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A NOTE ON THE INTERSECTIONS OF COUNTABLY GENERATED σ-ALGEBRAS

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Let $\mathscr{C}_1 \subseteq \mathscr{C}_2$ be countably generated σ-algebras on a set X. We present some conditions under which there exists a non-countably generated σ-algebra $\mathscr{A}$ such that $\mathscr{C}_1 \subseteq \mathscr{A} \subseteq \mathscr{C}_2$ and under which $\mathscr{A}$ can be obtained as an intersection $\mathscr{C}_2 \cap \mathscr{C}_3$ for some countably generated $\mathscr{C}_3$ on X.

1. Let $\mathscr{C}$ be a σ-algebra on a set X. The σ-algebra $\mathscr{C}$ is called *countably generated* (c.g.) if there exists a countable family $\mathscr{G}$ of subsets of X such that $\mathscr{C} = \sigma_X(\mathscr{G})$, where $\sigma_X(\mathscr{G})$ denotes the smallest σ-algebra on X containing $\mathscr{G}$.

This note* is the result of work inspired by a problem of B. V. Rao ([6], P 687) and some obvious questions arising from its solution [4]. The main problem we shall investigate may be stated as follows:

Suppose $\mathscr{C}_1 \subseteq \mathscr{C}_2$ are c.g. σ-algebras on a set X. Does there exist a c.g. σ-algebra $\mathscr{C}_3 \supseteq \mathscr{C}_1$ such that $\mathscr{C}_2 \cap \mathscr{C}_3$ is not c.g.?

We are mainly concerned with the case where $\mathscr{C}_2$ and $\mathscr{C}_3$ are generated from $\mathscr{C}_1$ by adding one set. In this situation Corollary 2 seems to provide a satisfactory answer to the above question. Also we would like to attract the reader's attention to Lemma 1′ which, we believe, may be of independent interest.

For a set X, $\mathscr{P}(X)$ denotes the family of all subsets of X. Whenever $Y \subseteq X$ and $\mathscr{A} \subseteq \mathscr{P}(X)$,

$$\mathscr{A}\restriction_Y = \{A \cap Y \colon A \in \mathscr{A}\}.$$

If X is a metric space, then $\mathscr{B}(X)$ denotes the family of all Borel subsets of X. In particular, if X is a subset of the reals $\boldsymbol{R}$, then $\mathscr{B}(X) = \mathscr{B}(\boldsymbol{R})\restriction_X$.

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Notice that if $\mathscr{C}$ is a σ-algebra on X and $V \subseteq X$, then

$$\begin{aligned}\sigma_X(\mathscr{C} \cup \{V\}) &= \{(E_1 \cap V) \cup (E_2 \setminus V)\colon E_1, E_2 \in \mathscr{C}\} \\ &= \{(C_1 \cap V) \cup (C_2 \setminus V) \cup C_3\colon \\ &\qquad C_1, C_2, C_3 \in \mathscr{C} \text{ and } C_1, C_2, C_3 \text{ are pairwise disjoint}\}.\end{aligned}$$

To see this, set

$$C_1 = E_1 \setminus E_2, \quad C_2 = E_2 \setminus E_1, \quad C_3 = E_1 \cap E_2.$$

MA stands for Martin's Axiom and CH for the Continuum Hypothesis.

2. The main point of this section is Lemma 2 which turns out to be a powerful tool when investigating the structure of c.g. σ-algebras.

LEMMA 1. *Let X be a set and let $\mathscr{C}$ be a σ-algebra on X. If*

$$V, W \subseteq X \quad \text{and} \quad A \in \sigma_X(\mathscr{C} \cup \{V\}) \cap \sigma_X(\mathscr{C} \cup \{W\}),$$

then there are D, $D' \in \mathscr{C}$ such that

$$D \subseteq V \Delta W \quad \text{and} \quad D' \cap (V \Delta W) = \emptyset$$

and

$$\sigma_X(\mathscr{C} \cup \{A\}) = \sigma_X(\mathscr{C} \cup \{D \cap V, D' \cap V\}).$$

Proof. Because $A \in \sigma_X(\mathscr{C} \cup \{V\}) \cap \sigma_X(\mathscr{C} \cup \{W\})$, we can find pairwise disjoint B_1, B_2, $B_3 \in \mathscr{C}$ and C_1, C_2, $C_3 \in \mathscr{C}$ such that

$$A = (B_1 \cap V) \cup (B_2 \setminus V) \cup B_3 = (C_1 \cap W) \cup (C_2 \setminus W) \cup C_3.$$

Set

$$\begin{aligned}D_1 &= B_1 \cap C_2, \quad & D_2 &= B_2 \cap C_1, \quad & D &= D_1 \cup D_2, \\ D'_1 &= B_1 \cap C_1, \quad & D'_2 &= B_2 \cap C_2, \quad & D' &= D'_1 \cup D'_2.\end{aligned}$$

Straightforward calculations show that

$$\begin{aligned}D_1 \cap A &= D_1 \cap V = D_1 \setminus W, \quad & D_2 \cap A &= D_2 \setminus V = D_2 \cap W, \\ D'_1 \cap A &= D'_1 \cap V = D'_1 \cap W, \quad & D'_2 \cap A &= D'_2 \setminus V = D'_2 \setminus W.\end{aligned}$$

It follows that

$$D \subseteq V \Delta W, \quad D' \cap (V \Delta W) = \emptyset, \quad D \cap V \in \sigma_X(\mathscr{C} \cup \{A\})$$

and

$$D' \cap V \in \sigma_X(\mathscr{C} \cup \{A\}),$$

so that

$$\sigma_X(\mathscr{C} \cup \{D \cap V, D' \cap V\}) \subseteq \sigma_X(\mathscr{C} \cup \{A\}).$$

Notice that

$$A = B_3 \cup C_3 \cup (D_1 \cap (D \cap V)) \cup (D_2 \setminus (D \cap V)) \cup (D_1' \cap (D' \cap V)) \cup (D_2' \setminus (D' \cap V));$$

hence $A \in \sigma_X(\mathscr{C} \cup \{D \cap V, D' \cap V\})$ and

$$\sigma_X(\mathscr{C} \cup \{D \cap V, D' \cap V\}) = \sigma_X(\mathscr{C} \cup \{A\}).$$

For $\mathscr{G} \subseteq \mathscr{P}(X)$ write $\varrho_X(\mathscr{G})$ for the subalgebra of $\mathscr{P}(X)$ generated by $\mathscr{G}$. Observe that in the above proof only finite set operations have been used, and therefore it actually shows that the following fact is true:

LEMMA 1'. *Let X be a set and let $\mathscr{C}$ be a subalgebra of $\mathscr{P}(X)$. If*

$$V, W \subseteq X \quad \textit{and} \quad A \in \varrho_X(\mathscr{C} \cup \{V\}) \cap \varrho_X(\mathscr{C} \cup \{W\}),$$

then there are D, $D' \in \mathscr{C}$ such that

$$D \subseteq V \Delta W, \quad D' \cap (V \Delta W) = \emptyset$$

and

$$\varrho_X(\mathscr{C} \cup \{A\}) = \varrho_X(\mathscr{C} \cup \{D \cap V, D' \cap V\}).$$

LEMMA 2. *Let X be a set, let V, $W \subseteq X$, and let $\mathscr{C}$ be a σ-algebra on X. If $\mathscr{D}$ is a c.g. σ-algebra with*

$$\mathscr{C} \subseteq \mathscr{D} \subseteq \sigma_X(\mathscr{C} \cup \{V\}) \cap \sigma_X(\mathscr{C} \cup \{W\}),$$

then

$$\mathscr{D} = \sigma_X(\mathscr{C} \cup \{(D \cup D') \cap V\})$$

for some D, $D' \in \mathscr{C}$ such that

$$D \subseteq V \Delta W \quad \textit{and} \quad D' \cap (V \Delta W) = \emptyset.$$

Proof. Let $\{A_n: n < \omega\} \subseteq \mathscr{D}$ be such that

$$\mathscr{D} = \sigma_X(\mathscr{C} \cup \{A_n: n < \omega\}).$$

For each $n < \omega$, by Lemma 1 choose D_n, $D_n' \in \mathscr{C}$ such that

$$D_n \subseteq V \Delta W, \quad D_n' \cap (V \Delta W) = \emptyset$$

and

$$(*) \qquad \sigma_X(\mathscr{C} \cup \{A_n\}) = \sigma_X(\mathscr{C} \cup \{D_n \cap V, D_n' \cap V\}).$$

Set

$$D = \bigcup_{n<\omega} D_n \quad \text{and} \quad D' = \bigcup_{n<\omega} D_n'.$$

Clearly, $D \cap V$, $D' \cap V \in \mathscr{D}$, so

$$\sigma_X(\mathscr{C} \cup \{(D \cup D') \cap V\}) \subseteq \mathscr{D}.$$

Since for every $n < \omega$

$$D_n \cap V,\ D'_n \cap V \in \sigma_X(\mathscr{C} \cup \{D \cap V,\ D' \cap V\}) = \sigma_X(\mathscr{C} \cup \{(D \cup D') \cap V\}),$$

by (*) we have $A_n \in \sigma_X(\mathscr{C} \cup \{(D \cup D') \cap V\})$, and the proof is complete.

Also we would like to recall a well-known fact (see [1], p. 8, (i)):

LEMMA 3. *If $\mathscr{G}$ is a family of subsets of X and $\sigma_X(\mathscr{G})$ is c.g., then there exists a countable family $\mathscr{G}' \subseteq \mathscr{G}$ such that $\sigma_X(\mathscr{G}') = \sigma_X(\mathscr{G})$.*

LEMMA 4. *If $\mathscr{C}$ is a σ-algebra on a set X and $\mathscr{F} \subseteq \mathscr{P}(X)\setminus\mathscr{C}$ is an uncountable disjoint family, then there exists $\mathscr{F}' \subseteq \mathscr{F}$ such that $\sigma(\mathscr{C} \cup \mathscr{F}')$ is not c.g.*

Proof. Suppose that $\mathscr{C}$ and $\sigma_X(\mathscr{C} \cup \mathscr{F})$ are both c.g. By Lemma 3 we have $\sigma_X(\mathscr{C} \cup \mathscr{F}) = \sigma_X(\mathscr{G})$ for some countable family $\mathscr{G} \subseteq \mathscr{C} \cup \mathscr{F}$. Set

$$E = X \setminus \bigcup(\mathscr{G} \cap \mathscr{F})$$

and enumerate $\mathscr{F}\setminus\mathscr{G}$ as $\{F_\alpha\colon \alpha < \varkappa\}$ for some cardinal $\varkappa$. Since $\sigma_X(\mathscr{G})\restriction_E = \mathscr{C}\restriction_E$, there is a disjoint family $\{C_\alpha\colon \alpha < \omega_1\} \subseteq \mathscr{C}$ with $C_\alpha \cap E = F_\alpha$ for $\alpha < \omega_1$. Now let

$$\mathscr{F}' = \{F_\alpha\colon \alpha < \omega_1\}.$$

Then $\sigma_X(\mathscr{C} \cup \mathscr{F}')$ is not c.g. Indeed, by Lemma 3 it suffices to show that whenever $\mathscr{G}' \subseteq \mathscr{C} \cup \mathscr{F}'$ with $\mathscr{C} \subseteq \sigma_X(\mathscr{G}')$ is countable, then $F_\alpha \in \sigma_X(\mathscr{G}')$ iff $F_\alpha \in \mathscr{G}'$. To use this, look at

$$E' = X \setminus \bigcup\{C_\alpha\colon F_\alpha \in \mathscr{G}'\}$$

and notice that $\sigma_X(\mathscr{G}')\restriction_{E'} = \mathscr{C}\restriction_E \subseteq \mathscr{C}$.

3. The main results

THEOREM 1. *Let X be a set, let $V \subseteq X$ and let $\mathscr{C}$ be a c.g. σ-algebra on X. Then the following conditions are equivalent:*

(i) *There exists a non-c.g. σ-algebra $\mathscr{A}$ on X such that*

$$\mathscr{C} \subseteq \mathscr{A} \subseteq \sigma_X(\mathscr{C} \cup \{V\}).$$

(ii) *There exists an uncountable disjoint family*

$$\mathscr{F} \subseteq \sigma_X(\mathscr{C} \cup \{V\})\setminus\mathscr{C}.$$

(iii) *There exists a σ-algebra $\mathscr{A}_1$ with $\mathscr{C} \subseteq \mathscr{A}_1 \subseteq \sigma_X(\mathscr{C} \cup \{V\})$ which is not expressible as $\sigma_X(\mathscr{C} \cup \{A\})$ for any $A \subseteq X$.*

Proof. Since $\sigma_X(\mathscr{C} \cup \{A\})$ is c.g. for every $A \subseteq X$, (ii) implies (iii) by

Lemma 4. By Lemma 2, (iii) implies (i), so it remains to show that (i) implies (ii).

Assume (i). Let $\mathscr{I} = \{C \in \mathscr{C}: \mathscr{A}\restriction_C \text{ is c.g.}\}$.

CLAIM. *Let $B \in \mathscr{A}\setminus\mathscr{C}$, $B = (B_1 \cap V) \cup (B_2 \setminus V)$ for some $B_1, B_2 \in \mathscr{C}$. Then*

$$B_1 \Delta B_2 \in \mathscr{I} \quad \textit{and} \quad \mathscr{C}\restriction_{B_1 \Delta B_2} \neq \mathscr{A}\restriction_{B_1 \Delta B_2}.$$

To see this, set $D = ((B_1 \setminus B_2) \cap V) \cup ((B_2 \setminus B_1) \setminus V)$ and notice that

$$\sigma(\mathscr{C} \cup \{V\})\restriction_{B_1 \Delta B_2} \subseteq \mathscr{A}\restriction_{B_1 \Delta B_2} \subseteq \sigma(\mathscr{C} \cup \{V\})\restriction_{B_1 \Delta B_2},$$

so $B_1 \Delta B_2 \in \mathscr{I}$. Since $D = B\setminus(B_1 \cap B_2)$, we have $D \in \mathscr{A}\restriction_{B_1 \Delta B_2}$. Also $B = D \cup (B_1 \cap B_2)$ so that

$$D \notin \mathscr{C} \supseteq \mathscr{C}\restriction_{B_1 \Delta B_2}.$$

$\mathscr{I}$ is a proper σ-ideal in $\mathscr{C}$, so inductively we may construct a disjoint sequence $\{C_\alpha: \alpha < \omega_1\}$ such that $C_\alpha \in \mathscr{I}$ and $\mathscr{A}\restriction_{C_\alpha} \neq \mathscr{C}\restriction_{C_\alpha}$. At the step α, apply the Claim to $X \setminus \bigcup \{C_\beta: \beta < \alpha\}$. For each $\alpha < \omega_1$ pick $F_\alpha \in \mathscr{A}\restriction_{C_\alpha} \setminus \mathscr{C}$, so the family $\mathscr{F} = \{F_\alpha: \alpha < \omega_1\}$ has the required properties.

COROLLARY 1. *Assume* MA. *Then there exists a set $X \subseteq \boldsymbol{R}$ with $|X| = \mathfrak{c}$ such that, for every $V \subseteq X$,*

$$\mathscr{A} = \sigma_X(\mathscr{B}(X) \cup \{A\}) \quad \textit{for some } A \subseteq X$$

whenever $\mathscr{A}$ is a σ-algebra such that

$$\mathscr{B}(X) \subseteq \mathscr{A} \subseteq \sigma_X(\mathscr{B}(X) \cup \{V\}).$$

Proof. Let $X \subseteq \boldsymbol{R}$ be a $\mathfrak{c}$-Lusin set (see [3], p. 43, 22D) so that $|X| = \mathfrak{c}$ and $|X \cap M| < \mathfrak{c}$ for every meagre $M \subseteq \boldsymbol{R}$.

Let $V \subseteq X$ and let $\{F_\alpha: \alpha < \omega_1\}$ be a disjoint family in $\sigma_X(\mathscr{B}(X) \cup \{V\})$. There are disjoint $\{B_\alpha: \alpha < \omega_1\}$ and $\{B'_\alpha: \alpha < \omega_1\}$ in $\mathscr{B}(X)$ such that

$$F_\alpha = (B_\alpha \cap V) \cup (B'_\alpha \setminus V).$$

Since all but perhaps countably many B_α's and B'_α's are meagre, we have

$$\left|\{\alpha < \omega_1: |F_\alpha| = \mathfrak{c}\}\right| \leqslant \omega,$$

and hence by [3], p. 60, 23C,

$$|\{\alpha < \omega_1: F_\alpha \notin \mathscr{B}(X)\}| \leqslant \omega.$$

Theorem 1 completes the proof.

A c.g. σ-algebra $\mathscr{C}$ on X is called *separable* if $\{x\} \in \mathscr{C}$ for every $x \in X$. Corollary 1 shows that it is consistent to suppose that there are separable σ-algebras $\mathscr{C}_1 \subsetneqq \mathscr{C}_2$ such that all the "intermediate" σ-algebras $\mathscr{A}$ with $\mathscr{C}_1 \subseteq \mathscr{A} \subseteq \mathscr{C}_2$ are c.g. We do not know whether such σ-algebras can be constructed within ZFC (**P 1357**). Notice that the requirement that $\mathscr{C}_1$ and $\mathscr{C}_2$ are separable is essential. Otherwise we could simply take any set X with $2 \leqslant |X| \leqslant \omega$, pick $x, y \in X$, $x \neq y$, and set

$$\mathscr{C}_1 = \{X' \subseteq X \colon x \in X' \text{ iff } y \in X'\} \quad \text{and} \quad \mathscr{C}_2 = \mathscr{P}(X).$$

LEMMA 5. *Let X be a set and let $\mathscr{C}$ be a c.g. σ-algebra on X. Suppose that $V \subseteq X$ is such that $\sigma_X(\mathscr{C} \cup \{V\}) \cap \sigma_X(\mathscr{C} \cup \{W\})$ is c.g. whenever $W \subseteq X$. If $\mathscr{U} \subseteq \mathscr{C}\restriction_V$, then there is a set $D \in \mathscr{C}$ such that*

$$D \cap V \subseteq \bigcup \mathscr{U} \quad \textit{and} \quad U \setminus D \in \mathscr{C} \textit{ for all } U \in \mathscr{U}.$$

Proof. For $U \in \mathscr{U}$ choose $B_U \in \mathscr{C}$ such that $U = B_U \cap V$. Set

$$W = \bigcup_{U \in \mathscr{U}} (B_U \setminus V) \cup (V \setminus \bigcup \mathscr{U}).$$

By hypothesis, $\mathscr{D}_W = \sigma_X(\mathscr{C} \cup \{V\}) \cap \sigma_X(\mathscr{C} \cup \{W\})$ is c.g., so by Lemma 2 take $D, D' \in \mathscr{C}$ such that

$$\mathscr{D}_W = \sigma_X(\mathscr{C} \cup \{(D \cup D') \setminus V\})$$

and

$$D \subseteq V \Delta W, \quad D' \cap (V \Delta W) = \emptyset.$$

If $U \in \mathscr{U}$, then

$$U \in \mathscr{C}\restriction_V \subseteq \sigma_X(\mathscr{C} \cup \{V\}) \quad \text{and} \quad U = B_U \setminus W \in \sigma_X(\mathscr{C} \cup \{W\}),$$

so $U \in \mathscr{D}_W$. It follows that $U \setminus (D \cup D') \in \mathscr{C}$. But also $U \subseteq V \setminus W$, so

$$U \cap D' = \emptyset \quad \text{and} \quad U \setminus (D \cup D') = U \setminus D.$$

Clearly, $\bigcup \mathscr{U} = V \setminus W$, thus $D \cap V \subseteq \bigcup \mathscr{U}$ as claimed.

THEOREM 2. *Let X be a set and let $\mathscr{C}$ be a c.g. σ-algebra on X. Suppose that $V \subseteq X$ is such that $\sigma_X(\mathscr{C} \cup \{V\}) \cap \sigma_X(\mathscr{C} \cup \{W\})$ is c.g. whenever $W \subseteq X$. If $\mathscr{F}$ is a disjoint family in $\sigma_X(\mathscr{C} \cup \{V\})$, then*

(i) $2^{|\mathscr{F} \setminus \mathscr{C}|} \leqslant \mathfrak{c}$;

(ii) *if there exists a complete separable metric on X such that $\mathscr{C} \subseteq \mathscr{B}(X)$, where $\mathscr{B}(X)$ is the family of all Borel sets with respect to that metric, then $|\mathscr{F} \setminus \mathscr{C}| \leqslant \omega$.*

Proof. (i) Let $\mathscr{F} \subseteq \sigma_X(\mathscr{C} \cup \{V\})$ be a disjoint family. For each $F \in \mathscr{F}$ let $B_F, C_F \in \mathscr{C}$ be such that

$$F = (B_F \cap V) \cup (C_F \setminus V).$$

If $F \notin \mathscr{C}$, then $B_F \cap V \notin \mathscr{C}$ or $C_F \setminus V \notin \mathscr{C}$. Putting

$$\mathscr{U} = \{B_F \cap V \colon F \in \mathscr{F}\} \quad \text{or} \quad \mathscr{U} = \{C_F \setminus V \colon F \in \mathscr{F}\}$$

we may assume that $\mathscr{U}$ is a disjoint family in $\mathscr{C}\restriction_{\hat{V}}$ such that $|\mathscr{U}\setminus\mathscr{C}| = |\mathscr{F}\setminus\mathscr{C}|$, where $\hat{V} = V$ or $X\setminus V$, respectively.

Now, if $\mathscr{U}' \subseteq \mathscr{U}\setminus\mathscr{C}$, by Lemma 5 there exists a set $D\in\mathscr{C}$ such that

$$\mathscr{U}' = \{U\in\mathscr{U}\setminus\mathscr{C}\colon D\cap U \neq \emptyset\}.$$

Therefore $2^{|\mathscr{F}\setminus\mathscr{C}|} = 2^{|\mathscr{U}\setminus\mathscr{C}|} \leqslant |\mathscr{C}| \leqslant \mathfrak{c}$.

(ii) Suppose the opposite. By the argument as in the proof of part (i) there is an uncountable disjoint family $\{U_\alpha\colon \alpha<\omega_1\}$ in $\mathscr{C}\restriction_{\hat{V}}\setminus\mathscr{C}$. Let $A\subseteq\{0,1\}^\omega$ be a non-Borel analytic set and let $\{B_\alpha\colon \alpha<\omega_1\}$ be a family of non-empty constituents of $\{0,1\}^\omega\setminus A$ (see [5], p. 499). For each $\alpha<\omega_1$ choose $s_\alpha\in B_\alpha$ and, by Lemma 5, for each $n<\omega$ choose $F_n, F'_n\in\mathscr{C}$ such that

$$F_n\cap U_\alpha = \emptyset \text{ if } s_\alpha(n) = 0, \qquad U_\alpha\setminus F_n\in\mathscr{C} \text{ if } s_\alpha(n) = 1$$

and

$$(\hat{V}\setminus\bigcup_{\alpha<\omega_1} U_\alpha)\cap F_n = \emptyset;$$

$$F'_n\cap U_\alpha = \emptyset \text{ if } s_\alpha(n) = 1, \qquad U_\alpha\setminus F'_n\in\mathscr{C} \text{ if } s_\alpha(n) = 0$$

and

$$(\hat{V}\setminus\bigcup_{\alpha<\omega_1} U_\alpha)\cap F'_n = \emptyset.$$

Set $F = \bigcap_{n<\omega}(F_n\cup F'_n)$. Clearly, $U_\alpha\setminus F\in\mathscr{C}$ for every $\alpha<\omega_1$ and

$$(\hat{V}\setminus\bigcup_{\alpha<\omega_1} U_\alpha)\cap F = \emptyset.$$

Define $f\colon X\to\{0,1\}^\omega$ by

$$(f(x))(n) = \begin{cases} 1 & \text{if } x\in F_n,\\ 0 & \text{if } x\notin F_n.\end{cases}$$

Notice that $f[F\cap U_\alpha] = \{s_\alpha\}$. Set

$$W = \bigcup_{\alpha<\omega_1}(F\cap U_\alpha)\cup f^{-1}[A].$$

Take $D, D'\in\mathscr{C}$ from Lemma 2 such that

$$D\subseteq \hat{V}\,\Delta\, W, \qquad D'\cap(\hat{V}\,\Delta\, W) = \emptyset$$

and

$$\mathscr{D}_W = \sigma_X(\mathscr{C}\cup\{V\})\cap\sigma_X(\mathscr{C}\cup\{W\}) = \sigma_X(\mathscr{C}\cup\{(D\cup D')\cap\hat{V}\}).$$

Fix $\alpha<\omega_1$. As

$$F\cap U_\alpha = W\cap f^{-1}[\{s_\alpha\}]\subseteq\sigma_X(\mathscr{C}\cup\{W\}),$$

we have

$$F \cap U_\alpha \in \mathcal{D}_W \quad \text{and} \quad (F \cap U_\alpha) \setminus (D \cup D') \in \mathcal{C}.$$

Also $U_\alpha \cap F \subseteq \hat{V} \cap W$, so $(U_\alpha \cap F) \cap D = \emptyset$, and hence

$$(U_\alpha \cap F) \setminus D' \in \mathcal{C}.$$

As $U_\alpha \notin \mathcal{C}$ and $U_\alpha \setminus F \in \mathcal{C}$, the intersection $(U_\alpha \cap F) \cap D' \notin \mathcal{C}$ and, in particular, is non-empty.

It follows that $\{s_\alpha \colon \alpha < \omega_1\} \subseteq f[F \cap D']$ and, as $f[F \cap D']$ is analytic,

$$f[F \cap D'] \cap A \neq \emptyset$$

(see [5], p. 501). Thus D' meets $F \cap f^{-1}[A]$. Furthermore, as $D' \cap (\hat{V} \Delta W) = \emptyset$ and $f^{-1}[A] \subseteq W$,

$$\emptyset \neq D' \cap F \cap f^{-1}[A] \subseteq \hat{V}.$$

But $F \cap \hat{V} \subseteq \bigcup_{\alpha < \omega_1} U_\alpha$, so

$$D' \cap F \cap f^{-1}[A] \cap \bigcup_{\alpha < \omega_1} U_\alpha \neq \emptyset,$$

which is impossible because $f[F \cap U_\alpha] \cap A = \emptyset$ for every $\alpha < \omega_1$.

By Theorems 1 and 2 we obtain the following

COROLLARY 2. *Let $\mathcal{C}$ be a c.g. σ-algebra on a set X. If* CH *holds or if X is a complete separable metric space and $\mathcal{C} \subseteq \mathcal{B}(X)$, then for every $V \subseteq X$ the following conditions are equivalent*:

(i) *There exists a set $W \subseteq X$ such that $\sigma_X(\mathcal{C} \cup \{V\}) \cap \sigma_X(\mathcal{C} \cup \{W\})$ is not c.g.*

(ii) *There exists a non-c.g. σ-algebra $\mathcal{A}$ such that*

$$\mathcal{C} \subseteq \mathcal{A} \subseteq \sigma_X(\mathcal{C} \cup \{V\}).$$

(iii) *There exists an uncountable disjoint family in $\sigma_X(\mathcal{C} \cup \{V\}) \setminus \mathcal{C}$.*

COROLLARY 3. *If $\mathcal{C}$ is a c.g. σ-algebra on a set X and $|X| \geqslant \mathfrak{c}$, then there exist c.g. σ-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ on X such that $\mathcal{C} \subseteq \mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2$ is not c.g.*

Proof. Enlarge $\mathcal{C}$ (if necessary) to a c.g. $\mathcal{C}' \supseteq \mathcal{C}$ with $\mathfrak{c}$ many disjoint sets $B_\alpha \in \mathcal{C}'$ of cardinality $\mathfrak{c}$. For each $\alpha < \mathfrak{c}$ pick $V_\alpha \in \mathcal{P}(B_\alpha) \setminus \mathcal{C}'$. Set

$$V = \bigcup_{\alpha < \mathfrak{c}} V_\alpha$$

and, by Theorem 2 (i), take $W \subseteq X$ such that

$$\mathcal{C}_1 = \sigma_X(\mathcal{C}' \cup \{V\}) \quad \text{and} \quad \mathcal{C}_2 = \sigma_X(\mathcal{C}' \cup \{W\})$$

have the required properties.

To show that the assumption in Theorem 2 (ii) is essential we would like to conclude with the following

EXAMPLE. Assume MA. Let $Y \subseteq \boldsymbol{R}$ with $|Y| < \mathfrak{c}$ and let $X \subseteq \boldsymbol{R}$ be a $\mathfrak{c}$-Lusin set.

By the Rothberger–Silver Theorem ([3], p. 60, 23B) it follows that $\mathscr{B}(Y) = \mathscr{P}(Y)$. Since whenever $D \in \mathscr{B}(X)$, then D is a $G_{\delta\sigma}$ in X (see, e.g., [3], p. 64, 23M(a)), by Theorem 3 of [2] we have $B \in \mathscr{B}(X \times Y)$ iff, for every $y \in Y$, $B^y \in \mathscr{B}(X)$, where, for $S \subseteq X \times Y$, $S^y = \{x\colon (x, y) \in S\}$.

Let V, $W \subseteq Z = X \times Y$.

CLAIM. *$C \in \sigma_Z(\mathscr{B}(Z) \cup \{V\})$ iff, for every $y \in Y$,*

$$C^y \in \sigma_X(\mathscr{B}(X) \cup \{V^y\}).$$

Suppose that $C \subseteq Z$ and, for every $y \in Y$,

$$C^y = (B_y \cap V^y) \cup (B'_y \setminus V^y) \quad \text{for some } B_y,\, B'_y \in \mathscr{B}(X).$$

Set

$$B = \bigcup_{y \in Y} (B'_y \times \{y\}) \quad \text{and} \quad B' = \bigcup_{y \in Y} (B'_y \times \{y\}).$$

Clearly, $C = (B \cap V) \cup (B' \setminus V)$ and, as B, $B' \in \mathscr{B}(Z)$,

$$C \in \sigma_Z(\mathscr{B}(X) \cup \{V\}).$$

The reverse implication is trivial.

Now, since $\sigma_X(\mathscr{B}(X) \cup \{V^y\}) \cap \sigma_X(\mathscr{B}(X) \cup \{W^y\})$ is c.g. (compare the proof of Corollary 1), by Lemma 2 for each $y \in Y$ there exists a set $D_y \subseteq X$ such that

$$\sigma_X(\mathscr{B}(X) \cup \{V^y\}) \cap \sigma_X(\mathscr{B}(X) \cup \{W^y\}) = \sigma_X(\mathscr{B}(X) \cup \{D_y\}).$$

Set $D = \bigcup_{y \in Y} (D_y \times \{y\})$ and notice that

$$\sigma_Z(\mathscr{B}(Z) \cup \{V\}) \cap \sigma_Z(\mathscr{B}(Z) \cup \{W\}) = \sigma_X(\mathscr{B}(Z) \cup \{D\}).$$

Obviously, if $\mathfrak{c} > \omega_1$, $|Y| > \omega$ and V is chosen such that $V^y \notin \mathscr{B}(X)$ for every $y \in Y$, then $\{V^y \times \{y\}\colon y \in Y\}$ is an uncountable disjoint family in $\sigma_Z(\mathscr{B}(Z) \cup \{V\}) \setminus \mathscr{B}(Z)$.

The above Example shows that it is consistent to suppose that there exist separable σ-algebras $\mathscr{C}_1 \subseteq \mathscr{C}_2$ on X with a non-c.g. σ-algebra $\mathscr{A}$ such that $\mathscr{C}_1 \subseteq \mathscr{A} \subseteq \mathscr{C}_2$ but for every $W \subseteq X$ the intersection $\mathscr{C}_2 \cap \sigma_X(\mathscr{C}_1 \cup \{W\})$ is c.g. As pointed out in Corollary 2 no such σ-algebras can be constructed in the presence of the CH.

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