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A GENERAL FORM OF THE VITALI THEOREM

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According to a result of Besicovitch ([1] and [2]), Vitali's covering theorem is valid in the following general form. Let μ be any measure which is defined on the Lebesgue measurable sets in R^n, and let μ^* be the exterior measure associated with μ. Let A be any set in R^n with $\mu^*(A) < \infty$ and, for any $x \in A$, let $\{B_k(x)\}$ be a sequence of closed Euclidean balls centered at x and with radii $r_k \downarrow 0$. Then it is possible to select from among $\{B_k(x): x \in A, k = 1, 2, \ldots\}$ a sequence $\{R_k\}$ of disjoint balls such that

$$\mu^*\left(A - \bigcup R_k\right) = 0.$$

This result has been given a more general form by Morse [6], in which balls are substituted by certain star-shaped and "nearly spherical" sets "centered" at the points x of A. In another direction the result of Besicovitch has been generalized by the author [4] to the case where the sequence $\{B_k(x)\}$ is substituted by closed intervals centered at x which need not be "nearly spherical", provided any $B_k(x)$ and $B_j(y)$ are comparable when translated to the origin.

The restriction on the balls $B_k(x)$ to be centered at x cannot be relaxed too much for R^n with $n > 1$, as Besicovitch himself has proved by means of several counterexamples at the end of [1]. However, Iseki [5] has shown that if one restricts all considerations to R^1 and if μ is any outer measure in the sense of Carathéodory (cf. Saks [8], p. 43), then the covering result remains valid.

An account of the above-mentioned results can be seen in the paper by Bruckner [3].

It is not necessary to emphasize here the usefulness of this type of covering lemmas. There are many important results in analysis which follow from them.

In this note it is shown that in R^1 Vitali's covering theorem is valid without any restriction on the measure and without requiring that the intervals $B_k(x)$ be centered in any way at x. More specifically, we prove the following result:

THEOREM. *Let μ be any measure in R^1 such that all intervals are μ-measurable, and let μ^* be the exterior measure associated with it (i.e., for any $E \subset R^1$, $\mu^*(E) = \inf\{\mu(M)$: M is μ-measurable, $M \supset E\})$. Let $A \subset R^1$ with $\mu^*(A) < \infty$ and, for each $x \in A$, let be given a sequence $\{I_k(x)\}$ of closed non-degenerate intervals containing x and such that their diameters $\delta(I_k(x))$ tend to zero as $k \uparrow \infty$. Then it is possible to select from among $\{I_k(x): x \in A, k = 1, 2, \ldots\}$ a sequence $\{R_k\}$ of disjoint intervals such that $\mu^*(A - \bigcup R_k) = 0$.*

In order to prove the Theorem we use the following modified form of Lindelöf's covering theorem:

LEMMA 1. *Let A be any set in R^1. Assume that, to each $x \in A$, there corresponds a closed, open, or half-closed finite interval $I(x)$ containing x. Then it is possible to select a sequence $\{I_k\}$ from $\{I(x)\}$ such that $A \subset \bigcup I_k$.*

Observe that if x is required to be in the interior of $I(x)$, then this is the theorem of Lindelöf. It is also easy to see that this lemma is false for R^2 if one interprets $I(x)$ to be a closed disc or a closed square containing x. To show this for discs, we take $A = \{x = (x_1, 0): 0 \leqslant x_1 \leqslant 1\}$ and,

$$I(x) = \{(y_1, y_2): (y_1 - x_1)^2 + (y_2 - 1)^2 \leqslant 1\},$$

i.e. the closed disc of center $(x_1, 1)$ passing through x.

Proof of Lemma 1. By Lindelöf's theorem, we know that $\bigcup \operatorname{Int} I(x)$ can be covered by a sequence of intervals $I(x)$. Therefore, we can assume that every x is the left end point of $I(x)$ (for right end points the argument would be analogous).

Write $A_k = \{x \in A: \delta(I(x)) > 1/k\}$ for $k = 1, 2, \ldots$, and consider

$$a_{kn} = \inf\{x: x \in A_k \cap [n/k, (n+1)/k]\} \quad \text{for } n \in Z.$$

Let $\{x_{kn}^j\} \subset A_k$, $x_{kn}^j \geqslant a_{kn}$, $\lim_j x_{kn}^j = a_{kn}$. Then

$$A_k \cap [n/k, (n+1)/k] \subset \bigcup_j I(x_{kn}^j).$$

Thus we have $A \subset \bigcup_{j,k,n \in Z} I(x_{kn}^j)$.

For the proof of the Theorem we also use the following lemma which goes back to Radó [7]:

LEMMA 2. *Let $I_1, I_2, \ldots, I_N$ be a finite sequence of closed bounded intervals in R^1. Then one can select two disjoint collections of disjoint intervals $\{J_1, J_2, \ldots, J_H\}$ and $\{L_1, L_2, \ldots, L_K\}$ such that*

$$\bigcup_1^N I_i = \Big(\bigcup_1^H J_i\Big) \cup \Big(\bigcup_1^K L_i\Big).$$

Proof. We look first at I_1. If

$$I_1 \subset \bigcup_2^N I_j,$$

then we discard I_1. If not, we retain it. Then we look at I_2. If it is contained in the union of the intervals different from I_2 not yet discarded, then we discard I_2. If not, we retain it, and so on. Having done this, we can assume, after renaming the intervals, if necessary, that the intervals which have been retained are $I_1, I_2, \ldots, I_M$. For each one of these intervals I_j there is a point $\xi_j \in I_j - \bigcup_{i \neq j} I_i$. Assume, by a rearrangement, if necessary, that $\xi_1 < \xi_2 < \ldots < \xi_M$. If we take $\{I_1, I_3, I_5, \ldots\}$ and $\{I_2, I_4, I_6, \ldots\}$, it is clear that these two collections satisfy the statement of the lemma.

The following lemma, which can be seen in Titchmarsh [9], will also be used in the proof of the Theorem.

LEMMA 3. *Let μ and μ^* be as in the statement of the Theorem. Let $\{E_k\}$ be a sequence of sets in R^1 such that $E_k \subset E_{k+1}$. Then*

$$\mu^*\Big(\bigcup E_k\Big) = \lim \mu^*(E_k) \qquad \text{as } k \uparrow \infty.$$

Proof of the Theorem. By Lemma 1, we can select a sequence $\{J_k\}$ from among all intervals $I_k(x)$ such that $\bigcup J_k \supset A$. Since, according to Lemma 3, we have

$$\infty > \mu^*(A) = \mu^*\Big(\Big(\bigcup_{k \geq 1} J_k\Big) \cap A\Big) = \lim_{N \to \infty} \mu^*\Big(\Big(\bigcup_1^N J_k\Big) \cap A\Big),$$

we can take N so that

$$\mu^*\Big(\Big(\bigcup_1^N J_k\Big) \cap A\Big) > \frac{3}{4}\mu^*(A).$$

Using Lemma 2, we can choose from among $\{J_k\}_{k=1}^N$ a subcollection of disjoint intervals $\{R_k\}_{k=1}^{h_1}$ such that

$$\mu^*\Big(\Big(\bigcup_1^{h_1} R_k\Big) \cap A\Big) > \frac{1}{2} \cdot \frac{3}{4}\mu^*(A).$$

Now we take a μ-measurable set M such that $M \supset A$ and $\mu(M) = \mu^*(A)$. So we can write

$$\mu\Big(\big(\bigcup_1^{h_1} R_k\big)\cap M\Big) \geqslant \mu^*\Big(\big(\bigcup_1^{h_1} R_k\big)\cap A\Big) > \frac{1}{2}\cdot\frac{3}{4}\,\mu(M),$$

$$\mu\Big(M - \bigcup_1^{h_1} R_k\Big) < \Big(1 - \frac{1}{2}\cdot\frac{3}{4}\Big)\,\mu(M),$$

$$\mu^*\Big(A - \bigcup_1^{h_1} R_k\Big) < \Big(1 - \frac{1}{2}\cdot\frac{3}{4}\Big)\,\mu^*(A) = a\mu^*(A).$$

Now, since $\bigcup_1^{h_1} R_k$ is compact, for each

$$x \in A - \bigcup_1^{h_1} R_k = A_1$$

we can take an interval from among $\{I_k(x)\}$ disjoint from $\bigcup_1^{h_1} R_k$ and proceed with A_1 as we have done with A, obtaining $\{R_k\}_{k=h_1+1}^{h_2}$ such that

$$\mu^*\Big(A - \bigcup_1^{h_2} R_k\Big) = \mu^*\Big(A_1 - \bigcup_{h_1+1}^{h_2} R_k\Big) < a\mu^*(A_1) < a^2\mu^*(A).$$

In this way we get $\{R_k\}$ such that $\mu^*\big(A - \bigcup R_k\big) = 0$.

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