\\u\le M}}(u^\beta)^2v+\frac{1}{v(B)}\int\limits_{\substack{tB\\u>M}}[M^\beta+\beta M^{\beta-1}(u-M)]^2v\right\}^{1/2}.$$

Here $u = u_{k_j} \to \tilde{u}$ in L_v^2. The left side of (3.9) can be treated in the limit as in [3]. Pick M large compared to $\|\tilde{u}\|_{L^\infty(B)}$. Since we may assume that $u = u_{k_j} \to \tilde{u}$ pointwise a.e., $(u^\beta)^2\chi_{(u\le M)} \to (\tilde{u}^\beta)^2$ a.e. too. Also, $(u^\beta)^2\chi_{(u\le M)} \le M^{2\beta}$, so by the dominated convergence theorem,

$$\int\limits_{\substack{tB\\u\le M}}\left[\frac{\beta}{2\beta-1}u^\beta\right]^2v\to\int\limits_{tB}\left[\frac{\beta}{2\beta-1}\tilde{u}^\beta\right]^2v,\qquad\int\limits_{\substack{tB\\u\le M}}(u^\beta)^2v\to\int\limits_{tB}(\tilde{u}^\beta)^2v.$$

It remains only to show that the remaining two integrals involving integration over $\{u > M\}$ tend to zero as $k_j \to \infty$. Calling the integrand f_j, we have the following situation: we want to show that

$$\int\limits_{u_j>M}f_j^2v\to 0$$

knowing that $f_j \to f$ in L_v^2 and that $u_j \to \tilde{u}$ a.e., $\|\tilde{u}\|_\infty \ll M$. Thus $\chi_{(u_j>M)} \to 0$ a.e., and

$$\int\limits_{u_j>M}f_j^2v\le 2\int(f_j-f)^2v+2\int f^2\chi_{(u_j>M)}v\to 0.$$

Thus, by passing to the limit, we obtain from (3.9) (cf. p. 1120, line 8 from the bottom)

$$\left[\frac{1}{v(sB)}\int\limits_{sB}H_M(\tilde{u})^qv\right]^{1/q}\le c\left[\frac{s}{t-s}\mu\frac{\beta}{2\beta-1}+1\right]\left[\frac{1}{v(tB)}\int\limits_{tB}\tilde{u}^{2\beta}v\right]^{1/2}$$

for large M. Since $H_M(\tilde{u}) \ge \tilde{u}^\beta\chi_{(\tilde{u}\le M)} = \tilde{u}^\beta$ for large M, we obtain the inequality on p. 1120, line 5 from the bottom, with the desired factor mentioned earlier.

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