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BOREL AND MONOTONE HIERARCHIES AND EXTENSION OF RÉNYI PROBABILITY SPACES

BY

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1. It is well known that in various standard cases (real line, Cantor set, Hilbert cube etc.) the Borel hierarchy contains ω_1 different families, i.e., $\mathscr{G}_\alpha \subsetneq \mathscr{G}_{\alpha+1}$ for each ordinal $\alpha < \omega_1$, where $\mathscr{G}_0$ is the family of all open sets in the considered space and

$$(1)\qquad \mathscr{G}_\alpha = \begin{cases} \big(\bigcup\limits_{\beta<\alpha} \mathscr{G}_\beta\big)_\sigma & \text{if } \alpha \text{ is even,} \\ \big(\bigcup\limits_{\beta<\alpha} \mathscr{G}_\beta\big)_\delta & \text{if } \alpha \text{ is odd,} \end{cases}$$

where $\mathscr{K}_\sigma$ and $\mathscr{K}_\delta$ mean, as usual, the families of all countable unions and intersections, respectively, of sets from a given family $\mathscr{K}$. On the other hand, under Continuum Hypothesis, for each $\alpha < \omega_1$ there exists a subset of the Cantor set such that for the topology induced on this subset by $\mathscr{G}_0$ we have

$$(2)\qquad \mathscr{G}_\beta \subsetneq \mathscr{G}_{\beta+1} \quad \text{for } \beta < \alpha$$

and

$$(3)\qquad \mathscr{G}_\alpha = \mathscr{G}_{\alpha+1} = \ldots$$

(see [4]).

Of course, we can consider the families $\mathscr{G}_\alpha$ of subsets of a given set X, defined by formula (1), without assuming that X is a topological space. The only assumptions we impose on the family $\mathscr{G}_0$ is that finite unions and finite intersections of sets from $\mathscr{G}_0$ are again in $\mathscr{G}_0$ (by induction, all the families $\mathscr{G}_\alpha$ possess this property). Also in this general case, by a *Borel hierarchy* (generated by a given $\mathscr{G}_0$) we shall mean the transfinite sequence $\mathscr{G}_\alpha$, $0 \leqslant \alpha \leqslant \omega_1$, defined by (1) and by its length the smallest ordinal α such that (2) and (3) hold.

In this paper, we shall show that the length of Borel hierarchies is strictly connected with the length of other hierarchies of families of sets.

It is evident that $\mathscr{K}_{\sigma\sigma} = \mathscr{K}_\sigma$ for an arbitrary family $\mathscr{K}$ of sets. An analogous property does not hold for the operation of forming unions of increasing sequences of sets from a given family.

For a given family $\mathscr{K}$ of subsets of a set X, let $\mathscr{K}_\Sigma$ denote the family of all sets of the form $\bigcup_{n=1}^{\infty} A_n$, where $A_n \in \mathscr{K}$ and $A_n \subset A_{n+1}$. Put

$$\mathscr{M}_1 = \mathscr{K}_\Sigma \quad \text{and} \quad \mathscr{M}_\alpha = \Big(\bigcup_{\beta<\alpha} \mathscr{M}_\beta\Big)_\Sigma \tag{4}$$

for an arbitrary ordinal $\alpha > 0$, assuming that the families $\mathscr{M}_\beta$ are defined for $\beta < \alpha$.

If $\mathscr{K}$ is an additive family, then we have $\mathscr{K}_{\Sigma\Sigma} = \mathscr{K}_\Sigma$ (cf. Theorem 1). On the other hand, it is not very difficult to construct a non-additive family $\mathscr{K}$ of sets such that the families $\mathscr{M}_\alpha$ defined above satisfy the relation

$$\mathscr{M}_\alpha \subsetneq \mathscr{M}_{\alpha+1} \quad \text{for all } \alpha < \omega_1 \tag{5}$$

(cf. [3]).

The situation is more interesting if the non-additive family $\mathscr{K}$ is postulated to be *hereditary*, i.e., $\mathscr{K} = \mathscr{K}_h$, where

$$\mathscr{K}_h = \{A \subset X\colon\ A \subset B \text{ for some } B \in \mathscr{K}\}.$$

If a family $\mathscr{K} = \mathscr{M}_0$ of subsets of X is hereditary, then the hierarchy $\{\mathscr{M}_\alpha\}$ generated by $\mathscr{K}$ (i.e., defined by (4)) will be called *monotone* (note that we can consider the dual hierarchies $\{\mathscr{M}'_\alpha\}$ defined by putting $\mathscr{M}'_1 = \mathscr{K}_\Delta$ and $\mathscr{M}'_\alpha = \big(\bigcup_{\beta<\alpha} \mathscr{M}'_\beta\big)_\Delta$ for $\alpha > 0$, where $\mathscr{K}$ is a family such that $A \supset B \in \mathscr{K}$ implies $A \in \mathscr{K}$ and $\mathscr{K}_\Delta$ consists of all sets of the form $\bigcap_{n=1}^{\infty} A_n$ with $A_n \in \mathscr{K}$, $A_n \supset A_{n+1}$).

In this paper, we shall show that there exist monotone hierarchies satisfying (5), i.e., of length ω_1. The construction used in Section 2 will be based on the Borel hierarchies of length ω_1. Namely, given a Borel hierarchy $\{\mathscr{G}_\alpha\}$ in X, we define the hereditary family $\mathscr{M}_0$ of all subsets of graphs of functions $f\colon X \to \{0, 1\}$ such that $f^{-1}(1) \in \mathscr{G}_1$. It appears that the monotone hierarchy $\{\mathscr{M}_\alpha\}$ in $Y = X \times \{0, 1\}$ generated by the family $\mathscr{K} = \mathscr{M}_0$ is of length ω_1 if and only if the length of the Borel hierarchy $\{\mathscr{G}_\alpha\}$ in X is ω_1 (see Theorem 3). In particular, if X is the Cantor set and $\mathscr{G}_1$ is the countable basis of open-and-closed subsets of X, then we find a countable family $\mathscr{K}$ of subsets of Y such that $\mathscr{M}_0 = \mathscr{K}_h$ generates the monotone hierarchy of length ω_1.

Moreover, we shall show (see Theorem 2) that if the length of the hierarchy $\{\mathscr{G}_\alpha\}$ ($\{\mathscr{M}_\alpha\}$) is known, then the length of the hierarchy $\{\mathscr{M}_\alpha\}$ ($\{\mathscr{G}_\alpha\}$) can be estimated and the difference of the lengths is finite.

Monotone hierarchies appear in connection with extensions of Rényi probability spaces (see Sections 3–4). It is proved in [2] that a given Rényi space $\mathscr{R}$ can be extended to Rényi spaces $\mathscr{R}^\circ$ and $\mathscr{R}^*$, whose definitions are related to the defined above extensions $\mathscr{M}_h$ and $\mathscr{M}_\Sigma$ of a given family $\mathscr{M}$ of subsets of some set (see Section 3). These extensions can be iterated. Denoting the αth iteration, $2 \leqslant \alpha \leqslant \omega_1$, of these extensions by $\mathscr{R}^\circ_\alpha$ and $\mathscr{R}^*_\alpha$, respectively, we have $\mathscr{R}^\circ_\alpha = \mathscr{R}^\circ$ and, in general, $\mathscr{R}^*_\alpha \subsetneq \mathscr{R}^*_{\alpha+1}$ for $\alpha < \omega_1$ (see [3]).

In [3], it is shown that the extension $\tilde{\mathscr{R}} = (\mathscr{R}^\circ)^*_{\omega_1}$ has the following property of minimality: $\tilde{\mathscr{R}}$ is the smallest Rényi space containing $\mathscr{R}$ such that $\tilde{\mathscr{R}}^\circ = \tilde{\mathscr{R}}$ and $\tilde{\mathscr{R}}^* = \tilde{\mathscr{R}}$.

The results about monotone hierarchies of families of sets allow us to construct (see Section 3) a Rényi space $\mathscr{R}$ such that $(\mathscr{R}^\circ)^* \subsetneq (\mathscr{R}^\circ)^*_{\alpha+1}$ for each $\alpha < \omega_1$, i.e. the extensions $(\mathscr{R}^\circ)^*_\alpha$ for $\alpha < \omega_1$ are not minimal, in general, in the above sense.

In Section 4, we shall give sufficient conditions for a given Rényi space $\mathscr{R}$ which guarantee that the Rényi space $\mathscr{R}^{\circ *}$ is minimal in that sense.

2. First we shall consider two particular families of sets which generate the monotone hierarchies of length 1. The second one is observed by A. Iwanik.

Let $\mathscr{K}$ be a family of subsets of a given set X and let $\mathscr{M}_0 = \mathscr{K}_h$.

THEOREM 1. *Suppose that* 1° $\mathscr{K}$ *is additive, or* 2° $\mathscr{K}$ *contains only countable sets. Then* $\mathscr{M}_1 = \mathscr{M}_2$, *i.e.*, $(\mathscr{K}_h)_\Sigma = (\mathscr{K}_h)_{\Sigma\Sigma}$.

Proof. It suffices to show that if $A \in \mathscr{M}_2$, then $A \in \mathscr{M}_1$. Assume that $A \in \mathscr{M}_2$, i.e. $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset A_{i+1}$ and $A_i = \bigcup_{j=1}^{\infty} A_{ij}$, where $A_{ij} \in \mathscr{M}_0$ and $A_{ij} \subset A_{i,j+1}$.

If $\mathscr{K}$ is additive, then so is $\mathscr{M}_0$. Let $B_n = \bigcup_{i,j \leqslant n} A_{ij}$, $B_n \in \mathscr{M}_0$. Since $B_n \subset B_{n+1}$ and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{i,j=1}^{\infty} A_{ij} = A$, we have $A \in \mathscr{M}_1$, as desired.

To prove the assertion in case 2° note that the set A is countable and arrange its elements in the sequence $x_1, x_2, \ldots$ For each $k \in N$ there exists an index i_k such that $x_1, \ldots, x_k \in A_{i_k j_k}$. Let

$$B_n = \bigcap_{k=n}^{\infty} A_{i_k j_k}.$$

Obviously, $B_n \subset B_{n+1}$ and $B_n \in \mathscr{M}_0$, because $\mathscr{M}_0$ is hereditary. Moreover, since $B_n \subset A$ and $B_n \supset \{x_1, \ldots, x_n\}$, we have $\bigcup_{n=1}^{\infty} B_n = A$. Consequently, $A \in \mathscr{M}_1$ and the assertion is proved.

Now, let X be a given set and $\mathscr{G}_0$ a family of its subsets closed with respect to finite unions and finite intersections. Consider the Borel hierarchy $\{\mathscr{G}_\alpha\}$ defined by formula (1). Define another hierarchy putting $\mathscr{H}_0 = \mathscr{G}_1$ and

$$\mathscr{H}_\alpha = \Big(\bigcup_{\beta<\alpha} \mathscr{H}_\beta\Big)_{\sigma\delta},$$

whenever the families $\mathscr{H}_\beta$ for $\beta < \alpha$ $(\alpha > 0)$ are defined.

It is easy to check, by induction, that

$$\mathscr{H}_{\alpha+n} = \mathscr{G}_{\alpha+2n+1} \tag{6}$$

for any limit ordinal α and $n < \omega_0$.

Let $\mathscr{S}_0$ denote the set of all functions $f\colon X \to \{0, 1\}$ such that $f^{-1}(1) \in \mathscr{H}_0$, i.e.

$$\mathscr{S}_0 = \{\chi_A\colon A \in \mathscr{H}_0\}.$$

Assuming that $\mathscr{S}_\beta$ for $\beta < \alpha$ are defined, we adopt for $\mathscr{S}_\alpha$ the set of all functions $f\colon X \to \{0, 1\}$ of the form

$$f(x) = \limsup_{n\to\infty} f_n(x) \qquad (x \in X),$$

where $f_n \in \bigcup_{\beta<\alpha} \mathscr{S}_\beta$ for $n = 1, 2, \ldots$

By $G(f)$ for $f \in \mathscr{S}_\alpha$ we shall denote the graph of the function f, i.e.

$$G(f) = \{(x, y)\colon x \in X,\ y = f(x)\}.$$

Put

$$\mathscr{M}_0 = \{G(f)\colon f \in \mathscr{S}_0\}_s,$$

i.e. a set $A \in X \times \{0, 1\}$ belongs to $\mathscr{M}_0$ iff $A \subset G(f)$ for some $f \in \mathscr{S}_0$.

Let $\mathscr{M}_\alpha$ be the families of the monotone hierarchy, defined by formula (4), generated by $\mathscr{M}_0$. Obviously, $\mathscr{M}_\alpha$ are hereditary.

We shall prove some lemmas about connections between the hierarchies of the families $\mathscr{G}_\alpha$, $\mathscr{H}_\alpha$ and $\mathscr{M}_\alpha$.

LEMMA 1. *If $A \in \mathscr{M}_\alpha$, then there exists $f \in \mathscr{S}_\alpha$ such that $A \subset G(f)$. In particular, if $G(g) \in \mathscr{M}_\alpha$ for $g\colon X \to \{0, 1\}$, then $g \in \mathscr{S}_\alpha$.*

Proof. The first assertion for $\alpha = 0$ follows from the definition of $\mathscr{M}_0$. Suppose that it is true for all ordinals $\beta < \alpha$, where $\alpha > 0$, and let $A \in \mathscr{M}_\alpha$. That means there exists a non-decreasing sequence of sets $A_n \in \bigcup_{\beta<\alpha} \mathscr{M}_\beta$ such that $A = \bigcup_{n=1}^{\infty} A_n$. By the induction hypothesis, we have $A_n \subset G(f_n)$ for some $f_n \in \bigcup_{\beta<\alpha} \mathscr{S}_\beta$ $(n = 1, 2, \ldots)$. Put $f = \limsup_{n\to\infty} f_n \in \mathscr{S}_\alpha$. Note that if $(x, y) \in A$, then $(x, y) \in A_n$ and $y = f_n(x)$ for sufficiently large n, so $y = f(x)$ and $(x, y) \in G(f)$. Thus $A \subset G(f)$ and the first assertion is proved for arbitrary α.

Since the inclusion $G(g) \subset G(f)$ for $f, g\colon X \to \{0, 1\}$ implies $f = g$, the second assertion follows from the first one.

LEMMA 2. *If* $f \in \mathscr{S}_\alpha$, *then* $f^{-1}(1) \in \mathscr{H}_\alpha$.

Proof. For $\alpha = 0$, the lemma is true by the definition of $\mathscr{S}_0$. Now let $\alpha > 0$ and assume that the assertion holds for all $\beta < \alpha$ and that $f \in \mathscr{S}_\alpha$. Therefore there exist functions $f_n \in \bigcup_{\beta<\alpha} \mathscr{S}_\beta$ such that $f = \limsup_{n\to\infty} f_n$. We have

$$f^{-1}(1) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(1)$$

and $f_n^{-1}(1) \in \bigcup_{\beta<\alpha} \mathscr{H}_\beta$ for $n = 1, 2, \ldots$ Hence $f^{-1}(1) \in \mathscr{H}_\alpha$ and the proof is completed.

Immediately from Lemmas 1 and 2, we obtain

LEMMA 3. *Let* $A \subset X$. *If* $G(\chi_A) \in \mathscr{M}_\alpha$, *then* $A \in \mathscr{H}_\alpha$.

On the other hand, we have

LEMMA 4. *If* $A \in \mathscr{G}_\alpha$, *then* $G(\chi_A) \in \mathscr{M}_\alpha$. *If* $A \in \mathscr{H}_{\gamma+n}$ *for a limit ordinal* γ *and* $n < \omega_0$, *then* $G(\chi_A) \in \mathscr{M}_{\gamma+2n+1}$.

Proof. The second statement results from the first one, in view of (6).

The first assertion is true for $\alpha = 0$, by the definition of $\mathscr{M}_0$. Suppose that it holds for all $\beta < \alpha$ $(\alpha > 0)$ and let $A \in \mathscr{G}_\alpha$.

If α is even, then $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \bigcup_{\beta<\alpha} \mathscr{G}_\beta$. Since $\mathscr{G}_0$ and, consequently, all $\mathscr{G}_\beta$ are closed with respect to finite unions, we can assume that $A_n \subset A_{n+1}$. Putting $B_n = (A^c \times \{0\}) \cup (A_n \times \{1\})$, we have $B_n \subset B_{n+1}$, $B_n \subset G(\chi_{A_n})$ and $\bigcup_{n=1}^{\infty} B_n = G(\chi_A)$. By the induction hypothesis and hereditariness of $\mathscr{M}_\beta$, we have $B_n \in \bigcup_{\beta<\alpha} \mathscr{M}_\beta$ for $n = 1, 2, \ldots$, so $G(\chi_A) \in \mathscr{M}_\alpha$, i.e. the assertion holds for α.

If $\alpha = \beta+1$ is odd, then $A = \bigcap_{n=1}^{\infty} A_n$ for $A_n \in \mathscr{G}_\beta$ such that $A_n \supset A_{n+1}$. Let $B_n = (A_n^c \times \{0\}) \cup (A \times \{1\})$. We have $B_n \subset B_{n+1}$, $B_n \subset G(\chi_{A_n})$ and $\bigcup_{n=1}^{\infty} B_n = G(\chi_A)$. Since $B_n \in \mathscr{M}_\beta$ $(n = 1, 2, \ldots)$, we infer that $G(\chi_A) \in \mathscr{M}_\alpha$, as desired.

Now, we shall show mutual relations between the lengths of the hierarchies $\{\mathscr{G}_\alpha\}$ and $\{\mathscr{M}_\alpha\}$.

THEOREM 2. *Let* α_0 *and* β_0 *be the lengths of the hierarchies* $\{\mathscr{G}_\alpha\}$ *and* $\{\mathscr{M}_\alpha\}$, *respectively. If* $\alpha_0 = \gamma+n$, *where* γ *is a limit ordinal and* $n < \omega_0$, *then*

$$\gamma+k \leqslant \beta_0 \leqslant \alpha_0 \tag{7}$$

where $k = [n/2]$. *If* $\beta_0 = \gamma + m$ (i.e., $k \leqslant m \leqslant n$), *then*

$$\beta_0 \leqslant \alpha_0 \leqslant \beta_0 + m + 1. \tag{8}$$

Proof. First we shall show that $\beta_0 \leqslant \alpha_0$, i.e.,

$$\mathscr{M}_{\alpha_0+1} = \mathscr{M}_{\alpha_0}. \tag{9}$$

Suppose that $A \in \mathscr{M}_{\alpha_0+1}$. By Lemma 1, $A \subset G(f)$ for some $f \in \mathscr{S}_{\alpha_0+1}$. In turn, we have $f = \chi_B$ for $B \in \mathscr{H}_{\alpha_0+1} = \mathscr{G}_{\alpha_0}$, in view of Lemma 2 and (6). Hence, by Lemma 4, $G(\chi_B) \in \mathscr{M}_{\alpha_0}$ and thus $A \in \mathscr{M}_{\alpha_0}$, because $\mathscr{M}_{\alpha_0}$ is hereditary. Equality (9) is shown.

To prove the inequality $\gamma + k \leqslant \beta_0$ suppose that $\mathscr{M}_{\gamma+k} = \mathscr{M}_\beta$ where
1° $\beta = \gamma + k - 1$ if $k \geqslant 1$ or
2° $\beta = \gamma' + l < \gamma$ (γ' is a limit ordinal, $l < \omega_0$) if $k = 0$.

Putting $\beta' = \gamma + n - 1$ in case 1° and $\beta' = \gamma' + 2l + 1$ in case 2°, we choose a set $A \in \mathscr{G}_{\alpha_0} \setminus \mathscr{G}_{\beta'}$. By Lemma 4, $G(\chi_A) \in \mathscr{M}_{\alpha_0} = \mathscr{M}_\beta$, which yields $A \in \mathscr{H}_\beta$, due to Lemma 3. But we have

$$\mathscr{H}_\beta = \mathscr{G}_{\gamma+2k-1} \subset \mathscr{G}_{\gamma+n-1} = \mathscr{G}_{\beta'}$$

in case 1° and

$$\mathscr{H}_\beta = \mathscr{G}_{\gamma'+2l+1} = \mathscr{G}_{\beta'}$$

in case 2°. This means the relation $A \in \mathscr{H}_\beta$ implies $A \in \mathscr{G}_{\beta'}$ which is impossible. The contradiction proves that $\mathscr{M}_{\gamma+k} \neq \mathscr{M}_\beta$ for any $\beta < \gamma + k$, i.e. $\gamma + k \leqslant \beta_0$. Thus inequality (7) is proved.

Now, let $\beta_0 = \gamma + m$ and assume that $A \in \mathscr{G}_{\beta_0+m+2}$. By Lemma 4,

$$G(\chi_A) \in \mathscr{M}_{\beta_0+m+2} = \mathscr{M}_{\beta_0}$$

whence

$$A \in \mathscr{H}_{\beta_0} = \mathscr{G}_{\beta_0+m+1},$$

by virtue of Lemma 3 and (6). Thus we have proved the relation $\mathscr{G}_{\beta_0+m+1} = \mathscr{G}_{\beta_0+m+2}$, which means that $\alpha_0 \leqslant \beta_0 + m + 1$, so inequalities (8) and the theorem are proved.

As an immediate corollary to Theorem 2, we get

THEOREM 3. *The length of the Borel hierarchy* $\{\mathscr{G}_\alpha\}$ *is* ω_1 *if and only if the length of the monotone hierarchy* $\{\mathscr{M}_\alpha\}$ *is* ω_1.

Remark 1. Together with the Borel hierarchy $\{\mathscr{G}_\alpha\}$, generated by the family of all open subsets of a given X, defined by (1), another hierarchy is usually considered by letting for $\mathcal{F}_0$ the family of all closed subsets of X and

$$(10) \qquad \mathcal{F}_\alpha = \begin{cases} (\bigcup_{\beta<\alpha} \mathcal{F}_\beta)_\delta & \text{if } \alpha \text{ is odd,} \\ (\bigcup_{\beta<\alpha} \mathcal{F}_\beta)_\sigma & \text{if } \alpha \text{ is even} \end{cases}$$

for $\alpha > 0$. Of course, the hierarchy $\{\mathcal{F}_\alpha\}$ defined by formula (10), where $\mathcal{F}_0$ is an arbitrary family of subsets of X, closed with respect to finite unions and finite intersections, is a Borel hierarchy in the sense of Section 1. It suffices to put $\mathcal{G}_0 = \mathcal{F}_1$. Conversely, given a Borel hierarchy $\{\mathcal{G}_\alpha\}$, by putting $\mathcal{F}_0 = \mathcal{G}_1$ we get the hierarchy described by (10).

Remark 2. It is worth noting that exactly the same construction of the monotone hierarchy in $Y = X \times \{0, 1\}$ as above can be repeated if the induction definitions of the sets S_α and $\mathcal{H}_\alpha$ are modified in some way. Namely, we can assume that $f \in S_\alpha$ iff $f(x) = \liminf_{n\to\infty} f_n(x)$ $(x \in X)$ for some functions $f_n \in \bigcup_{\beta<\alpha} S_\beta$ $(n = 1, 2, \ldots)$ and

$$\mathcal{H}_\alpha = (\bigcup_{\beta<\alpha} \mathcal{H}_\beta)_{\delta\sigma}.$$

Remark 3. If X is the Cantor set and $\mathcal{G}_1$ is the countable basis of open-and-closed sets in X, then

$$\mathcal{K} = \{G(\chi_A)\colon\ A \in \mathcal{G}_1\}$$

is a countable family such that the monotone hierarchy $\{\mathcal{M}_\alpha\}$ generated by $\mathcal{M}_0 = \mathcal{K}_h$ has length ω_1.

Remark 4. Note that Lemmas 1–4, Theorems 2, 3 and Remarks 2, 3 are true for monotone hierarchies $\{\mathcal{M}'_\alpha\}$, dual to $\{\mathcal{M}_\alpha\}$, defined by setting $\mathcal{M}'_1 = K_\Delta$ and $\mathcal{M}'_\alpha = (\bigcup_{\beta<\alpha} \mathcal{M}_\beta)_\Delta$ for some family $\mathcal{K}$ fulfilling the condition: $A \supset B \in \mathcal{K}$ implies $A \in \mathcal{K}$ (see Section 1). Namely, it suffices to consider complements of the sets used for the families in the hierarchies $\{\mathcal{M}_\alpha\}$ and the respective assertions follow from the facts proved above.

On the other hand, let us notice that Lemmas 1–4 and all their consequences formulated for a hierarchy $\{\mathcal{M}'_\alpha\}$ can be proved immediately, if we take for $\{\mathcal{M}'_\alpha\}$ the hierarchy generated by the family:

$$\mathcal{M}'_0 = \{A \in X \times \{0, 1\}\colon\ A \supset G(f) \text{ for some } f \in \mathcal{H}_0\}.$$

3. Now, we shall give the definition of Rényi spaces, being a generalization of the classical probability spaces of Kolmogorov, and recall some facts concerning extensions of Rényi spaces.

By a *Rényi space* we shall mean in this paper a system $\mathcal{R} = [\Omega, \mathcal{A}, \mathcal{B}, P]$, where Ω is an arbitrary set, $\mathcal{A}$ is a σ-algebra of its subsets,

$\mathscr{B}$ is a non-empty subfamily of $\mathscr{A}$ and $P(\cdot|\cdot)$ is a mapping from $\mathscr{A} \times \mathscr{B}$ into $[0, 1]$ (*conditional probability*) such that $P(\cdot|B)$ is a probability measure on $\mathscr{A}$ for each $B \in \mathscr{B}$ and moreover

(i) $P(B|B) = 1$ for $B \in \mathscr{B}$;

(ii) $P(A|B) = \dfrac{P(A \cap B|B')}{P(B|B')}$ whenever $A \in \mathscr{A}$; $B, B' \in \mathscr{B}$; $B \subset B'$; $P(B|B') > 0$;

(iii) $P(A_1|B_1) \cdot P(A_2|B_2) = P(A_2|B_1) \cdot P(A_1|B_2)$ for $A_1, A_2 \in \mathscr{A}$ and $B_1, B_2 \in \mathscr{B}$ such that $A_1 \cup A_2 \subset B_1 \cap B_2$ (cf. [5], pp. 289, 291, [6], p. 70).

Let us formulate some properties of Rényi spaces, following immediately from the definition:

$$P(A|B') = P(A|B) \cdot P(B|B') \tag{11}$$

for $A \in \mathscr{A}$ and $B, B' \in \mathscr{B}$ such that $A \subset B \subset B'$;

$$P(B_1|B_n) = \prod_{i=1}^{n-1} P(B_i|B_{i+1})$$

for $B_1, \ldots, B_n \in \mathscr{B}$ such that $B_1 \subset \ldots \subset B_n$;

$$\lim_{n\to\infty} P(A \cap A_n|B) = P\big(A \cap \bigcup_{n=1}^{\infty} A_n|B\big) \tag{12}$$

for $A, A_n \in \mathscr{A}$ with $A_n \subset A_{n+1}$ and $B \in \mathscr{B}$;

$$\lim_{n\to\infty} P(B_1|B_n) = \prod_{i=1}^{\infty} P(B_i|B_{i+1}) \tag{13}$$

for $B_i \in \mathscr{B}$ such that $B_i \subset B_{i+1}$ (cf. [5] and [2]).

From the definition of Rényi spaces it also follows that $\emptyset \notin \mathscr{B}$, so the family $\mathscr{B}$ is not multiplicative. On the other hand, $\mathscr{B}$ need not be additive, in general.

In a given Rényi space $\mathscr{R}$, the following two types of extensions of the family $\mathscr{B}$ (by adding sets from $\mathscr{A}$) can be considered.

Let $\mathscr{B}^\circ$ denote the family of all sets $B^\circ \in \mathscr{A}$ such that $B^\circ \subset B$, $P(B^\circ|B) > 0$ for some $B \in \mathscr{B}$, and let $\mathscr{B}^*$ be the family of all sets of the form $\bigcup_{n=1}^{\infty} B_n$, where $\{B_n\}$ is a $\mathscr{B}$-*increasing sequence*, i.e. $B_n \in \mathscr{B}$, $B_n \subset B_{n+1}$ for $n = 1, 2, \ldots$ and

$$\prod_{i=1}^{\infty} P(B_i|B_{i+1}) > 0.$$

We define a conditional probability for the extended families $\mathscr{B}^\circ$ and $\mathscr{B}^*$ as follows:

$$P^\circ(A|B^\circ) = \frac{P(A \cap B^\circ|B)}{P(B^\circ|B)} \quad (A \in \mathscr{A},\ B^\circ \in \mathscr{B}^\circ), \tag{14}$$

where $B \in \mathscr{B}$, $B^\circ \subset B$, $P(B^\circ|B) > 0$, and

$$(15) \qquad P^*(A|B^*) = \lim_{n\to\infty} P(A|B_n) \quad (A \in \mathscr{A}, B^* \in \mathscr{B}^*),$$

where $B^* = \bigcup_{n=1}^{\infty} B_n$ and $\{B_n\}$ is $\mathscr{B}$-increasing.

It is proved in [2] (Theorems 3.2 and 5.3) that definitions (14) and (15) are consistent and the systems $\mathscr{R}^\circ = [\Omega, \mathscr{A}, \mathscr{B}^\circ, P^\circ]$ and $\mathscr{R}^* = [\Omega, \mathscr{A}, \mathscr{B}^*, P^*]$ are Rényi spaces containing $\mathscr{R}$ (i.e. $\mathscr{B} \subset \mathscr{B}^\circ$, $\mathscr{B} \subset \mathscr{B}^*$ and $P^\circ = P^* = P$ on $\mathscr{A} \times \mathscr{B}$).

Extensions of Rényi spaces can be iterated. Put $\mathscr{R}_1^* = \mathscr{R}^*$ and let α be an ordinal greater than 1. Assume that $\mathscr{R}_\beta^*$ are Rényi spaces for each $\beta < \alpha$ such that $\mathscr{R}_\beta^* \subset \mathscr{R}_{\beta+1}^*$ for $\beta+1 < \alpha$ (i.e. $\mathscr{B}_\beta^* \subset \mathscr{B}_{\beta+1}^*$ and $P_\beta^* = P_{\beta+1}^*$ on $\mathscr{A} \times \mathscr{B}_\beta^*$). Note that the system $\mathscr{R}_\alpha = [\Omega, \mathscr{A}, \mathscr{B}_\alpha, P_\alpha]$, where $\mathscr{B}_\alpha = \bigcup_{\beta<\alpha} \mathscr{B}_\beta^*$, $P_\alpha = P_\beta^*$ on $\mathscr{A} \times \mathscr{B}_\beta^*$, is a Rényi space. We define $\mathscr{R}_\alpha^* = [\Omega, \mathscr{A}, \mathscr{B}_\alpha^*, P_\alpha^*]$, where $\mathscr{B}_\alpha^* = (\mathscr{B}_\alpha)^*$ and $P_\alpha^* = (P_\alpha)^*$. By Theorem 5.3 in [2], $\mathscr{R}_\alpha^*$ is a Rényi space.

It can be proved that the system $\bar{\mathscr{R}} = \mathscr{R}_{\omega_1}^*$ is the smallest Rényi space containing $\mathscr{R}$ such that $\bar{\mathscr{R}}^* = \bar{\mathscr{R}}$ (see Theorem 5.6 in [2]).

On the other hand, we have $\mathscr{R}^{\circ\circ} = \mathscr{R}^\circ$, i.e. $\mathscr{B}^{\circ\circ} = \mathscr{B}^\circ$ and $P^{\circ\circ} = P^\circ$ on $\mathscr{A} \times \mathscr{B}^\circ$ (see Theorem 3.2 in [2]).

In [3] (Theorem 2), it is shown that $\tilde{\mathscr{R}} = (\mathscr{R}^\circ)_{\omega_1}^*$ is the smallest Rényi space containing $\mathscr{R}$ and invariant with respect to both the operations of extensions: $\tilde{\mathscr{R}}^\circ = \tilde{\mathscr{R}}$ and $\tilde{\mathscr{R}}^* = \tilde{\mathscr{R}}$.

Applying the results of Section 2, we shall prove that the Rényi space $(\mathscr{R}^\circ)_\alpha^*$ for $\alpha < \omega_1$ has not the above property, in general. Namely

THEOREM 4. *There exists a Rényi space $\mathscr{R}$ such that*

$$(\mathscr{R}^\circ)_\alpha^* \subsetneq (\mathscr{R}^\circ)_{\alpha+1}^*$$

for each ordinal α, $1 \leqslant \alpha < \omega_1$.

Proof. Let $X = [0, 1]$ and $Y = X \times \{0, 1\}$. Let $\mathscr{K}$ be the family of all graphs $G(f) \subset Y$ of functions $f\colon X \to \{0, 1\}$ such that $f^{-1}(1)$ is an open set in X. Since the hierarchy of Borel sets in [0, 1] has length ω_1, the monotone hierarchy $\{\mathscr{M}_\alpha\}$, generated by $\mathscr{M}_0 = \mathscr{K}_h$ is also of length ω_1, in view of Theorem 3. That means

$$(16) \qquad \mathscr{M}_\alpha \subsetneq \mathscr{M}_{\alpha+1}$$

for each $\alpha < \omega_1$.

Let $\mathscr{C}$ be the family of all Borel subsets of $X = [0, 1]$ and let λ be the Lebesgue measure on $\mathscr{C}$. Let $\mathscr{C}_0 \subset \mathscr{C}$ be the family of all Borel subsets A of X such that $\lambda(A) > 0$. Put $\Omega = X \cup Y$, $\mathscr{A} = \{A \subset \Omega\colon A \cap X \in \mathscr{C}\}$, $\mathscr{B} = \{B \subset \Omega\colon B = X \cup D, D \in \mathscr{K}\}$ and $P(A|B) = \lambda(A \cap X)$ for any $A \in \mathscr{A}$ and

$B \in \mathscr{B}$. Note that $X \cap Y = \emptyset$ and $P(\cdot|B)$ is a probability measure on $\mathscr{A}$ for each $B \in \mathscr{B}$. Moreover,

$$\frac{P(A \cap B|B')}{P(B|B')} = \frac{\lambda(A \cap B \cap X)}{\lambda(B \cap X)} = \lambda(A \cap X) = P(A|B)$$

for $A \in \mathscr{A}$ and $B, B' \in \mathscr{B}$ such that $B \subset B'$, and

$$P(A_1|B) \cdot P(A_2|B) = \lambda(A_1 \cap X) \cdot \lambda(A_2 \cap X) = P(A_2|B_1) \cdot P(A_1|B_2)$$

for $A_1, A_2 \in \mathscr{A}$ and $B_1, B_2 \in \mathscr{B}$ such that $A_1 \cup A_2 \subset B_1 \cup B_2$. This means the system $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is a Rényi space.

Let $\tilde{\mathscr{B}}_0^* = \mathscr{B}^\circ$ and $\tilde{\mathscr{B}}_\alpha^* = (\mathscr{B}^\circ)_\alpha^*$ for $\alpha \geqslant 1$. Moreover, let $\tilde{P}_0^* = P^\circ$ and $\tilde{P}_\alpha^* = (P^\circ)_\alpha^*$. To prove the theorem, it suffices to show that

$$\tilde{\mathscr{B}}_\alpha^* \subsetneq \tilde{\mathscr{B}}_{\alpha+1}^* \tag{17}$$

for all α, $0 \leqslant \alpha < \omega_1$.

We shall show that

$$B \in \tilde{\mathscr{B}}_\alpha^* \quad \text{iff} \quad B = C \cup D, \quad \text{where } C \in \mathscr{C}_0,\ D \in \mathscr{M}_\alpha \tag{18}$$

and

$$\tilde{P}^*(A|B) = \frac{\lambda(A \cap C)}{\lambda(C)} \quad \text{for any } A \in \mathscr{A} \text{ and } B \in \tilde{\mathscr{B}}_\alpha^*. \tag{19}$$

First note that $B^\circ \in \tilde{\mathscr{B}}_0^* = \mathscr{B}^\circ$ iff $B^\circ = C \cup D$ for some $C \in \mathscr{C}_0$ and $D \in \mathscr{M}_0$. Moreover,

$$P^\circ(A|B^\circ) = \frac{\lambda(A \cap B^\circ \cap X)}{\lambda(B^\circ \cap X)} = \frac{\lambda(A \cap C)}{\lambda(C)}$$

for $A \in \mathscr{A}$.

Suppose now that $\alpha \geqslant 1$ and let (18) and (19) hold for all $\beta < \alpha$. Assume that $B \in \tilde{\mathscr{B}}_\alpha^*$, i.e. $B = \bigcup_{n=1}^{\infty} B_n$ for some $\tilde{\mathscr{B}}_\alpha$-increasing sequence $\{B_n\}$. By induction hypothesis, we have $B_n = C_n \cup D_n$, where $C_n \in \mathscr{C}_0$ and $D_n \in \bigcup_{\beta<\alpha} \mathscr{M}_\beta$ for $n = 1, 2, \ldots$ Of course, $C_n = B_n \cap X$ and $D_n = B_n \cap Y$, so $C_n \subset C_{n+1}$ and $D_n \subset D_{n+1}$. Putting $C = \bigcup_{n=1}^{\infty} C_n$ and $D = \bigcup_{n=1}^{\infty} D_n$, we have $B = C \cup D$, $C \in \mathscr{C}_0$ and

$$D \in \Big(\bigcup_{\beta<\alpha} \mathscr{M}_\beta\Big)_\Sigma = \mathscr{M}_\alpha.$$

Now, suppose that $B = C \cup D$, where $C \in \mathscr{C}_0$, $D \in \mathscr{M}_\alpha$, and let $D = \bigcup_{n=1}^{\infty} D_n$ for an increasing sequence of sets $D_n \in \bigcup_{\beta<\alpha} \mathscr{M}_\beta$. Put $B_n = C \cup D_n$. Of course,

$B_n \subset B_{n+1}$ and $\bigcup_{n=1}^{\infty} B_n = B$. Moreover $B_n \in \bigcup_{\beta<\alpha} \tilde{\mathscr{B}}_\beta^* = \tilde{\mathscr{B}}_\beta$ and

$$\tilde{P}_\alpha(A|B_n) = \frac{\lambda(A \cap C)}{\lambda(C)} \quad \text{for } A \in \mathscr{A},$$

by induction hypothesis. Hence $\prod_{n=1}^{\infty} \tilde{P}_\alpha(B_n|B_{n+1}) = 1$, because $\lambda(B_n \cap C) = \lambda(C) = 1$. Consequently, $B \in (\tilde{\mathscr{B}}_\alpha)^* = \tilde{\mathscr{B}}_\alpha^*$ and

$$\tilde{P}_\alpha^*(A|B) = \lim_{n\to\infty} \tilde{P}_\alpha(A|B_n) = \frac{\lambda(A \cap C)}{\lambda(C)},$$

which completes the proof of (18) and (19).

Relations (16) and (18) imply (17) and the proof of the theorem is finished.

4. We shall give in this section, for Rényi spaces of a special type, a condition guaranteeing that $\mathscr{R}^{\circ *}$ is the minimal Rényi space invariant with respect to the extension operations $^\circ$ and *.

First note that

$$\mathscr{R}^{\circ * \circ} = \mathscr{R}^{\circ *} \tag{20}$$

for an arbitrary Rényi space $\mathscr{R}$. In fact, we have, in general, $\mathscr{R}^{\circ *} \subset \mathscr{R}^{\circ * \circ}$ (see [2], Theorem 3.2) and, on the other hand,

$$\mathscr{R}^{\circ * \circ} \subset \mathscr{R}^{\circ \circ *} = \mathscr{R}^{\circ *},$$

in view of Theorem 1 in [3] and Theorem 3.2 in [2].

The equation

$$\mathscr{R}^{\circ **} = \mathscr{R}^{\circ *} \tag{21}$$

does not hold, in general, as Theorem 4 shows. It does under additional assumptions.

Following Á. Császár [1], we say that the Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *is generated by the ordered* (with respect to dimension) *family* $\{\mu_\alpha\}$ *of* (non-negative, bounded or not) *measures on* $\mathscr{A}$ if

1° indices α form a linearly ordered set,

2° if $A \in \mathscr{A}$, $\mu_\alpha(A) < \infty$ and $\alpha < \beta$, then $\mu_\beta(A) = 0$,

3° for each α there exists a $B \in \mathscr{B}$ such that $0 < \mu_\alpha(B) < \infty$,

4° if $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $0 < \mu_\alpha(B) < \infty$, then

$$P(A|B) = \frac{\mu_\alpha(A \cap B)}{\mu_\alpha(B)}.$$

In particular, a Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *is generated by one* (non-negative, bounded or not) *measure* μ *on* $\mathscr{A}$ if $0 < \mu(B) < \infty$ for $B \in \mathscr{B}$ and

$$P(A|B) = \frac{\mu(A \cap B)}{\mu(B)}$$

for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

In [1] (Theorems 3.3 and 3.5), it is proved that a Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is generated by an ordered family of measures on $\mathscr{A}$ iff $\mathscr{R}$ fulfils the condition:

$$\prod_{i=1}^{n} P(A_i|B_i) = \prod_{i=1}^{n} P(A_i|B_{i+1})$$

for arbitrary $n = 1, 2, \ldots$ and sets $A_i \in \mathscr{A}$, $B_i \in \mathscr{B}$ such that $A_i \subset B_i \cap B_{i+1}$ for $i = 1, 2, \ldots, n$, where $B_{n+1} = B_1$ (cf. condition (ii)).

Remark 5. Conditions 1°–3° imply that for each $B \in \mathscr{B}$ there exists exactly one index α such that $0 < \mu_\alpha(B) < \infty$.

Remark 6. If $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is generated by the ordered family $\{\mu_\alpha\}$ of measures on $\mathscr{A}$, then $\mathscr{R}^\circ = [\Omega, \mathscr{A}, \mathscr{B}^\circ, P^\circ]$ is also generated by $\{\mu_\alpha\}$. Moreover, if $B^\circ \in \mathscr{B}^\circ$, $B \in \mathscr{B}$, $B^\circ \subset B$ and $P(B^\circ|B) > 0$, then $0 < \mu_\alpha(B^\circ) < \infty$ iff $0 < \mu_\alpha(B) < \infty$.

In fact. Suppose that $\mathscr{R}$ fulfils 1°–4°. If $0 < \mu_\alpha(B) < \infty$ then $\mu_\alpha(B^\circ) < \mu_\alpha(B) < \infty$ and

$$0 < P(B^\circ|B) = \frac{\mu_\alpha(B^\circ)}{\mu_\alpha(B)},$$

so $0 < \mu_\alpha(B^\circ) < \infty$ and the first implication is shown. Moreover, this means that $\mathscr{R}^\circ$ satisfies conditions 1°–3°. Hence, in view of 3°, the implication just shown and Remark 5, we have also the converse implication. It remains to see that if $0 < \mu_\alpha(B^\circ) < \infty$ (which yields $0 < \mu_\alpha(B) < \infty$), then

$$P^\circ(A|B^\circ) = \frac{P(A \cap B^\circ|B)}{P(B^\circ|B)} = \frac{\mu_\alpha(A \cap B^\circ)}{\mu_\alpha(B^\circ)},$$

i.e. $\mathscr{R}^\circ$ fulfils condition 4°, too.

Remark 7. If $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is generated by the ordered family $\{\mu_\alpha\}$ of measures on $\mathscr{A}$, then $\mathscr{R}^* = [\Omega, \mathscr{A}, \mathscr{B}^*, P^*]$ is also generated by $\{\mu_\alpha\}$. Moreover, given a set $B^* = \bigcup_{n=1}^{\infty} B_n \in \mathscr{B}^*$, where $\{B_n\}$ is a $\mathscr{B}$-increasing sequence, we have $0 < \mu_\alpha(B_n) < \infty$ for all $n = 1, 2, \ldots$ iff $0 < \mu_\alpha(B^*) < \infty$.

To show this note first that if $\{B_n\}$ is a $\mathscr{B}$-increasing sequence and $0 < \mu_{\alpha_n}(B_n) < \infty$, then $\alpha_1 = \alpha_2 = \ldots = 0$. Indeed, if $\alpha_n > \alpha_{n+1}$ for some n, then

$$\mu_{\alpha_n}(B_n) \leqslant \mu_{\alpha_n}(B_{n+1}) = 0$$

and if $\alpha_n < \alpha_{n+1}$ for some n, then

$$\prod_{n=1}^{\infty} P(B_n|B_{n+1}) = \prod_{n=1}^{\infty} \frac{\mu_{\alpha_{n+1}}(B_n)}{\mu_{\alpha_{n+1}}(B_{n+1})} = 0$$

and both cases are impossible. Thus to every $\mathscr{B}$-increasing sequence $\{B_n\}$ there corresponds exactly one index α such that

$$(22) \qquad 0 < \mu_\alpha(B_n) < \infty \quad \text{for } n = 1, 2, \ldots$$

(cf. Remark 5).

Now, if $B^* = \bigcup_{n=1}^{\infty} B_n$ for a $\mathscr{B}$-increasing sequence $\{B_n\}$ satisfying (22), then also $0 < \mu_\alpha(B^*) < \infty$, since the condition

$$0 < \prod_{n=1}^{\infty} P(B_n|B_{n+1}) = \lim_{m\to\infty} \frac{\mu_\alpha(B_1)}{\mu_\alpha(B_m)}$$

(cf. (13)) implies $\mu_\alpha(B^*) < \infty$. This means $\mathscr{R}^*$ fulfils conditions 1°–3°. In turn, if $0 < \mu_\alpha(B^*) < \infty$, then $0 < \mu_\alpha(B_n) < \infty$ for $n = 1, 2, \ldots$, in view of 3° and the above observations. It remains to notice that $0 < \mu_\alpha(B^*) < \infty$ implies, for each $A \in \mathscr{A}$, the relations

$$P^*(A|B^*) = \lim_{n\to\infty} P(A|B_n) = \lim_{n\to\infty} \frac{\mu_\alpha(A \cap B_n)}{\mu_\alpha(B_n)} = \frac{\mu_\alpha(A \cap B^*)}{\mu_\alpha(B^*)},$$

i.e. $\mathscr{R}^*$ satisfies condition 4°, too.

Now, we formulate conditions for Rényi spaces to fulfil identity (21).

THEOREM 5. *Suppose that the Rényi space* $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *is generated by the ordered family* $\{\mu_\alpha\}$ *of measures on* $\mathscr{A}$ *and, moreover, the family* $\mathscr{B}$ *fulfils the following condition of completeness*:

(c) *if* $B \in \mathscr{B}$, $B \subset A \in \mathscr{A}$, $0 < \mu_\alpha(B) < \infty$ *and* $\mu_\alpha(A \setminus B) = 0$, *then* $A \in \mathscr{B}$.

Then $\mathscr{R}$ *satisfies* (21).

In particular, if $\mathscr{R}$ *is generated by one measure* μ *on* $\mathscr{A}$ *and*

(c′) $B \in \mathscr{B}$, $B \subset A \in \mathscr{A}$, $\mu(A \setminus B) = 0$ *imply* $A \in \mathscr{B}$,

then $\mathscr{R}$ *satisfies* (21).

In the proof of the above theorem we shall need the following lemma:

LEMMA 5. *Let* $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *be a Rényi space such that* $\mathscr{R} = \mathscr{R}^\circ$. *If we have*

$$(23) \qquad B = \bigcup_{i=1}^{\infty} B_i^*,$$

where $\{B_i^*\}$ *is a* $\mathscr{B}^*$*-increasing sequence, then there exists a* $\mathscr{B}$*-increasing sequence* $\{B_i\}$ *such that*

$$(24) \qquad B_i \subset B_i^*$$

and

$$\lim_{i\to\infty} P^*(B_i|B_i^*) = 1. \tag{25}$$

Proof. Suppose that (23) holds for some $\mathscr{B}^*$-increasing sequence $\{B_i^*\}$ and let $\{c_i\}$ be a given sequence of numbers such that $0 < c_i < 1$ and $c_i \to 1$. Since $\{B_i^*\}$ is $\mathscr{B}^*$-increasing, we have

$$P^{**}(B_1^*|B) = \prod_{i=1}^{\infty} P^*(B_i^*|B_{i+1}^*) > 0,$$

by (13). Let ε be a positive number such that

$$P^{**}(B_1^*|B) > \varepsilon.$$

Since $B_i^* \in \mathscr{B}^*$, there exist $\mathscr{B}$-increasing sequences $\{B_{ij}\}_{j=1,2,\ldots}$ for $i = 1, 2, \ldots$ such that

$$B_i^* = \bigcup_{j=1}^{\infty} B_{ij}. \tag{26}$$

We shall construct, by induction, indices $k_1, k_2, \ldots$ such that

$$P^*(B_{m,k_m} \cap \ldots \cap B_{n,k_n}|B_m^*) > c_m \quad \text{for } m \leqslant n \tag{27}$$

and

$$P^{**}(B_{1,k_1} \cap \ldots \cap B_{n,k_n}|B) > \varepsilon. \tag{28}$$

By (26) (for $i = 1$), we have

$$P^*(B_{1j}|B_1^*) \to 1 \quad \text{and} \quad P^{**}(B_{1j}|B) \to P^{**}(B_1^*|B)$$

as $j \to \infty$. Thus there exists a k_1 such that

$$P^*(B_{1,k_1}|B_1^*) > c_1 \quad \text{and} \quad P^{**}(B_{1,k_1}|B) > \varepsilon,$$

i.e. (27) and (28) hold for $n = 1$.

Assume that we have constructed indices $k_1, \ldots, k_n$ satisfying (27) and (28). Due to relations (26) (for $i = n+1$), (12) and

$$B_{m,k_m} \cap \ldots \cap B_{n,k_n} \subset B_n \subset B_{n+1},$$

we have

$$P^*(B_{n+1,k}|B_{n+1}^*) \to 1,$$

$$P^*\Big(\bigcap_{p=m}^{n} B_{p,k_p} \cap B_{n+1,k}|B_m^*\Big) \to P^*\Big(\bigcap_{p=m}^{n} B_{p,k_p}|B_m^*\Big) \tag{29}$$

for $m \leqslant n$, and

$$P^{**}\Big(\bigcap_{p=1}^{n} B_{p,k_p} \cap B_{n+1,k} | B\Big) \to P^{**}\Big(\bigcap_{p=1}^{n} B_{p,k_p} | B\Big) \tag{30}$$

as $k \to \infty$. Since the limits in (29) and (30) are greater than c_m (for $m \leqslant n$) and ε, respectively, and since $1 > c_{n+1}$, we can find an index k_{n+1} such that (27) and (28) hold for $n+1$ and $m \leqslant n+1$.

Now, let $B_i = B_{i,k_i} \cap B_{i+1,k_{i+1}} \cap \ldots$ We have $B_i \subset B_{i,k_i} \subset B_i^*$, so (24) holds and, by (11) and (27),

$$P(B_i | B_{i,k_i}) \geqslant P^*(B_i | B_i^*) \geqslant c_i > 0. \tag{31}$$

Hence $B_i \in \mathscr{B}^\circ = \mathscr{B}$. Moreover, $B_i \subset B_{i+1}$ and

$$P(B_1 | B_n) \geqslant P^*(B_1 | B) \geqslant \varepsilon$$

by virtue of (11) and (28), so

$$\prod_{i=1}^{\infty} P(B_i | B_{i+1}) = \lim_{n \to \infty} P(B_1 | B_n) > 0.$$

This means $\{B_i\}$ is a $\mathscr{B}$-increasing sequence such that relation (24) and, which is a consequence of (31), relation (25) hold.

Thus the lemma is proved.

Proof of Theorem 5. Suppose that $\mathscr{B}$ fulfils (c). We shall show first that also the families $\mathscr{B}^\circ$ and $\mathscr{B}^*$ satisfy (c).

Assume that $B^\circ \in \mathscr{B}^\circ$, $B^\circ \subset A \in \mathscr{A}$, $0 < \mu_\alpha(B^\circ) < \infty$ and $\mu_\alpha(A \setminus B^\circ) = 0$. By Remark 6, there exists a set $B \in \mathscr{B}$ such that $B^\circ \subset B$, $0 < \mu_\alpha(B) < \infty$ and

$$0 < P(B^\circ | B) = \frac{\mu_\alpha(B^\circ)}{\mu_\alpha(B)}.$$

Since $\mu_\alpha((A \cup B) \setminus B) \leqslant \mu_\alpha(A \setminus B^\circ) = 0$, we have $A \cup B \in \mathscr{B}$, by (c). Moreover,

$$0 < \mu_\alpha(A \cup B) = \mu_\alpha(B) < \infty$$

and

$$P(A | A \cup B) = \frac{\mu_\alpha(A)}{\mu_\alpha(A \cup B)} = \frac{\mu_\alpha(B^\circ)}{\mu_\alpha(B)} > 0,$$

so $A \in \mathscr{B}^\circ$. This means that $\mathscr{B}^\circ$ fulfils (c).

Now, assume that $B^* \in \mathscr{B}^*$, $B^* \subset A \in \mathscr{A}$, $0 < \mu_\alpha(B^*) < \infty$, $\mu_\alpha(A \setminus B^*) = 0$ and $B^* = \bigcup_{i=1}^{\infty} B_i$ for a $\mathscr{B}$-increasing sequence $\{B_i\}$. By Remark 7, we have $0 < \mu_\alpha(B_i) < \infty$ for all $i = 1, 2, \ldots$ Denoting $C = A \setminus B^*$ and $A_i = B_i \cup C$, we see that $A_i \in \mathscr{B}$, by virtue of (c). Moreover, $A_i \subset A_{i+1}$ and

$$\prod_{i=1}^{\infty} P(A_i|A_{i+1}) = \prod_{i=1}^{\infty} \frac{\mu_\alpha(B_i)}{\mu_\alpha(B_{i+1})} = \frac{\mu_\alpha(B_1)}{\mu_\alpha(B^*)} > 0.$$

This means $\{A_i\}$ is $\mathscr{B}$-increasing. Since, additionally, $\bigcup_{i=1}^{\infty} A_i = A$, we infer that $A \in \mathscr{B}^*$, so $\mathscr{B}^*$ fulfils (c).

In view of the above observations, it suffices to prove the equality

$$\mathscr{B}^* = \mathscr{B}^{**} \tag{32}$$

under the assumptions that $B = \mathscr{B}^\circ$ and $\mathscr{B}^*$ satisfies (c).

Suppose that $B \in \mathscr{B}^{**}$, i.e. (23) holds for some $\mathscr{B}^*$-increasing sequence $\{B_i^*\}$. By Remark 7, we have $0 < \mu_\alpha(B_i^*) < \infty$ $(i = 1, 2, \ldots)$ and

$$0 < \mu_\alpha(B) = \lim_{i\to\infty} \mu_\alpha(B_i^*) < \infty \tag{33}$$

for some α. By Lemma 5, there exists a $\mathscr{B}$-increasing sequence $\{B_i\}$ such that (24) and (25) hold, i.e., $\bigcup_{i=1}^{\infty} B_i \subset B$ and

$$\lim_{i\to\infty} \frac{\mu_\alpha(B_i)}{\mu_\alpha(B_i^*)} = 1.$$

Hence, by (33), $\mu_\alpha(B) = \lim_{i\to\infty} \mu_\alpha(B_i)$ or, equivalently,

$$\mu_\alpha(B \setminus \bigcup_{i=1}^{\infty} B_i) = 0.$$

Since $\bigcup_{i=1}^{\infty} B_i \in \mathscr{B}^*$ and $\mathscr{B}^*$ fulfils (c), we conclude that $B \in \mathscr{B}^*$. Consequently, (32) holds and the theorem is proved.

In view of (20), we obtain the following consequence of Theorem 5:

COROLLARY. *Suppose that* $\mathscr{R}$ *is generated by the ordered family* $\{\mu_\alpha\}$ *of measures (by one measure* μ*) on* $\mathscr{A}$ *satisfying condition* (c) *(condition* (c′)*). Then* $\mathscr{R}^{\circ *}$ *is the smallest Rényi space such that* $\mathscr{R} \subset \mathscr{R}^{\circ *}$, $(\mathscr{R}^{\circ *})^\circ = \mathscr{R}^{\circ *}$ *and* $(\mathscr{R}^{\circ *})^* = \mathscr{R}^{\circ *}$.

Remark 8. Note that an arbitrary Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ generated by the family $\{\mu_\alpha\}$ of measures on $\mathscr{A}$ can be extended to a Rényi space (generated by $\{\mu_\alpha\}$), satisfying (c). Namely, we can define the family $\mathscr{B}_c$ of all sets B_c such that $B \subset B_c$, $0 < \mu_\alpha(B) < \infty$ and $\mu_\alpha(B_c \setminus B) = 0$ for some α and $B \in \mathscr{B}$. Moreover, put

$$P_c(A|B_c) = \frac{\mu_\alpha(A \cap B_c)}{\mu_\alpha(B_c)} = \frac{\mu_\alpha(A \cap B)}{\mu_\alpha(B)}.$$

It is easy to see that $\mathscr{R}_c = [\Omega, \mathscr{A}, \mathscr{B}_c, P_c]$ is a Rényi space generated by $\{\mu_\alpha\}$, for which condition (c) holds. Another example of such an extension is the Rényi space $\tilde{\mathscr{R}} = [\Omega, \mathscr{A}, \tilde{\mathscr{B}}, \tilde{P}]$, where $\tilde{\mathscr{B}}$ consists of all $\tilde{B} \in \mathscr{A}$ such that $0 < \mu_\alpha(\tilde{B}) < \infty$ for some α (by 1° and 2°, α is the unique index with this property) and $\tilde{P}$ is defined by the formula

$$\tilde{P}(A|\tilde{B}) = \frac{\mu_\alpha(A \cap \tilde{B})}{\mu_\alpha(\tilde{B})}$$

for $A \in \mathscr{A}$ and $\tilde{B} \in \tilde{\mathscr{B}}$.

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