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ON FILLING AN IRREDUCIBLE CONTINUUM WITH THE CARTESIAN PRODUCT OF 1-DIMENSIONAL CONTINUA

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In this paper, a *continuum* is a compact connected metric space.

As defined in [6], $\mathscr{K}$ denotes the class of all continua K such that there exists an upper semicontinuous decomposition G of an irreducible continuum M with each element of G homeomorphic to K and with decomposition space M/G an arc. In [2] it is shown that every n-cell is in $\mathscr{K}$. In [3] it is shown that the Knaster indecomposable continuum (see [4], p. 209) and the curve $\sin x^{-1}$ are in $\mathscr{K}$. Also, in [1] it is shown that the universal Sierpiński curve in the plane (cf. [5], pp. 275–276) is in $\mathscr{K}$. In this paper it is shown that the Cartesian product of each of these continua with an arc belongs to $\mathscr{K}$.

In this paper, T will denote the Cantor middle third set on $[0, 1]$ and $I_1, I_2, I_3, \ldots$ will denote the components of $[0, 1]-T$. For each positive integer n, let c_n denote the left endpoint of $\bar{I}_n$, the closure of I_n, and d_n denote the right endpoint of $\bar{I}_n$. Put

$$E = \bigcup_{n=1}^{\infty} (c_n \cup d_n).$$

Observation 1. There exists a closed subset M' of $T \times T$ such that
(1) if P is a point of E, then $M' \cap (P \times T)$ is an isometric copy of T;
(2) if P is a point of $T-E$, then $M' \cap (P \times T)$ is degenerate;
(3) if t is a subset of T such that

$$(c_n \times t) = M' \cap (c_n \times T),$$

then

$$(d_n \times t) = M' \cap (d_n \times T);$$

(4) if, for each positive integer n, P_n is a point of $M' \cap (c_n \times T)$ and Q_n is a point of $M' \cap (d_n \times T)$, then $\bigcup_{n=1}^{\infty} \{Q_n\}$ is dense in M';

(5) $M' \cap [(T-E) \times T]$ is dense in M'.

A set M' satisfying all of these conditions may be obtained by taking a sequence $t_1, t_2, \ldots$ of subsets of T in such a way that

$$\lim_{n \to \infty} \operatorname{diam}(t_n) = 0 \quad \text{and} \quad \overline{\bigcup_{n=1}^{\infty} [(c_n \cup d_n) \times t_n]} = M'.$$

OBSERVATION 2. By Theorem 1 of [2], there exists a compact metric continuum M such that

(1) M is irreducible and a subset of $[0, 1] \times [0, 1]$;

(2) there exists an upper semicontinuous collection G of arcs filling up M such that M/G is an arc;

(3) there exists a countable subcollection H of G such that

(a) if h is in H, h contains an arc Z_h such that each point of Z_h is a separating point of M, and Z_h contains every separating point of M in h,

(b) if E' denotes the set of all points P such that P is an endpoint of Z_h for some h in H, then E' is dense in $M - \bigcup_{h \in H} Z_h$,

(c) if $\delta > 0$, then only finitely many members h of H have $\operatorname{diam}(Z_h) > \delta$,

(d) $\bigcup_{h \in H} Z_h$ contains all separating points of M.

As in the proof of Theorem 2 of [2], let K denote the collection to which k belongs if and only if, for some element h of H, k is the closure of a component of $h - Z_h$. Now

$$(G-H)^* \cup K^* = \overline{M - \bigcup_{h \in H} Z_h}$$

and $(G-H) \cup K$ is an upper semicontinuous collection of mutually exclusive arcs filling up $\overline{M - \bigcup_{h \in H} Z_h}$. Furthermore,

$$(\overline{M - \bigcup_{h \in H} Z_h})/[(G-H) \cup K]$$

is a copy of the Cantor set T.

Let C denote indecomposable continuum due to Knaster (see [4], p. 209).

THEOREM 1. *$[0, 1] \times C$ is an element of $\mathscr{K}$.*

Proof. Let f denote a continuous mapping from M onto $[0, 1]$ such that f restricted to $\bigcup_{h \in H} Z_h$ is one-to-one and onto $\bigcup_{n=1}^{\infty} \bar{I}_n$ and if $h \in H$, then there is one and only one n such that $f(Z_h) = \bar{I}_n$. Let g denote the projection mapping of M' onto T.

Let L_1 denote the set to which x belongs if and only if, for some point P of T, x is a point of $g^{-1}(P)\times f^{-1}(P)$ and let

$$L_1(P) = g^{-1}(P)\times f^{-1}(P).$$

Let C_1 denote the closure of the set of all points of C above the x-axis and let C_2 denote the closure of the set of all points of C below the x-axis. If P is a point of

$$E = \bigcup_{n=1}^{\infty} (c_n \cup d_n),$$

the endpoints of $I_1, I_2, \ldots$, then $L_1(P)$ is a copy of $T\times[0, 1]$, and if P is a point of $T-E$, then $L_1(P)$ is an arc.

If P is a point of $c_1, c_2, \ldots$, then there is an arc h_p of H, as in Observation 2, such that $f(h_p)$ contains P. The set $\overline{h_p-Z_{h_p}}$ has two components, r_p and s_p, where $f(r_p) = c_n$ and $f(s_p) = d_n$. Let a_p and a'_p denote the endpoints of r_p, where a_p is an endpoint of h_p; and let b_p and b'_p denote the endpoints of s_p, where b_p is an endpoint of h_p.

For each positive integer n, let $L'_1(c_n)$ denote the subset $g^{-1}(c_n)\times a_{c_n}$ of $L_1(c_n)$ and let $L'_1(d_n)$ denote the subset $g^{-1}(d_n)\times b_{c_n}$ of $L_1(d_n)$. Also, let $L''_1(c_n)$ denote $g^{-1}(c_n)\times a'_{c_n}$ and let $L''_1(d_n)$ denote $g^{-1}(d_n)\times b'_{c_n}$.

There exists an upper semicontinuous collection U_1 filling up L_1 such that

(1) if x is a point of $L_1 - \bigcup_{P\in E} L'_1(P)$, then x is in U_1;

(2) one element of U_1 which intersects $\bigcup_{n=1}^{\infty} L'_1(d_n)$ is a point of $\bigcup_{n=1}^{\infty} L'_1(d_n)$ and each of the other elements of U_1 which intersect $\bigcup_{P\in E} L'_1(P)$ is a pair of points of $L'_1(P)$ for some point P of E;

(3) L_1/U_1 is the sum of an upper semicontinuous collection of arcs such that, for each n, $L_1(c_n)/U_1$ is homeomorphic to C_1 and $L_1(d_n)/U_1$ is homeomorphic to C_2.

In order to see that such a collection U_1 exists it is sufficient to observe that if P is in E and $u_1, u_2, \ldots$ are elements of U_1 in $L'_1(P)$, then

$$\lim_{n\to\infty} \operatorname{diam}(u_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \operatorname{diam}[L'_1(c_n)\cup L'_1(d_n)] = 0.$$

Note that if P is in $T-E$, then $L_1(P)/U_1$ is an arc. Let L_2 denote $(L_1/U_1)\times C$.

Let $\alpha_1, \alpha_2, \ldots$ denote a sequence of arcs in C such that, for each n, $\alpha_n \subset \alpha_{n+1}$ and let $g_1, g_2, \ldots$ denote a sequence of continuous mappings such that, for each n, g_n maps C onto α_n and if P is a point of C, then the distance from P to $g_n(P)$ is less than $1/n$.

Let U_2 denote the collection to which x belongs if and only if

(1) for some point P of $T-E$, x is a point of $(L_1(P)/U_1)\times C$;

(2) for some positive integer n, some point P of $L_1(c_n)/U_1$ and some point Q of α_n, x is $P\times g_n^{-1}(Q)$; or

(3) for some positive integer n, some point P of $L_1(d_n)/U_1$ and some point Q of α_n, x is $P\times g_n^{-1}(Q)$.

U_2 is an upper semicontinuous decomposition of $(L_1/U_1)\times C$ since if, for each n, Q_n is a point of α_n and P_n is a point of $L_1(c_n\cup d_n)$, then

$$\lim_{n\to\infty} \operatorname{diam}[P_n\times g_n^{-1}(Q_n)] = 0.$$

Let L_3 denote $[(L_1/U_1)\times C]/U_2$.

There exists a sequence of points $Q_1, Q_2, Q_3, \ldots$ such that, for each n, Q_n is a point of α_n. The set

$$R_c = \bigcup_{n=1}^{\infty} [(L_1''(c_n)/U_1)\times g_n^{-1}(Q_n)]$$

is dense in L_3, and the set

$$R_d = \bigcup_{n=1}^{\infty} [(L_1''(d_n)/U_1)\times g_n^{-1}(Q_n)]$$

is also dense in L_3.

Suppose R_c' is a subset of R_c such that if n is a positive integer, then only one point of R_c' is in $(L_1''(c_n)/U_1)\times g_n^{-1}(Q_n)$, and assume that R_d' is a subset of R_d such that, for each n, only one point of R_d' is in $(L_1''(d_n)/U_1)\times g_n^{-1}(Q_n)$. Then from the construction of M' in Observation 1 and of M in Observation 2 we see that R_c' and R_d' are both dense in L_3.

Let $\theta_1, \theta_2, \ldots$ denote a sequence of arcs such that, for each n, θ_n is a subset of α_n that contains Q_n and $\operatorname{diam}(\theta_n) < 1/n$.

Suppose P is a point of $L_1''(c_n)$ for some n. Let a'_{c_n} and b'_{c_n} be defined as in paragraph 4 of the proof of this theorem. There is a point (c_n, y) of M' such that P is $(c_n, y)\times a'_{c_n}$. The point $(d_n, y)\times b'_{c_n}$ is a point of $L_1''(d_n)$. Let

$$m_n(P) = (d_n, y)\times b'_{c_n}.$$

m_n is a continuous one-to-one map from $L_1''(c_n)$ onto $L_1''(d_n)$. Furthermore, P is also an element of U_1, and so is $m_n(P)$; therefore, m_n is a continuous one-to-one map of $L_1''(c_n)/U_1$ onto $L_1''(d_n)/U_1$. Also, if P_1 and P_2 are points of $L_1''(c_n)/U_1$, then

$$\operatorname{diam}(P_1, m_n(P_1)) = \operatorname{diam}(P_2, m_n(P_2)).$$

If Q is a point of $[(L_1''(c_n)/U_1)\times C]/U_2$, then there are a point P of $L_1''(c_n)/U_1$ and a point q of α_n such that $P\times g_n^{-1}(q)$ is Q. Let

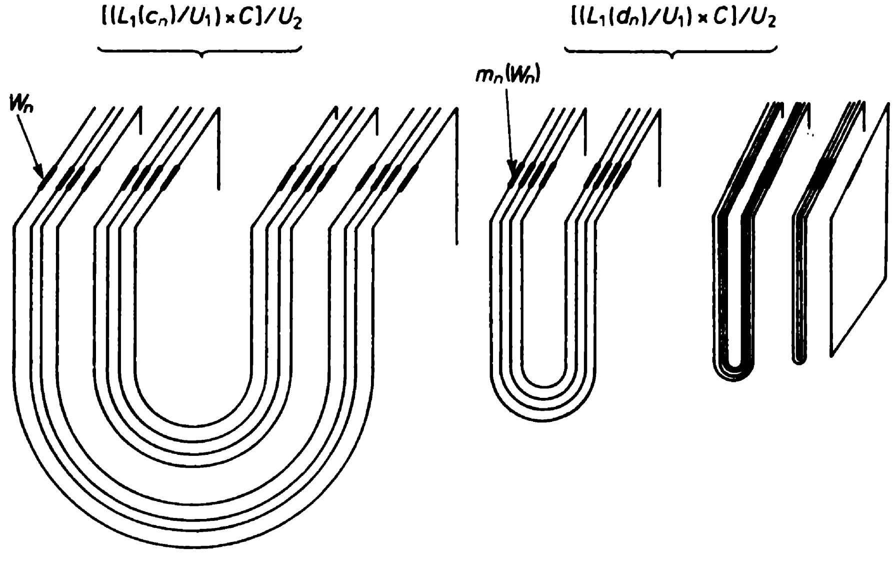
$$\hat{m}_n(Q) = m_n(P)\times g_n^{-1}(q).$$

Again, if Q_1 and Q_2 are points of $[(L_1''(c_n)/U_1)\times C]/U_2$, then

$$\operatorname{diam}(Q_1, \hat{m}_n(Q_1)) = \operatorname{diam}(Q_2, \hat{m}_n(Q_2))$$

and $\hat{m}_n$ is a continuous one-to-one map from $[(L_1''(c_n)/U_1)\times C]/U_2$ onto $[(L_1''(d_n)/U_1)\times C]/U_2$.

For each n, let W_n denote the subset of $[(L_1''(c_n)/U_1)\times C]/U_2$ to which Q belongs if and only if, for some point P of $L_1''(c_n)/U_1$ and some point q of θ_n, Q is $P\times g_n^{-1}(q)$. Note that W_n is the sum of an upper semicontinuous collection w_n of mutually exclusive straight-line closed intervals all of the same length such that W_n/w_n is a copy of T.



Let U_3 denote the collection to which x belongs if and only if

(1) for some positive integer n and some point Q of W_n,

$$x = Q\cup \hat{m}_n(Q)$$

or

(2) x is a point of the set

$$L_3 - \bigcup_{n=1}^{\infty} [W_n \cup \hat{m}_n(W_n)].$$

U_3 is an upper semicontinuous decomposition of $[(L_1/U_1)\times C]/U_2$ since, for each n, Q_n is a point of W_n,

$$\lim_{n\to\infty} \operatorname{diam}(Q_n \cup \hat{m}_n(Q_n)) = 0.$$

Finally, let L_4 denote $[(L_1/U_1)\times C]/U_2/U_3$. Let U denote the collection to which x belongs if and only if

(1) for some point P of $T-E$, x is a point of the set

$$[(L_1(P)/U_1)\times C]/U_2/U_3$$

or

(2) for some n, x is $\{[(L_1(c_n) \cup L_1(d_n))/U_1] \times C\}/U_2/U_3$.

The elements of U are all homeomorphic to $[0, 1] \times C$ and L_4/U is an arc.

Suppose L_4 is not irreducible from

$$l_A = [(L_1(0)/U_1) \times C]/U_2/U_3$$

to

$$l_\beta = [(L_1(1)/U_1) \times C]/U_2/U_3.$$

Then there is a subcontinuum L_4' of L_4 which intersects l_A and l_β. For each n, L_4' must intersect a point

$$l_n = P_n \cup \hat{m}_n(P_n)$$

for some P_n of W_n in $[(L_1''(c_n)/U_1) \times C]/U_2$. The set $\bigcup_{n=1}^{\infty} \{l_n\}$ is dense in L_4. Therefore L_4 is irreducible. This completes the proof of Theorem 1.

Let $s(x) = \sin x^{-1}$, $0 \leqslant x \leqslant 1$. Also, let s denote the graph of $s(x)$ and A denote the vertical interval from $(0, -1)$ to $(0, 1)$. Let $S = A \cup s$.

THEOREM 2. *$[0, 1] \times S$ belongs to $\mathscr{K}$.*

Proof. Let M, G, H, and K be defined as in Observation 2 and, for each h in H, let Z_h be defined as before. Also, let the function f be defined as in the proof of Theorem 1 and, for each positive integer n, let a'_{c_n} and b'_{c_n} be defined as in paragraph 4 of the proof of Theorem 1.

Let

$$L_1 = \overline{M - \bigcup_{h \in H} Z_h} \times S.$$

For each positive integer n, let $\alpha_1, \alpha_2, \ldots$ denote a sequence of arcs in S all containing the point $(1, s(1))$, $\alpha_n \subset \alpha_{n+1}$, and $g_1, g_2, \ldots$ denote a sequence of maps from S onto α_n such that for each point P of α_n,

$$\operatorname{diam}(g_n^{-1}(P)) < 1/n \quad \text{and} \quad g_n(P) = P.$$

Let U_1 denote the collection to which x belongs if and only if

(1) x is a point of $L_1 - \bigcup_{n=1}^{\infty} (f^{-1}(d_n) \times S)$

(2) for some positive integer n, some point P of α_n, and some point Q of $f^{-1}(d_n)$, x is $Q \times g_n^{-1}(P)$.

U_1 is an upper semicontinuous collection of mutually exclusive closed point sets filling up L_1, since if, for each n, Q_n is a point of $f^{-1}(d_n)$ and P_n is a point of α_n, then

$$\lim_{n \to \infty} \operatorname{diam}(Q_n \times g_n^{-1}(P_n)) = 0.$$

If n is a positive integer, $(f^{-1}(d_n)\times S)/U_1$ is homeomorphic to a disc; and if Q is a point of $f^{-1}(d_n)$, then $(Q\times S)/U_1$ is an arc. Let L_2 denote L_1/U_1.

Let $p_1, p_2, \ldots$ denote a sequence of points in S such that, for each n, p_n is in α_n and $\bigcup_{n=1}^{\infty} p_n$ is dense in S; and let $\theta_1, \theta_2, \ldots$ denote a sequence of arcs such that, for each n, θ_n is a subset of α_n that contains p_n and

$$\lim_{n\to\infty} \operatorname{diam}(\theta_n) = 0.$$

If n is a positive integer, q is a point of θ_n and Q is $a'_{c_n}\times q$, let $m_n(Q)$ denote $b'_{c_n}\times g_n^{-1}(q)$. Let U_2 denote the collection to which x belongs if and only if

(1) x is a point of the set

$$L_1/U_1 - \bigcup_{n=1}^{\infty}\left[(a'_{c_n}\times\theta_n)\cup(b'_{c_n}\times m_n(\theta_n))\right]$$

or

(2) for some positive integer n and some point Q of $a'_{c_n}\times\theta_n$, x is $Q\cup m_n(Q)$.

U_2 is an upper semicontinuous collection of mutually exclusive closed point sets filling up L_2 since

$$\lim_{n\to\infty} \operatorname{diam}\left[(a'_{c_n}\times\theta_n)\cup(b'_{c_n}\times m_n(\theta_n))\right] = 0.$$

Let L_3 denote $L_1/U_1/U_2$.

Suppose L_3 is not irreducible from $f^{-1}(0)\times S$ to $f^{-1}(1)\times S$. Let L'_3 denote a proper subcontinuum of L_3 intersecting $f^{-1}(0)\times S$ and $f^{-1}(1)\times S$. For each n, there is a point q_n of θ_n such that L'_3 contains a point

$$p'_n = (a'_{c_n}\times q_n)\cup(b'_{c_n}\times m_n(q_n))$$

in L_3. Since $\bigcup_{n=1}^{\infty}\{a'_{c_n}\}$ is dense in $\overline{M-\bigcup_{h\in H} Z_h}$, $\bigcup_{n=1}^{\infty}\{p_n\}$ is dense in S, $\lim_{n\to\infty}\operatorname{diam}(\theta_n)=0$ and θ_n contains p_n, it follows that $\bigcup_{n=1}^{\infty}\{p'_n\}$ is dense in L_3. This involves a contradiction. Therefore, L_3 is irreducible.

Let U_3 denote the collection to which x belongs if and only if

(1) for some point P of $T-\bigcup_{n=1}^{\infty}(c_n\cup d_n)$, x is $f^{-1}(P)\times S$

or

(2) for some positive integer n, x is

$$[(f^{-1}(c_n)\times S)\cup(f^{-1}(d_n)\times S)]/U_1/U_2.$$

Each element of U_3 is homeomorphic to $[0, 1] \times S$ and L_3/U_3 is an arc. This completes the proof.

THEOREM 3. *The Cartesian product of* $[0, 1]$ *with the universal plane continuum is in* $\mathscr{K}$.

Proof. Let M, H, Z_h for each h in H, f, a'_{c_n} and b'_{c_n} be defined as before. Let N denote the universal Sierpiński curve in the plane (cf. [5], pp. 275–276) bounded by the unit circle.

Let L_1 denote $\overline{(M - \bigcup_{h \in H} Z_h)} \times N$. There exists a sequence $J_1, J_2, \ldots$ of single closed curves lying in N such that

(1) for each n, J_n does not intersect the boundary of a complementary domain of N;

(2) $\lim_{n \to \infty} \operatorname{diam}(J_n) = 0$;

(3) $\bigcup_{n=1}^{\infty} (a'_{c_n} \times J_n)$ and $\bigcup_{n=1}^{\infty} (b'_{c_n} \times J_n)$ are dense in L_1.

For each n, let

$$J'_n = \overline{N \cap \operatorname{Int}(J_n)}.$$

If, for some point n and some point q of J'_n, P is $a'_{c_n} \times q$, let $m_n(P)$ denote $b'_{c_n} \times q$.

Let U_1 denote the collection to which x belongs if and only if

(1) for some point P of $L_1 - \bigcup_{n=1}^{\infty} [(a'_{c_n} \cup b'_{c_n}) \times J'_n]$, x is P

or

(2) for some n, P is a point of $a'_{c_n} \times J'_n$; then x is $P \cup m_n(P)$.

Let U_2 denote the collection to which x belongs if and only if

(1) for some point P of $T - E$, x is $(f^{-1}(P) \times N)/U_1$

or

(2) for some n, x is $[(f^{-1}(c_n \cup d_n)) \times N]/U_1$.

As before, L_1/U_1 is irreducible, $L_1/U_1/U_2$ is an arc, and the elements of U_2 are all homeomorphic to $[0, 1] \times N$.

QUESTIONS. Is every chainable continuum in $\mathscr{K}$? **(P 1339)**

Does the Cartesian product of $[0, 1]$ with a chainable member of $\mathscr{K}$ belong to $\mathscr{K}$? **(P 1340)**

Does the Cartesian product of $[0, 1]$ with a member of $\mathscr{K}$ belong to $\mathscr{K}$? **(P 1341)**

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