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A PROBLEM OF A. MONTEIRO CONCERNING RELATIVE COMPLEMENTATION OF LATTICES

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1. Introduction. In [3] Monteiro points out that in a relatively complemented distributive lattice every ideal is the intersection of maximal ideals, and, consequently, the dual statement for filters. Further investigation by Monteiro revealed that if L is a complete distributive lattice such that every ideal is the intersection of maximal ideals, and dually for filters, then L is relatively complemented. The subject of this paper is the problem he then considered, namely whether it is possible to drop the restriction that L be complete (Problem 39 in Grätzer [2]).

We give two different examples of restrictions that can be imposed on distributive lattices satisfying the ideal and filter conditions that force relative complementation. However, we show by the construction of a counter-example that some extra condition is always necessary. That is, we give an example of a distributive lattice such that every ideal is the intersection of maximal ideals, similarly for filters, which is not relatively complemented.

The problem is tackled from a topological aspect and as such we deal only with $(0, 1)$ distributive lattices. Any generalizations to lattices without a zero or unit are straightforward.

I would like to thank my supervisor Dr. Brian Rotman.

2. The topology. We use the topological representation of distributive lattices introduced by Priestley in [4] and begin by restating some of those basic definitions and theorems.

Definition 1. (1) An *ordered space* is a topological space with a partial order ($\leqslant^*$).

(2) A subset S of an ordered space is *increasing* when, for any $x \in S$, if $y \geqslant^* x$, then $y \in S$ (*decreasing* is defined similarly).

(3) The space S is *totally order disconnected* when, for any $x, y \in S$, $x \not\geqslant^* y$, there exist disjoint clopen sets X, Y such that $x \in X$, $y \in Y$, X is decreasing and Y is increasing.

Distributive lattices have a representation in topological terms. The interpretation of a particular lattice notion or *vice versa* is referred to as its dual. Topological duals will be denoted by a prime symbol.

THEOREM 1 (THE REPRESENTATION THEOREM). *Every* $(0, 1)$ *distributive lattice* L *has a dual space* L' *which is compact and totally order disconnected.* L *being isomorphic to the lattice of clopen decreasing sets in* L'. *Conversely, every compact totally order disconnected space is the dual space of some* $(0, 1)$ *distributive lattice.*

LEMMA 1. *For a* $(0, 1)$ *distributive lattice* L, *there is a duality between ideals (filters) in* L *and open decreasing (closed decreasing) sets in* L'. *For an ideal* I *(filter* F*),* $d \in I$ $(d \in F)$ *iff* $d' \subseteq I'$ $(d' \supseteq F')$.

The problem in hand concerns maximal ideals and filters, together with their intersections. We develop the topological representation accordingly. As is usual, we denote the closure of a set X by ^{-}X.

LEMMA 2. (1) (a) *The dual of a maximal ideal* I *is an open decreasing set* $I' = L' - \{d\}$ *for some* $d \in L'$.

(b) *The dual of a maximal filter* F *is a closed decreasing set* $F' = \{d\}$ *for some* $d \in L'$.

(2) (a) *If an ideal* $I = \bigcap_{k \in K} I_k$, *where* I_k *is an ideal for* $k \in K$, *then*

$$I' = \bigcup \{X \mid X \text{ is an open decreasing set and } X \subseteq \bigcap_{k \in K} I'_k\}.$$

(b) *If a filter* $F = \bigcap_{k \in K} F_k$, *where* F_k *is a filter for* $k \in K$, *then*

$$F' = \{x \mid x \leqslant^* y,\ y \in {}^{-}(\bigcup_{k \in K} F'_k)\}.$$

Proof. (1) (a) Suppose I is maximal and $d_1, d_2 \in L' - I'$, $d_1 \neq d_2$, either $d_1 \not\leqslant^* d_2$ or $d_2 \not\leqslant^* d_1$. If $d_1 \not\leqslant^* d_2$, there exists a clopen decreasing set D_2 such that $d_2 \in D_2$ and $d_1 \notin D_2$. But then $I' \cup D_2$ is open decreasing, $d_1 \notin I' \cup D_2$ and $I' \subset I' \cup D_2$, contradicting the maximality of I.

(2) (a) $x \in I$ iff $x' \subseteq I'$ iff $x' \subseteq \bigcap_{k \in K} I'_k$.

Similarly (1) (b) and (2) (b).

LEMMA 3. *For a* $(0, 1)$ *distributive lattice* L, *let* $\{I_m \mid m \in M\}$ *be the maximal ideals and* $\{F_n \mid n \in N\}$ *the maximal filters.*

(a) *Every ideal* $I = \bigcap_{m \in M' \subseteq M} I_m$ *for some* M' *iff* $d \in L'$ *means that* d *is maximal* $(\leqslant^*)$ *or* $\exists x_r \in L'$, $r \in R$, *such that* $d \in {}^{-}(\bigcup_{r \in R} \{x_r\})$, $x_r >^* d$ *and* x_r *is maximal* $(\leqslant^*)$.

(b) *Every filter* $F = \bigcap_{n \in N' \subseteq N} F_n$ *for some* N' *iff* $d \in L'$ *means that* d *is minimal* $(\leqslant^*)$ *or* $\exists x_r \in L'$, $r \in R$, *such that* $d \in {}^{-}(\bigcup_{r \in R} \{x_r\})$, $x_r <^* d$ *and* x_r *is minimal* $(\leqslant^*)$.

Proof. (a) We begin by showing the condition is necessary. Suppose there exists a $d \in L'$, d not maximal ($\leqslant^*$) and there is no R such that $d \in {}^{-}\big(\bigcup_{r \in R}\{x_r\}\big)$ with $x_r >^* d$ and x_r maximal ($\leqslant^*$).

Consider $D = \{x \mid x \geqslant^* d\}$. We claim that D is closed. If not, there exists a $y \in {}^{-}D$ such that $y \not\geqslant^* d$, and, consequently, a clopen decreasing set X, $y \in X$ and $d \notin X$. But $d \notin X$ implies $D \cap X = \emptyset$, a contradiction.

Since D is closed increasing, $L' - D$ is open decreasing and represents an ideal. We show that the dual ideal of $L' - D$ is not an intersection of maximals.

Let $D_1 = {}^{-}\{x \mid x >^* d \text{ and } x \text{ is maximal } (\leqslant^*)\}$. By hypothesis, $d \notin D_1$. Thus, for $x \in D_1$, there are a clopen increasing X_x and a clopen decreasing Y_x such that $X_x \cap Y_x = \emptyset$, $x \in X_x$ and $d \in Y_x$. $\{X_x\}_{x \in D_1}$ forms an open cover for D_1, which, by compactness, has a finite subcover $X_{x_1}, \ldots, X_{x_n}$. Then $Y = \bigcap_{1 \leqslant i \leqslant n} Y_{x_i}$ is a clopen decreasing set, yet it contains no maximal ($\leqslant^*$) point $p \geqslant^* d$; hence Y is contained in every maximal decreasing open set that contains $L' - D$. But $d \in Y$ thus, by Lemma 2, the ideal with dual $L' - D$ is not the intersection of maximal ideals containing it.

Next is the sufficiency. Given an open decreasing set X, then, clearly, for $d \in L' - X$, if d is maximal ($\geqslant^*$), d is not a member of any open set contained in $L' - \{d\}$. If d is not maximal ($\leqslant^*$), then $\{x_r\}_{r \in R}$ exists but $x_r >^* d$ implies $x_r \in L' - X$. Hence, d is not a member of any open set in $\bigcap_{r \in R} L' - \{x_r\}$.

By Lemma 2, the ideal with dual X satisfies part (a) of the lemma.

Part (b) is similar.

By strengthening the hypothesis of Monteiro's problem, we deduce the next two theorems.

THEOREM 2. *If L is a $(0, 1)$ distributive lattice, every filter is the intersection of maximal filters and L is pseudo-complemented, then it is a boolean lattice.*

Proof. Suppose, to the contrary, that L is not a boolean lattice. Then there are $d_1, d_2 \in L'$, $d_1 <^* d_2$. Choose a clopen decreasing set D, $d_1 \in D$ and $d_2 \notin D$. By hypothesis, D has a pseudo-complement, say C, in L'.

If $y \in L'$ and $y \geqslant^* d$ for some $d \in D$, then $y \notin C$; otherwise $d \in D \cap C$, a contradiction.

Alternately suppose $y \in L'$ and $y \not\geqslant^* d$ for any $d \in D$. Then, for $d \in D$, there exist a clopen increasing X_d and a clopen decreasing Y_d, $X_d \cap Y_d = \emptyset$, $d \in X_d$ and $y \in Y_d$. D is closed implies the open cover $\{X_d\}$, $d \in D$, has a finite subcover $X_{d_1}, \ldots, X_{d_n}$. Thus

$$y \in Y = \bigcap_{1 \leqslant i \leqslant n} Y_{d_i}$$

which is a clopen decreasing set. But $D \cap Y = \emptyset$ implies $Y \subseteq C$; hence $y \in C$.

This means that clopen $C = \{y \mid y \not\geqslant^* d \text{ for any } d \in D\}$. Hence $d_2 \in \{y \mid y \geqslant^* d$ for some $d \in D\} - D$, a clopen set. But d_2 is not minimal ($\leqslant^*$), so, by Lemma 3 any clopen set containing d_2 contains minimal ($\leqslant^*$) points x_r. A contradiction, since $x_r \not\geqslant^* d$ for any $d \in D$.

A lattice L is *scattered* providing the chain of rationals cannot be embedded in it.

THEOREM 3. *Let L be a* $(0, 1)$ *distributive lattice satisfying the maximal ideal and filter conditions of Lemma* 3. *If L is scattered, then it is a boolean lattice.*

Proof. For $p_1, p_2 \in L$, $p_1 < p_2$, we say that $\langle p_1, p_2\rangle$ is a *nice pair* providing in the topology there are points $d_1, d_2 \in p_2' - p_1'$, $d_1 <^* d_2$. Suppose L satisfies the maximal ideal and filter conditions and $\langle p_1, p_2\rangle$ is a nice pair. Choose a clopen decreasing set C such that $d_1 \in C$ and $d_2 \notin C$, and let $q' = (C \cap p_2') \cup p_1'$. Now, $d_1 \in q' - p_1'$ and d_1 is not a maximal ($\leqslant^*$), hence, by Lemma 3, there is an $x_1 \in q' - p_1'$ and $d_1 <^* x_1$. Similarly, $d_2 \in p_2' - q'$ is not minimal ($\leqslant^*$). Hence there exists an $x_2 \in p_2' - q'$, $x_2 <^* d_2$. That is to say that, for a nice pair $\langle p_1, p_2\rangle$, $p_1 < p_2$, there is a triple $p_1 < q < p_2$ with $\langle p_1, q\rangle$ and $\langle q, p_2\rangle$, nice pairs.

Now suppose L is not a boolean lattice; then there are $d_1, d_2 \in L'$, $d_1 <^* d_2$. Therefore, $\langle 0, 1\rangle$ is a nice pair.

Using the generating procedure just described, it is possible to embed the rationals, a contradiction.

3. The construction. We begin with some basic definitions for a set A with a total order ($\leqslant$).

Definition 2 (e.g., Sierpiński [5]). (1) If $A = A_1 \cup A_2$, $A_i \neq \emptyset$ $(i = 1, 2)$, $A_1 \cap A_2 = \emptyset$ and, for any $a_1 \in A_1$ and $a_2 \in A_2$ we have $a_1 < a_2$, then $\langle A_1, A_2\rangle$ is called a *cut* for A.

(2) If in a cut $\langle A_1, A_2\rangle$ either A_1 has a last element or A_2 has a first element, then we say that cut gives a *jump*.

(3) If a cut $\langle A_1, A_2\rangle$ is such that A_1 has no last element and A_2 has no first element, then that cut gives a *gap*.

(4) An *initial segment* of A is a set A' such that $a_1 \in A'$ and $a_2 < a_1$ implies $a_2 \in A'$.

Consequently, there is a natural correspondence between initial segments and cuts. So that in a set whose members are initial segments a particular element is either $\emptyset$, A, a jump point or a gap point depending on the type of cut it gives.

Definition 3 (e.g., Birkhoff [1]). (1) An *open interval* in A is a set of one of the forms: (i) A, (ii) $(, a) = \{x \mid x < a\}$, (iii) $(a ,) = \{x \mid x > a\}$, (iv) $(a, b) = \{x \mid a < x < b\}$ for $a, b \in A$.

(2) The *interval topology* on A is the topology with the open intervals as base.

Consider now the chain C formed when the initial segments of the rationals in the real interval $(0, 1)$ are ordered by inclusion. The interval topology on C is a compact totally disconnected space and is well known as a representation of the countable atomless boolean algebra. C is a natural candidate amongst the class of compact totally ordered spaces that might harbour a suitable counter-example as a sublattice, since Theorem 3 states that the lattice C is always embeddable in the minimal boolean extension of a proper distributive lattice satisfying Lemma 3.

We now impose a partial order $(\leqslant^*)$ on C by induction.

Definition 4 (1). Let $\langle r_i \rangle$, $1 \leqslant i < \omega$, be an enumeration of the rationals. Choose a gap point $p \in C$ (Definition 2, (3)).

I. (a) Choose a sequence of gap points p_i $(i < \omega)$ satisfying the following conditions:

(i) $p_i < p_j$ for $i < j$; $p_i \neq p$;

(ii) $p \in {}^{-}\{p_i \mid i < \omega\}$.

Let $p_i <^* p$ for $i < \omega$.

(b) Choose clopen intervals P_i $(i < \omega)$ such that

(i) $p_i \in P_i$;

(ii) $P_i \cap P_j = \emptyset$ $(i \neq j)$;

(iii) length $(P_i) \leqslant 1/2$ in the pseudo-metric imposed on C by the real metric on $(0, 1)$;

(iv) $(, r_1)$, $(, r_1] \notin P_i$.

II. (a) For each $n < \omega$, choose a sequence of gap points p_{ni} $(i < \omega)$ such that

(i) $p_{ni} < p_{nj}$ for $i > j$; $p_{ni} \neq p_n$;

(ii) $p_n \in {}^{-}\{p_{ni} \mid i < \omega\}$;

(iii) $p_{ni} \in P_n$.

Let $p_{ni} >^* p_n$.

(b) For each $n < \omega$, choose clopen intervals P_{ni} $(i < \omega)$ such that

(i) $p_{ni} \in P_{ni}$;

(ii) $P_{ni} \cap P_{nj} = \emptyset$ $(i \neq j)$;

(iii) length $(P_{ni}) \leqslant 1/2^2$;

(iv) $(, r_2)$, $(, r_2] \notin P_{ni}$;

(v) $P_{ni} \subseteq P_n$.

We now inductively repeat stages I and II with inserted clauses dual to II (a), (iii), and II (b), (v), in I:

(a) (iii) $p_{n_1 \ldots n_{2r} i} \in P_{n_1 \ldots n_{2r}}$;

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(1) I would like to thank B. Davey for his criticism of an earlier presentation.

(b) (v) $P_{n_1 \ldots n_{2r}i} \subseteq P_{n_1 \ldots n_{2r}}$.

THEOREM 4. *The space $(C, \leqslant^*)$ is the dual space of a $(0, 1)$ distributive lattice L in which every ideal is the intersection of maximal ideals and dually for filters, but is not boolean.*

Proof. The space is compact and totally disconnected so that we must show that (α) ($\leqslant^*$) is a partial order, (β) the space is totally order disconnected, (γ) ($\leqslant^*$) satisfies the conditions of Lemma 3.

(α) The points involved in the relation ($\leqslant^*$) are the following:

(i) $p_n <^* p$, $n < \omega$;

(ii) $p_{n_1 \ldots n_{2r+1}} <^* p_{n_1 \ldots n_{2r}}$, $1 \leqslant r < \omega$;

(iii) $p_{n_1 \ldots n_{2r}} >^* p_{n_1 \ldots n_{2r-1}}$, $1 \leqslant r < \omega$.

After a point has been chosen to be a member of the relation ($<^*$), it is only directly involved in one more induction stage, hence we see that the relation is reflexive, antisymmetric and, by default, transitive.

(β) Given $d_1, d_2 \in C$ we must construct appropriate clopen D_1 and D_2. We begin by making the following observations:

At an odd stage $2r+1$, $r \geqslant 0$, for $x \in P_{n_1 \ldots n_{2r+1}}$ and $y \in C - P_{n_1 \ldots n_{2r+1}}$, we have

$$(x \text{ comparable } (\leqslant^*)\ y) \to (x = p_{n_1 \ldots n_{2r+1}},\ y = p_{n_1 \ldots n_{2r}} \text{ and } x <^* y).$$

At an even stage $2r$, $r \geqslant 1$, for $x \in P_{n_1 \ldots n_{2r}}$ and $y \in C - P_{n_1 \ldots n_{2r}}$, we have

$$(x \text{ comparable } (\leqslant^*)\ y) \to (x = p_{n_1 \ldots n_{2r}},\ y = p_{n_1 \ldots n_{2r-1}} \text{ and } x >^* y).$$

A new pair $x <^* y$ is formed after a stage r only if $x, y \in P_{n_1 \ldots n_r}$ for some sequence $\langle n_1 \ldots n_r \rangle$. Thus we always have $P_{n_1 \ldots n_{2r+1}}$ is decreasing and $P_{n_1 \ldots n_{2r}}$ is increasing. Thus we always have one of $P_{n_1 \ldots n_r}$ and $P_{n_1 \ldots n_{r+1}}$ is clopen increasing and the other clopen decreasing. Together with the fact that $P_{n_1 \ldots n_{r+1}} \subseteq P_{n_1 \ldots n_r}$ we have the following statement:

(A) If $d_1 \in P_{n_1 \ldots n_{r+1}}$ and $d_2 \notin P_{n_1 \ldots n_r}$, then d_1 and d_2 may be separated by the desired D_1 and D_2.

Let $P = \{p_{n_1 \ldots n_r} \mid r \geqslant 1,\ n_i < \omega\} \cup \{p\}$. Noting that if X is increasing, $C - X$ is decreasing (similarly decreasing), we now consider the possible values of d_i.

(i) $d_1, d_2 \in P$.

Since $(\leqslant^*) \subseteq (\leqslant)$, if $d_1 <^* d_2$, we can choose a rational s such that $d_1 < (, s) < d_2$. Then $D_1 = [\emptyset, (, s)]$ is a clopen decreasing separating set.

For convenience, if one of d_1, d_2 is defined at a later stage than the other, let this point be d_1. Suppose we are at an odd stage and $d_1 = p_{n_1 \ldots n_{2r+1}}$, $r \geqslant 0$. Then, by (A), we need only consider $d_2 \in P_{n_1 \ldots n_{2r}}$ (if $r = 0$, we would interpret this as C) but this means that $d_2 = p_{n_1 \ldots n_{2r}}$ (i.e., $d_1 <^* d_2$) or

$d_2 = p_{n_1 \dots n_{2r} m}$. In the event of the second case occurring we see that $P_{n_1 \dots n_{2r+1}}$ and $P_{n_1 \dots n_{2r} m}$ are both decreasing clopen sets. Thus, by Definition 4 (b), (ii) and (i), one of them will serve as an appropriate D_i.

Similarly at an even stage with $d_1 = p_{n_1 \dots n_{2r}}$, $r \geqslant 1$. If d_1 and d_2 are not comparable ($\leqslant^*$) and condition (A) fails, we choose appropriately from the two clopen increasing sets $P_{n_1 \dots n_{2r}}$ and $P_{n_1 \dots n_{2r-1} m}$. Thus we now need only to consider the case

(ii) Either d_1 or $d_2 \notin P$.

Suppose $d_1 \notin P$ and at a stage r we have $d_1 \in P_{n_1 \dots n_r}$ and $d_1 \notin P_{n_1 \dots n_r m}$ for any $m < \omega$. By construction for $x \in P_{n_1 \dots n_r}$, if x belongs to the boundary of $\bigcup_{m<\omega} P_{n_1 \dots n_r m}$, then $x = p_{n_1 \dots n_r}$.

Thus, there is a clopen interval I such that $I \subseteq P_{n_1 \dots n_r}$, $I \cap P_{n_1 \dots n_r m} = \emptyset$ and $d_1 \in I$. Hence $x \in I$ implies $x \notin P$.

Let I' be a clopen set in the space C such that $d_1 \in I'$ and $d_2 \notin I'$. Then $D_1 = I \cap I'$ is clopen increasing and decreasing and will always serve.

Next we consider the case where there exists a sequence n_r, $r < \omega$, such that $d_1 \in P_{n_1 \dots n_r}$ for any $r < \omega$. By Definition 4 (b), (iv), d_1 must be a gap point, and thus the pseudo-metric and Definition 4 (b), (iii), ensure that $d_2 \notin P_{n_1 \dots n_r}$ for some r. Continuing the induction step one more stage gives condition (A) and we are through.

(γ) We see from the relations in (α) that p is maximal ($\leqslant^*$), $p_{n_1 \dots n_{2r}}$ is maximal ($\leqslant^*$) and $p_{n_1 \dots n_{2r+1}}$ is minimal ($\leqslant^*$). Thus, by Definition 4 (a), (ii), the conditions of Lemma 3 are satisfied. Hence $(C, \leqslant^*)$ is a genuine dual space that satisfies the maximal ideal and filter conditions, yet, by virtue of the relation ($\leqslant^*$) fails to be boolean.

Addendum. Monteiro's problem has been independently solved by R. Balbes using an alternative method.

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