COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

A GENERAL FIXED POINT THEOREM*

BY

L. E. WARD, JR. (EUGENE, OREGON)

1. Introduction. There is a well-known theorem of Borsuk [1] which asserts that a Peano continuum has the fixed point property if and only if each cyclic element of the continuum has the fixed point property. A proof for the slightly more general case of a semi-locally connected continuum can be found in [9]. The interesting half of this theorem (extending the fixed point property from the cyclic elements to the whole continuum) depends very strongly on the „acyclic" structure of the set of cyclic elements. Now it happens that this acyclic structure is reflected in a natural partial order possessed by all connected spaces, and the use of this partial order permits a proof of Borsuk's theorem in a somewhat more general setting.

In establishing this generalization we shall also widen the class of mappings under consideration. Our results will be stated for the class of all upper semicontinuous, continuum-valued mappings which possess a property Q which is preserved when the maps are composed with order-preserving retractions. This degree of generality will permit us to infer as corollaries several previously known results.

It is a pleasure to acknowledge my great debt to Professor A. J. Ward of Emmanuel College, Cambridge, for several corrections and improvements upon the original manuscript of this paper.

2. A class of partially ordered spaces. Let X be a Hausdorff space. A partial order $\leqslant$ on X is *continuous* if its graph is a closed subset of $X\times X$. The space X, together with a continuous partial order, is termed a *continuously partially ordered space.* We also define

$$L(x) = \{y\in X : y \leqslant x\}, \qquad M(x) = \{y\in X : x \leqslant y\},$$
$$C(x) = \big\{y\in X : M(x) \cap L(y) = \{x, y\}\big\},$$

for each $x\in X$. If $A\subset X$ it is convenient to write

$$L(A) = \bigcup\{L(x) : x\in A\}, \qquad M(A) = \bigcup\{M(x) : x\in A\}.$$

* This research was supported by the National Science Foundation.

For the continuously partially ordered spaces it is known [6] that all of the sets $L(x)$ and $M(x)$ are closed, as is each maximal chain of X. Moreover, if X is compact, then X contains maximal and minimal elements.

We shall be especially concerned with partially ordered spaces satisfying the conditions

(i) *if* $x, y \in X$, *then the set* $L(x) \cap L(y)$ *is a non-empty compact chain,*

(ii) *if* $x \in X$, *then the sets* $M(x) - \{x\}$ *and* $M(C(x) - \{x\})$ *are open.*

(iii) *if* $x \in X$ *and if* F *is a closed subset of* $C(x)$, *then* $M(F - \{x\}) \cup \{x\}$ *is closed.*

Note that if X satisfies (i), then $M(x)$ and $M(y)$ are disjoint whenever x and y are non-comparable elements of X. Further, if X is compact, then X contains a *zero* i.e., an element 0 such that $M(0) = X$.

The sets $C(x)$ are, roughly speaking, order-theoretic analogues of the cyclic elements of a continuum. In this section we establish a few of their elementary properties.

LEMMA 1. *If X is a continuously partially ordered space satisfying* (ii) *and if* $x \in X$, *then* $C(x)$ *is a closed set.*

Proof. Let $y \in X - C(x)$. Since $\overline{C(x)} \subset \overline{M(x)} = M(x)$ it is clear that $y \in X - \overline{C(x)}$ if $y \in X - M(x)$. On the other hand, if $y \in M(x) - C(x)$, then there exists $z \in X$ such that $x < z < y$. By (ii), $M(z) - \{z\}$ is an open set containing y, and by definition $M(z) - \{z\}$ and $C(x)$ are disjoint. Therefore $y \in X - \overline{C(x)}$ and thus $C(x)$ is a closed set.

The next lemma is implicit in a result from [8], but we include a proof here to make the treatment as self-contained as possible.

LEMMA 2. *If F is a compact subset of the continuously partially ordered space X, then $M(F)$ is a closed set.*

Proof. If $y \in X - M(F)$, then for each $x \in F$ there are disjoint open sets U_x and V_x such that $(x, y) \in U_x \times V_x \subset X \times X - \Gamma$, where Γ denotes the graph of the partial order. Since F is compact, it is covered by finitely many of the open sets U_x, say by $U_{x_1}, \ldots, U_{x_n}$. If

$$V = V_{x_1} \cap \ldots \cap V_{x_n}, \qquad U = U_{x_1} \cup \ldots \cup U_{x_n},$$

then

$$F \times V \subset U \times V \subset X \times X - \Gamma,$$

that is to say, $V \cap M(F)$ is empty. Since V is a neighborhood of y, it follows that $M(F)$ is a closed set.

If A and B are partially ordered sets, then a function $f: A \to B$ is *order-preserving* provided $f(a_1) \leqslant f(a_2)$ whenever $a_1 \leqslant a_2$.

LEMMA 3. *If X is a continuously partially ordered space satisfying* (i), (ii) *and* (iii), *and if* $x \in X$ *such that* $C(x)$ *is compact, then there exists an order-preserving retraction* $p_x : X \to C(x)$.

Proof. Let $p_x(y) = y$ if $y \epsilon C(x)$. If $y \epsilon X - M(x)$, or if $y \epsilon M(x) - C(x)$ and there is no $t \epsilon C(x) - \{x\}$ such that $t < y$, let $p_x(y) = x$. If there exists $t_y \epsilon C(x) - \{x\}$ such that $t_y < y$, then by (i) and the definition of $C(x)$, the element t_y is unique and we define $p_x(y) = t_y$. It is easy to verify that p_x is order-preserving, so that it remains only to show that p_x is continuous. This is obvious if $C(x) = \{x\}$, so we assume that $C(x)$ is non-degenerate and let F be a closed subset of $C(x)$. If $x \notin F$, then $p_x^{-1}(F) = M(F)$, which is closed by Lemma 2. If $x \epsilon F$, then

$$p_x^{-1}(F) = [X - [M(C(x) - \{x\})]] \cup [M(F - \{x\}) \cup \{x\}].$$

By (ii) the set $X - [M(C(x) - \{x\})]$ is closed, and by (iii) the set $M(F - \{x\}) \cup \{x\}$ is closed. Thus $p_x^{-1}(F)$ is closed so that the continuity of p_x is established.

LEMMA 4. *Let X be a compact, continuously partially ordered space satisfying* (i) *and let $x \epsilon X$. If $z \epsilon M(x) - C(x)$, if there is no $t \epsilon C(x) - \{x\}$ such that $t < z$, and if U is a neighborhood of x, then*

$$(M(x) - \{x\}) \cap L(z) \cap U$$

is not empty.

Proof. The lemma is obvious if $z \epsilon U$. Otherwise, since X is compact and the partial order is continuous, the set

$$K = (M(x) \cap L(z)) - U$$

is compact and non-empty. Because $x \epsilon U$ we may write

$$K = (M(x) - \{x\}) \cap L(z) - U.$$

The set K is also a chain, and therefore K has a zero, y. By hypothesis $y \notin C(x)$ and hence there exists $w \epsilon X$ such that $x < w < y$. Since y is minimal in K, it follows that $w \epsilon U$ and hence

$$w \epsilon (M(x) - \{x\}) \cap L(z) \cap U.$$

3. The fixed point theorem. Let Q be a property of set-valued functions. If Y is a set, then $Q(Y)$ denotes the family of all set-valued functions from Y into Y which possess the property Q. If g is a set-valued function and F is a family of set-valued functions all of whose ranges lie in the domain of g, we define

$$gF = \{gf : f \epsilon F\},$$

i.e., the family of all compositions of g with members of F. Similarly, if the range of g lies in the domain of each member of F, we write

$$Fg = \{fg : f \epsilon F\}.$$

Now suppose X and Y are spaces and that $f: X \to Y$ is a set-valued function. Then f is said to be *upper semi-continuous* (hereafter, u.s.c.) provided each point image $f(x)$ is a closed set and, whenever U is an open set containing $f(x)$, there exists an open set V containing x such that $f(t) \subset U$ for each $t \in V$. Finally, if F is a family of set-valued functions from Y into Y, we say that Y has *the fixed point property for F* (or, simply, the F-f.p.p.) provided for each $f \in F$ there exists at least one $y \in Y$ such that $y \in f(y)$.

The next lemma is an obvious generalization of the well-known proposition that the fixed point property (for single-valued mappings) is a retraction invariant. Lemma 6 is also trivial, and the proofs are omitted.

LEMMA 5. *Let Y be a set, let $B \subset Y$ and let $\sigma: Y \to B$ be a function such that $\sigma \mid B$ is the identity. Let Q be a property of set-valued functions and suppose Y has the $Q(B)\sigma$-f.p.p. Then B has the $Q(B)$-f.p.p.*

LEMMA 6. *Let X and Y be spaces and let $g: X \to Y$ be a closed continuous function. If $f: X \to X$ is a set-valued function which is u.s.c. (has connected point images), then gf is u.s.c. (has connected point-images). Similarly, if $f: Y \to Y$ is u.s.c. (has connected point images), then fg is u.s.c. (has connected point images).*

LEMMA 7. *Let X be a continuously partially ordered space with zero and let f be a u.s.c. set-valued map of X into itself with compact point-images. Then the set*

$$P = \{x \in X : f(x) \cap M(x) \text{ is non-empty}\}$$

is closed and non-empty.

Proof. Let 0 be the zero of X; then $f(0) \cap M(0) = f(0)$ is non-empty so that P is non-empty. If $x \in X - P$, then

$$\{x\} \times f(x) \subset X \times X - \Gamma,$$

where Γ denotes the graph of the partial order. Since Γ is a closed set, the compactness of $f(x)$ insures the existence of open sets U and V such that

$$\{x\} \times f(x) \subset U \times V \subset X \times X - \Gamma,$$

and since f is u.s.c., we may choose U so that $f(t) \subset V$ for each $t \in U$. Therefore $U \subset X - P$ so that P is a closed set.

LEMMA 8. *If X, f and P satisfy the hypotheses of Lemma 7, and if, in addition, X is compact, then P contains a maximal element.*

Proof. By Lemma 7, P is closed, hence compact. As noted in section 2, such a set has a maximal element.

THEOREM 1. *Let X be a compact, continuously partially ordered space satisfying* (i), (ii) *and* (iii), *and let Q be a property of u.s.c. mappings with connected point-images, and which is preserved under composition with order-preserving retractions. Then X has the $Q(X)$-f.p.p. if and only if each set of the form $C(x)$ has the $Q(C(x))$-f.p.p.*

Proof. Suppose X has the $Q(X)$-f.p.p. and $x \in X$. By Lemma 3 there exists an order-preserving retraction p_x of X onto $C(x)$. By hypothesis and Lemma 6, $Q(C(x)p_x \subset Q(X)$. Therefore, by Lemma 5, $C(x)$ has the $Q(C(x))$-f.p.p.

Now suppose each set $C(x)$ has the $Q(C(x))$-f.p.p., and let $f \in Q(X)$. By Lemma 8, there exists $x_1 \in X$, maximal with respect to $f(x_1) \cap M(x_1)$ being non-empty. If $x_1 \in f(x_1)$ the proof is complete, so we assume $x_1 \in X - f(x_1)$. Since $M(x_1) - \{x_1\}$ is open, x_1 is a cutpoint; and since $f(x_1)$ meets $M(x_1)$, is connected, and does not contain x_1, it follows that $f(x_1) \subset M(x_1) - \{x_1\}$. If $C(x_1) = \{x_1\}$, or if there exists $z_1 \in f(x_1)$ and there is no $t \in C(x_1) - \{x_1\}$ such that $t < z_1$, then by Lemma 4 there exists y_1 such that $x_1 < y_1 < z_1$ and y_1 lies in the open set $X - f(x_1)$. Let U denote the open set $M(y_1) - \{y_1\}$. Again, since $f(x_1)$ meets U but does not contain y_1, it follows that $f(x_1) \subset U$. By the upper semi-continuity of f, there exists a neighborhood V of x_1 such that $f(t) \subset U$ for each $t \in V$. By Lemma 4 there exists

$$y_2 \in \big(M(x_1) - \{x_1\}\big) \cap L(y_1) \cap V.$$

Thus $f(y_2)$ meets $M(y_2)$, contrary to the maximality of x_1. Hence $C(x_1)$ is non-degenerate and either $f(x_1) \subset C(x_1) - \{x_1\}$ or, for each $y \in f(x_1) - C(x_1)$, there exists $t \in C(x_1) - \{x_1\}$ such that $t < y$. Letting $p_1 = p_{x_1}$ be the retraction of Lemma 3, it follows from these asumptions that $x_1 \notin p_1 f(x)$. From Lemma 6, the mapping $(p_1 f) \mid C(x_1)$ is a member of $Q(C(x_1))$ and hence there exists

$$a \in p_1 f(a) \cap \big(C(x_1) - \{x_1\}\big).$$

Now $p_1^{-1}(a) = M(a)$, so that $f(a) \cap M(a)$ is non-empty, contrary to the maximality of x_1. Therefore $x_1 \in f(x_1)$ and the theorem is proved.

4. Application to semi-locally connected continua; Borsuk's theorem. A *continuum* is a compact, connected Hausdorff space. A connected space is said to be *semi-locally connected* (hereafter, s.l.c.) provided each point is contained in arbitrarily small open sets whose complements have finitely many components. The s.l.c. continua are discussed in [9] where it is shown that every locally connected continuum is semi-locally connected. We recall also that if Y is an s.l.c. continuum and if x, y and z are elements of Y such that z does not separate x and y, then x and y

lie in a subcontinuum N such that $N \subset Y-\{z\}$. (Although these results are stated for *metrizable* continua, the extension to the non-metrizable situation offers no difficulty.)

Every connected topological space admits a natural partial order which has been termed the *cutpoint* order [6]. In such a space, fix a point 0 and define $x \leqslant y$ if and only if $x = 0$ or $x = y$ or x separates 0 and y. It is a simple matter to verify that this cutpoint order is, in fact, a partial order.

THEOREM 2. *If X is an s.l.c. continuum, then the cutpoint order on X is continuous and satisfies* (i), (ii) *and* (iii).

Proof. Let Γ denote the graph of the cutpoint order with minimal element 0, and suppose $(x, y) \in X \times X - \Gamma$. Then there exists a continuum N with $\{0, y\} \subset N \subset X-\{x\}$, and hence there exists an open set U such that $x \in U \subset X-N$, $y \in X-\overline{U}$, and $X-U$ has finitely many components, $C_0, C_1, \ldots, C_n$, with $N \subset C_0$. Now each of the sets $C_1, \ldots, C_n$ is closed and hence there is an open set V such that $y \in V \subset C_0$. Now if $x' \in U$ and $y' \in V$, we have $x \in X-C_0$, $\{0, y'\} \subset C_0$ and thus $(x', y') \in X \times X - \Gamma$. Thus Γ is a closed set, i.e., the cutpoint order is continuous.

It is known [6] that each set $L(x) \cap L(y)$ is a non-empty chain. Further, for each $x \in X$, consider all decompositions of the form

$$X-\{x\} = A_a \cup B_a$$

where A_a and B_a are separated sets and $0 \in A_a$. Then $M(x)-\{x\}$ is precisely the union of the sets B_a and hence $M(x)-\{x\}$ is open. To complete the verification of (ii) we note that if $C(x) = \{x\}$, then $M\big(C(x)-\{x\}\big)$ is empty and hence open, so it remains to show that $M\big(C(x)-\{x\}\big)$ is open when $C(x)$ is non-degenerate. If $y \in M\big(C(x)-\{x\}\big)-C(x)$, then there exists $y_1 \in C(x)-\{x\}$ such that $y_1 < y$ and hence the open set $M(y_1)-\{y_1\}$ contains y and is contained in $M\big(C(x)-\{x\}\big)$. Therefore we need only show that if $y \in C(x)-\{x\}$, then some neighborhood of y lies in the set $M\big(C(x)-\{x\}\big)$.

Let R be a neighborhood of x such that $y \in X-\overline{R}$ and $X-R$ has finitely many components. If K is that component of $X-R$ which contains y, then there is neighborhood S of y such that $S \subset K-\overline{R}$.

We note that $S \subset M(x)$. For if not there is a point $t \in S$ and a continuum M which contains 0 and t but which does not contain x. But then $K \cup M$ is a continuum containing 0 and y but not x, and this contradicts the hypothesis that $x < y$.

Now suppose there exists $t \in S - M\big(C(x)-\{x\}\big)$. Since $t \in M(x)-\{x\}$, the set $L(t) \cap \big(M(x)-\{x\}\big)$ is a non-empty chain. If

$$L(t) \cap \big(M(x)-\{x\}\big) \subset X-R,$$

then

$$L(t) \cap \big(M(x)-\{x\}\big) = \big(L(t) \cap M(x)\big)-R$$

which is compact and hence has a zero $z_0 \in C(x)-\{x\}$. But this implies $t \in M\big(C(x)-\{x\}\big)$, contrary to the hypotheses. Hence there exists an element $z_1 \in R$ such that $x < z_1 < t$. Since $M(z_1)$ is closed and $M(z_1)-\{z_1\}$ is open, the point z_1 separates t and y. And since K is a connected set containing t and y, we conclude that $z_1 \in K$, and hence that $K \cap R$ is non-empty. But K was defined to be a component of $X-R$. Therefore the open set S is contained in $M\big(C(x)-\{x\}\big)$.

To establish (iii) let F be a closed subset of $C(x)$. The result follows at once from Lemma 2 if $x \notin F$, so we assume $x \in F$ and suppose there exists a point z in the closure of $M(F-\{x\}) \cup \{x\}$ but not in $M(F-\{x\}) \cup \{x\}$. If $z \notin C(x)$, then there exists t such that $x < t < z$ and $M(t)-\{t\}$ is an open set containing x. If $t \notin F$, then $M(t)-\{t\}$ is disjoint from $M(F-\{x\}) \cup \{x\}$, contrary to the assumptions on z. If $t \in F$, then $z \in M(F-\{x\})$. Thus we may assume $z \in C(x)$.

Let z_a be a net in $M(F-\{x\})$ which converges to z_a and let $x_a < z_a$ with $x_a \in F-\{x\}$. Without loss of generality we may assume that x_a converges, say, to $y \in F$. By the continuity of Γ, $y \leqslant z$ and, since $L(z) \cap C(x) = \{x, z\}$, either $y = x$ or $y = z$. If $y = z$, then $z \in F$ and hence $z \in M(F-\{x\}) \cup \{x\}$, a contradiction. Therefore $y = x$. Let U be an open set such that $x \in U$ and $z \in X-\overline{U}$. Then the sets $M(x_a)-U$ are disjoint and are both open and closed in the relative topology of $X-U$. Since infinitely many of the sets $M(x_a)-U$ are non-empty, the semi-local connectedness of X is contradicted. This completes the proof.

We recall a few definitions and results from [9]. If X is an s.l.c. continuum, then by a *cyclic element* of X is meant an endpoint of X, a cutpoint of X or an E_0-set of X, i.e., a non-degenerate connected subset of X which is maximal with respect to the property of having no cutpoint. The non-degenerate cyclic elements of X are termed *true cyclic elements*, and they can be characterized as follows. Each true cyclic element C of X is a subcontinuum which contains a non-cutpoint p of X; moreover C consists of p and all elements $x \in X$ which are not separated from p by some point. Further, any two distinct true cyclic elements of X can intersect in at most a point which is necessarily a cutpoint of X.

THEOREM 3. *Let X be an s.l.c continuum. With respect to the cutpoint order each set $C(x)$ is either a single point or is the union of a family of cyclic elements each of which contains x. Conversely, each true cyclic element of X lies in a unique $C(x)$*

We omit the proof of Theorem 3 which stems directly from the definition of $C(x)$ and the above remarks about cyclic elements.

Our next result is the generalized Borsuk theorem alluded to in section 1.

THEOREM 4. *If X is an s.l.c. continuum, then X has the fixed point property for u.s.c. continuum-valued maps if and only if each cyclic element of X has this property.*

Proof. Let P denote the fixed point property for u.s.c. continuum-valued maps. By Lemmas 5 and 6 and Theorems 1 and 2, X has P if and only if each set $C(x)$ has P. Thus we need only show that a non-degenerate $C(x)$ has P if and only if each of the true cyclic elements comprising $C(x)$ has P. The "only if" part of this assertion is clear, since these cyclic elements are retracts of $C(x)$. But the converse is equally clear, for if $f: C(x) \to C(x)$ is u.s.c. and continuum-valued and $x \notin f(x)$, then $f(x)$ lies in some true cyclic element E. Letting $p: C(x) \to E$ be the natural retraction, pf maps E into E and hence pf has a fixed point $y \in E$. But then $y \in pf(y) \subset f(y)$.

COROLLARY 4.1 (Wallace). *If X is a tree, i.e., a continuum in which each pair of distinct points is separated by some third point, then X has the fixed point property for u.s.c. continuum-valued maps.*

Proof. It is well known [7] that a tree is locally connected and hence is s.l.c. Further, each cyclic element is degenerate so that the corollary follows at once from Theorem 4.

5. Application to the Eilenberg-Montgomery fixed point theorem. This notable result asserts that if X is an acyclic ANR, then X has the f.p.p. for u.s.c. mappings with acyclic point-images. (Here *acyclic* means having the homology of a point. An ANR is a compact metric space which is a neighborhood retract of any metric space in which it can be imbedded.) We may use Theorem 4 to obtain an immediate extension of this result to Peano continua all of whose cyclic elements are acyclic ANR's. The natural retractions p_x clearly preserve the property of having acyclic point-images, and so we may assert

COROLLARY 4.2. *If X is a Peano continuum each cyclic element of which is an acyclic ANR, then X has the fixed point property for u.s.c. maps whose point-images are acyclic.*

It should be noted that this result truly generalizes the Eilenberg-Montgomery theorem, i.e., there exists a continuum satisfying the hypotheses of the theorem which is not an ANR. To see this let Σ denote the "Poincaré sphere" which is described in [3], p. 218, and [10], p. 245, and let Δ denote the set which results when a single open 3-cell is deleted from Σ. It can be shown that Δ has trivial homology but is not contractible. Thus Δ is an acyclic ANR which is not an AR. We construct a continuum X by letting $\Delta_1, \Delta_2, \ldots$ be a sequence of disjoint copies of Δ each meeting an arc $[a, b]$ in a single point p_n, such that $\lim \operatorname{diam} \Delta_n = 0$ and $\lim p_n = a$. Let

$$X = \bigcup_{n=1}^{\infty} \{\Delta_n\} \cup [a, b].$$

The sets Δ_n are the true cyclic elements of X, but X is not an ANR, since otherwise a closed ball $B(a, \varepsilon)$ would contain some Δ_n as a retract, and this would imply Δ_n is an AR.

6. An unsolved problem. The theorems in this paper have dealt with the following general question. If X is a connected space which is the union of subspaces X_i, if $X_i \cap X_j$ is at most a point when $i \neq j$, and if there are no "circular chains" among the family $\{X_i\}$, when does a certain fixed point property extend from the subspaces X_i to the space X?

It is natural to inquire under what circumstances the condition that $X_i \cap X_j$ be at most a point can be relaxed.

For example, suppose $X = A \cup B$ where A and B are continua with the f.p.p. Štanko [4] has shown that if $\dim A = \dim B = 1$ and if $A \cap B$ is a tree, then X has the f.p.p. On the other hand, Yandl [11] has exhibited two-dimensional continua A and B in E^3 such that $A \cap B$ is a retract of $X = A \cup B$, A and B have the f.p.p., yet X does not have the f.p.p.

Suppose $X = A \cup B$ where A and B are *locally connected* continua with the f.p.p., and suppose $A \cap B$ is a retract of X (or even that $A \cap B$ is an arc). Does X have the f.p.p.? (**P 560**) This problem appears to be quite difficult.

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UNIVERSITY OF OREGON
EUGENE, OREGON

Reçu par la Rédaction le 8. 9. 1965