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MONOTONICITY ASSUMPTION IN UNIQUENESS CRITERIA FOR DIFFERENTIAL EQUATIONS

BY

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Introduction. Many authors [1, 2, 4, 7-12] have discussed the problem of the convergence of successive approximations to a solution of a differential equation, when the latter is unique in virtue of a uniqueness criterion. In these papers, a comparison function $\omega(t, u)$ appears which, in addition to the assumptions required by a uniqueness criterion, is assumed to be non-decreasing in u. In the present paper, we investigate the following problem. Suppose for a given differential equation there holds a uniqueness criterion with a comparison function $\omega(t, u)$. Does then exist another comparison function $\omega^*(t, u)$ non-decreasing in u which also supplies a uniqueness criterion for the equation in question? The answer is negative in general (Theorem 4) but it can be positive in a special case (Theorem 3). Both mentioned theorems are given in Section 2, which also includes a discussion of the problem of convergence of successive approximations. The theorems of Section 2 are consequences of the main two results (Theorem 1 and 2) presented in Section 1. The first concerns a class of real continuous functions of one real variable and is perhaps of some interest by itself, while the second is a kind of peculiar example of an ordinary differential equation. Proofs of Theorems 1 and 2 are given in Sections 3 and 4, respectively.

1. Statements of the basic results. The class of real functions we have mentioned in the introduction is the following:

Class K. We say that a real function $\omega \in K$ if it is continuous on $[0, a]$, $a > 0$, and satisfies two conditions:

$$\omega(0) = 0, \quad \omega(u) \geqslant 0 \text{ if } u > 0, \tag{1.1}$$

$$|\omega(u_1) - \omega(u_2)| \leqslant \omega(|u_1 - u_2|) \quad \text{for each } u_1, u_2 \in [0, a]. \tag{1.2}$$

If $\omega \in K$, then by ω^* we denote the best non-decreasing majorant of ω, that is the function defined by

$$\omega^*(u) = \max_{0 \leqslant v \leqslant u} \omega(v), \quad u \in [0, a]. \tag{1.3}$$

It is easy to check that if $\omega \in K$, then so does ω^*.

Our first result reads as follows:

THEOREM 1. *Suppose $\omega \in K$ and let ω^* be given by* (1.3). *Let C be in the range of ω. Let $u(C)$ be defined by*

$$\omega\big(u(C)\big) = C \quad \text{and} \quad \omega(u) < C \text{ if } u < u(C). \tag{1.4}$$

Then we have

$$\int_0^{u(C/3)} du/\omega(u) \leqslant M \int_0^{u(C)} du/\omega^*(u) \tag{1.5}$$

for each C in the range of ω, where M is a constant which depends neither on ω nor on C.

Note that the integrand of the left-hand side integral of (1.5) is always greater than or equal to that of the right-hand side integral. From Theorem 1 we have the following

COROLLARY 1. *Suppose $\omega \in K$ and ω^* is given by* (1.3). *Then both integrals*

$$\int_0 du/\omega(u) \quad \text{and} \quad \int_0 du/\omega^*(u) \tag{1.6}$$

are simultaneously convergent or divergent.

As it is well known the divergence of the integrals in (1.6) is equivalent to the statement that $u(t) \equiv 0$ on any interval $[0, \varepsilon)$, $\varepsilon > 0$, is the only solution of the initial value problem

$$u' = \omega(u), \quad u(0) = 0, \tag{1.7}$$

or

$$u' = \omega^*(u), \quad u(0) = 0, \tag{1.8}$$

respectively. Thus from Theorem 1 (or Corollary 1) we have the following

COROLLARY 2. *Suppose a function $\omega \in K$ and let ω^* be defined by* (1.3). *Then $u(t) \equiv 0$ is simultaneously either a unique or a non-unique solution of both equations* (1.7) *and* (1.8).

The second result is in the effect that the non-autonomous counterpart of Corollary 2 is no more valid. For that purpose consider a real function ω continuous on

$$P_0 = \{(t, u):\ 0 \leqslant t \leqslant a,\ 0 \leqslant u \leqslant b\}$$

with the properties:

$$\omega(t, 0) \equiv 0, \quad \omega(t, u) \geqslant 0 \text{ if } u > 0, \tag{1.9}$$

$$|\omega(t, u_1) - \omega(t, u_2)| \leqslant \omega(t, |u_1 - u_2|) \tag{1.10}$$

$$\text{for each} \quad (t, u_1), (t, u_2) \in P_0.$$

Let ω^* stand for the best majorant of ω non-decreasing in u, that is

$$(1.11) \qquad \omega^*(t, u) = \max_{0 \leqslant v \leqslant u} \omega(t, y), \qquad (t, u) \in P_0.$$

Then we have the following result:

THEOREM 2. *There exists a real function ω continuous on P_0, satisfying (1.9) and (1.10) and such that $u(t) \equiv 0$ on any interval $[0, \varepsilon)$, $0 < \varepsilon \leqslant a$, is the only solution to the initial value problem*

$$(1.12) \qquad u' = \omega(t, u), \quad u(0) = 0,$$

while the corresponding problem

$$(1.13) \qquad u' = \omega^*(t, u), \quad u(0) = 0$$

for ω^* given by (1.11) admits a solution $u(t)$ which is positive for $t > 0$.

Let us call the reader's attention to the importance of assumptions (1.2) and (1.10) in Theorems 1 and 2, respectively. If those assumptions were droped, then Theorem 1 would be false while the example involved in Theorem 2 would be easy.

Theorem 1 has been obtained in the early fifties by the second of the authors without being published. The proof of it presented here is due to the first of the authors and, though it follows the main lines of the original proof due to Pliś, it has been considerably simplified.

2. Successive approximations. In this section we are concerned with the problem whether or not a solution of

$$(2.1) \qquad y' = f(t, y), \quad y(t_0) = y_0$$

can be obtained as the limit of the sequence of successive approximations defined by

$$(2.2) \qquad y_n(t) = y_0 + \int_{t_0}^{t} f\big(t, y_{n-1}(t)\big)\,dt \quad \text{if} \quad n \geqslant 1,$$

where $y_0(t)$ is arbitrary but continuous and $y_0(t_0) = y_0$.

For the sake of simplicity we assume in the sequel that $t_0 = y_0 = 0$ and we restrict ourselves to the case where y and $f(t, y)$ in (2.1) are real. We will also assume throughout this section that f, in (2.1), is a continuous function on P, where

$$P = \{(t, y):\ 0 \leqslant t \leqslant a,\ |y| \leqslant b\}, \qquad a > 0,\ b > 0.$$

A well known example due to Müller [5] shows that the sequence (2.2) as well as all its subsequences may be not convergent to a solution of (2.1) even if the latter is known to be unique. On the other hand, it is also known that (2.2) tends to the unique solution of (2.1) if the function f in (2.1) satisfies the Lipschitz condition with respect to y.

A great deal of attention has been paid to the above problem in the case where the solution of (2.1) is unique and the uniqueness is a consequence of a uniqueness criterion [1, 2, 7-12]. Now the most known uniqueness criteria involve a generalized Lipschitz condition for the function f which consists of the existence of a function λ defined on

$$P_0 = \{(t, u):\ 0 < t \leqslant a,\ 0 \leqslant u \leqslant 2b\}$$

such that, for any (t, y_1), $(t, y_2) \in P$, we have

$$|f(t, y_1) - f(t, y_2)| \leqslant \lambda(t, |y_1 - y_2|), \qquad t > 0. \tag{2.3}$$

For brevity of the subsequent discussion we introduce now certain classes of functions λ defined on P_0.

Class Λ. We say that $\lambda \in \Lambda$ if λ is continuous on P_0 with the properties: $\lambda(t, 0) \equiv 0$, $\lambda(t, u) \geqslant 0$ if $u > 0$ and $u(t) \equiv 0$ is the only solution $u = u(t)$ of the differential equation

$$u' = \lambda(t, u) \tag{2.4}$$

on any interval $(0, \varepsilon]$, $0 < \varepsilon \leqslant a$, with the property

$$u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{t} \to 0 \text{ as } t \to 0+. \tag{2.5}$$

Class Λ_f. This class is assigned to a fixed continuous function f on P and consists of those elements of Λ for which inequality (2.3) is satisfied.

Classes Λ^* and Λ_f^*. These are subclasses of Λ and Λ_f, respectively, composed of functions λ which, in addition, are non-decreasing in u for any fixed t.

Now we state a uniqueness criterion for solutions of (2.1).

THEOREM A. *If f in* (2.1) *is continuous on P and the class Λ_f is not empty, then the solution $y = y(t)$ of* (2.1) *is unique on any interval* $[0, \varepsilon]$, $\varepsilon \leqslant a$.

Theorem A is often known as Kamke's general uniqueness theorem (cf. [3]) and many other uniqueness criteria involving inequality (2.3) are either special cases of Theorem A or can be easily reduced to it (cf. [8]).

Concerning the problem whether or not the solution of (2.1), unique in virtue of a uniqueness criterion, can be obtained as the limit of (2.2), seemingly the most general known result is due to Coddington and Levinson [1] and it reads as follows:

THEOREM B. *Let f in* (2.1) *be continuous on P, let $|f(t, y)| \leqslant M$ if $(t, y) \in P$ and put $\alpha = \min(a, b/M)$. If the class Λ_f^* is not empty, then the sequence* (2.2) *is defined and converges uniformly on* $[0, \alpha]$ *to the unique solution of* (2.1).

Let us remark that in the known proofs of Theorem B the monotonicity assumption (the non-emptiness of Λ_f^*) is essentially used. A problem (cf. Ważewski [11], Problème LP), which seems to be still unsolved, is whether or not Theorem B remains true without any monotonicity assumption; that is, whether or not Theorem B is true if the assumption of non-emptiness of Λ_f^* is replaced by the non-emptiness of Λ_f. A natural question to be answered first, if one wishes to approach the above problem, is whether or not the class Λ_f^* is not empty if the class Λ_f is known to be not empty, because by Theorem B the positive answer to this question would settle the problem.

The aim of this section is to settle the above particular question. We are going to show by an example (Theorem 4) that the answer is, in general, in the negative but we also show (Theorem 3) that if the non-emptiness of Λ_f is effected by a function λ of the form

$$\lambda(t, u) = \varphi(t)\psi(u), \tag{2.6}$$

where ψ is continuous and non-negative on $(0, a]$ and

$$\int_0^a \varphi(t)\,dt < \infty, \tag{2.7}$$

then the monotonicity assumption in Theorem B is superfluous.

It is a simple matter to check that the necessary and sufficient condition for λ of the form (2.6) with $\varphi(t)$ satisfying (2.7) to be in the class Λ is that ψ is continuous on $[0, 2b]$, $\psi(0) = 0$, $\psi(u) \geqslant 0$ if $u > 0$ and

$$\int_0 du/\psi(u) = \infty. \tag{2.8}$$

For the result which follows we will need the following

PROPOSITION 1. *Suppose $\omega_t \in K$ for $t \in T$ and $\omega_t(u) \leqslant \psi(u)$ for $t \in T$ and $u \in [0, a]$, where T is an arbitrary set and $\psi(u)$ is continuous on $[0, a]$ and $\psi(0) = 0$. Then the function*

$$\omega(u) = \sup_{t \in T} \omega_t(u) \tag{2.9}$$

also belongs to K.

Proof. It is easy to see that $\omega(0) = 0$ and $\omega(u) \to 0$ as $u \to 0+$. In order to prove that $\omega \in K$ it is enough to check (1.2), since (1.1) is obvious and the continuity of ω follows from (1.2) and the fact that $\omega(0) = 0$ and ω is continuous for $u = 0$. Let us fix u_1, u_2 and suppose $\omega(u_1) - \omega(u_2) \geqslant 0$. By (2.9) there is a $t \in T$ such that $\omega(u_1) \leqslant \omega_t(u_1) + \varepsilon$, where ε is an arbitrary positive but fixed number. We have $-\omega(u) \leqslant -\omega_t(u)$ for any t and u. Therefore, using the fact that $\omega_t \in K$, we get

$$\omega(u_1) - \omega(u_2) \leqslant \omega_t(u_1) - \omega_t(u_2) + \varepsilon \leqslant \omega_t(|u_1 - u_2|) + \varepsilon \leqslant \omega(|u_1 - u_2|) + \varepsilon.$$

Since ε is arbitrary, (1.2) holds for ω defined by (2.9). This completes the proof of Proposition 1.

The following result is a consequence of Theorem 1:

THEOREM 3. *If, for a fixed function f continuous on P, Λ_f contains a function λ of the form* (2.6) *with $\varphi(t)$ continuous and non-negative on $(0, a]$ and satisfying* (2.7), *then Λ_f^* is not empty, too.*

Proof. Without any loss of generality we may assume that $\varphi(t) > 0$, if $t \in (0, a]$. For any fixed $t \in (0, a]$ let us set

$$\omega_t(u) = \frac{1}{\varphi(t)} \max |f(t, y_1) - f(t, y_2)|, \tag{2.10}$$

where the maximum is taken for $|y_1 - y_2| = u$, $|y_1|$, $|y_1| \leqslant b$. We have $\lambda = \varphi(t)\psi(u) \in \Lambda_f$. Therefore by (2.3), (2.6) and (2.10) we get

$$\omega_t(u) \leqslant \psi(u) \quad \text{for} \quad t \in (0, a] \text{ and } u \in [0, 2b]. \tag{2.11}$$

We prove now that $\omega_t \in K$. The continuity of ω_t as well as condition (1.1) is clear. To prove (1.2) let us fix u_1, u_2 and suppose $\omega_t(u_1) - \omega_t(u_2) \geqslant 0$. By (2.10) there exist $y_1, y_2 \in [-b, b]$ such that $|y_1 - y_2| = u_1$ and

$$\varphi(t)\,\omega(u_1) = |f(t, y_1) - f(t, y_2)|. \tag{2.12}$$

There exists a $y_3 \in [-b, b]$ such that either $|y_1 - y_3| = |u_1 - u_2|$ and $|y_2 - y_3| = u_2$ or $|y_1 - y_3| = u_2$ and $|y_2 - y_3| = |u_1 - u_2|$. To fix ideas we assume that the first possibility is the case. The other can be handled in exactly the same way. We have then

$$|f(t, y_2) - f(t, y_3)| \leqslant \varphi(t)\,\omega_t(u_2) \tag{2.13}$$

and

$$|f(t, y_1) - f(t, y_3)| \leqslant \varphi(t)\,\omega_t(|u_1 - u_2|). \tag{2.14}$$

From (2.12), (2.13) and (2.14) we get

$$\begin{aligned} 0 \leqslant \varphi(t)\big(\omega_t(u_1) - \omega_t(u_2)\big) &\leqslant |f(t, y_1) - f(t, y_2)| - |f(t, y_2) - f(t, y_3)| \\ &\leqslant |f(t, y_1) - f(t, y_3)| \leqslant \varphi(t)\,\omega_t(|u_1 - u_2|), \end{aligned}$$

which proves condition (1.2) for ω_t. Therefore $\omega_t \in K$ for every $t \in (0, a]$. Let us put now

$$\omega(u) = \sup_t \omega_t(u), \qquad t \in (0, a]. \tag{2.15}$$

It follows from (2.11) that $\omega(u) \leqslant \psi(u)$ for $u \in [0, 2b]$, whence, by Proposition 1, we conclude that $\omega \in K$. Since $\lambda = \varphi(t)\psi(u)$ belongs to Λ, (2.8) holds true and by (2.11) and (2.15) we get that $\int\limits_0 du/\omega(u) = \infty$. Hence $\lambda_1 = \varphi(t)\,\omega(u) \in \Lambda$, and by (2.10) and (2.15) it is easy to see that also $\lambda_1 \in \Lambda_f'$.

Consider now the function $\lambda_2 = \varphi(t)\omega^*(u)$, where ω^* is the best non-decreasing majorant of ω (cf. (1.3)). Since $\omega \epsilon K$, it follows from Theorem 1 (or Corollary 1) that $\int\limits_0 du/\omega^*(u) = \infty$, Hence, by (2.7), $\lambda_2 \epsilon \Lambda^*$. But $\lambda_2(t, u) \geqslant \lambda_1(t, u)$, thus $\lambda_2 \epsilon \Lambda_f^*$, which completes the proof of Theorem 3.

As an obvious consequence of Theorem 3 and Theorem B we have

COROLLARY 3. *If $f(t, y)$ in (2.1) is continuous on P and there exist continuous and non-negative functions φ on $(0, a]$ and ψ on $[0, 2b]$ such that $\psi(0) = 0$ and conditions (2.7) and (2.8) hold, and for any (t, y_1) and (t, y_2) in P $(t > 0)$ we have the inequality*

$$|f(t, y_1) - f(t, y_2)| \leqslant \varphi(t)\psi(|y_1 - y_2|),$$

then (without any monotonicity assumption on ψ) the sequence (2.2) is defined and converges uniformly on a certain interval $[0, a]$ to the unique solution of (2.1).

Remark 1. LaSalle [4] and earlier Wintner [12] (for the case $\varphi(t) \equiv 1$) have obtained Corollary 3 under an additional assumption that $\psi(u)$ is non-decreasing with respect to u. In Wintner's paper there is a statement that the monotonicity assumption is superfluous. However, there is no proof of it but only a hint, which, in our opinion, is misleading. Judging from this hint, we do surmise that Wintner overlooked some difficulties which arise if one drops the monotonicity assumption.

Our next result is a consequence of Theorem 2.

THEOREM 4. *There exists a function f continuous on P such that Λ_f is not empty, while Λ_f^* is empty.*

Proof. Let $\omega(t, u)$ be the function whose existence is given by Theorem 2. The function ω is continuous on P_0 and satisfies conditions (1.9) and (1.10), which also means that, for each fixed t, $\omega = \omega(t, u)$ as function of u belongs to K. Moreover, $\omega \epsilon \Lambda$ while $\omega^* \notin \Lambda^*$, where ω^* is the best non-decreasing in u majorant of ω defined by (1.11). A function f, the existence of which is asserted by Theorem 4, is to be defined as follows:

$$f(t, y) = \begin{cases} \omega(t, y) & \text{if} \quad 0 \leqslant y \leqslant b, \\ \min\limits_{0 \leqslant u \leqslant b} \big(\omega(t, u) + \omega(t, u - y)\big) & \text{if} \quad -b \leqslant y < 0. \end{cases} \tag{2.16}$$

The continuity of f is easy to see and we omit the details. Consider now the functions

$$\Omega(t, u) = \max_{|y_1 - y_2| = u} |f(t, y_1) - f(t, y_2)| \tag{2.17}$$

and

$$\Omega^*(t, u) = \max_{|y_1 - y_2| \leqslant u} |f(t, y_1) - f(t, y_2)|, \tag{2.18}$$

where $|y_i| \leqslant b$, $i = 1, 2$, and $t \in [0, a]$. The functions Ω and Ω^* are continuous on P_0 and clearly we have

$$(2.19) \qquad \Omega^*(t, u) = \max_{0 \leqslant v \leqslant u} \Omega(t, u).$$

We shall prove that

$$(2.20) \qquad \Omega(t, u) = \omega(t, u) \quad \text{if} \quad 0 \leqslant u \leqslant b.$$

The remarks concerning the uniqueness criteria contained in [6] are in the effect that if Λ_f is not empty, then $\Omega \in \Lambda_f$, as well as that if Λ_f^* is not empty, then $\Omega^* \in \Lambda_f$. Hence to show that Λ_f^* is empty we must prove that Ω^* does not belong to Λ. Now we set that this is the case by taking into account the properties of ω, if (2.20) holds true. Thus to complete the proof we need to prove (2.20).

It follows from (2.16) and (2.17) that

$$(2.21) \qquad f(t, y) - f(t, 0) = \omega(t, y) \leqslant \Omega(t, y) \quad \text{if} \quad 0 \leqslant y \leqslant b.$$

On the other hand, we will prove that for any (t, y_1) and (t, y_2) in P we have

$$(2.22) \qquad |f(t, y_1) - f(t, y_2)| \leqslant \omega(t, |y_1 - y_2|) \quad \text{if} \quad |y_1 - y_2| \leqslant b.$$

Inequality (2.22) is to be proved for each fixed t. Thus, for brevity's sake we may forget about t and argue on function of one variable u. To be more precise, we have to prove the following statement: if $\omega = \omega(u)$ defined on $[0, 2b]$ belongs to the class K and a function f defined on $[-b, b]$ is given by

$$(2.16') \qquad f(y) = \omega(y) \text{ if } 0 \leqslant y \leqslant b$$
$$\text{and} \quad f(y) = \min_{0 \leqslant u \leqslant b} \big(\omega(u) + \omega(u - y)\big) \text{ if } -b \leqslant y < 0,$$

then

$$(2.22') \qquad |f(y_1) - f(y_2)| \leqslant \omega(|y_1 - y_2|) \quad \text{if} \quad |y_1 - y_2| \leqslant b,\ |y_i| \leqslant b,\ i = 1, 2.$$

For this purpose, let us fix y_1 and y_2 in (2.22') and suppose that $f(y_1) - f(y_2) \geqslant 0$. Then by (2.16') we have the following cases:

$$(2.23a) \qquad f(y_1) - f(y_2) = \omega(y_1) - \omega(y_2) \quad \text{if} \quad y_1 \geqslant 0,\ y_2 \geqslant 0,$$

$$(2.23b) \qquad = \omega(u_1) + \omega(u_1 - y_1) - \omega(y_2) \quad \text{if} \quad y_1 < 0,\ y_2 \geqslant 0,$$

$$(2.23c) \qquad = \omega(u_1) + \omega(u_1 - y_1) - \omega(u_2) - \omega(u_2 - y_2) \quad \text{if} \quad y_1 < 0,\ y_2 < 0,$$

$$(2.23d) \qquad = \omega(y_1) - \omega(u_2) - \omega(u_2 - y_2) \quad \text{if} \quad y_1 \geqslant 0,\ y_2 < 0,$$

where u_1 and u_2 are values at which the minimum in (2.16') is attained.

In the first case, inequality (2.22′) is obvious, since $\omega \in K$. In the second case, by (2.26′) we increase the right-hand side of (2.23b) if we put $u_1 = y_2$ and we get (2.22′) (note that since $|y_1 - y_2| \leqslant b$, $y_1 < 0$ and $y_2 \geqslant 0$, we have $y_2 \leqslant b$). In the third case, replacing u_1 by u_2 we again increase the right-hand side of (2.23c) and since $\omega \in K$, we get $\omega(u_2 - y_1) - \omega(u_2 - y_2) \leqslant \omega(|y_1 - y_2|)$, hence also (2.22′) holds. In the last case, adding and subtracting $\omega(u_2 + y_1 - y_2)$ from the right-hand side of (2.23d) (note that $u_2 + y_1 - y_2 \leqslant 2b$) and making use of inequalities $\omega(y_1) - \omega(u_2 + y_1 - y_2) \leqslant \omega(u_2 - y_2)$ and $\omega(u_2 + y_1 - y_2) - \omega(u_2) \leqslant \omega(y_1 - y_2)$ we obtain (2.22′). Therefore we have proved (2.22′), hence also (2.22).

By (2.22) and (2.17) we get the inequality $\Omega(t, y) \leqslant \omega(t, y)$, if $y \leqslant b$, which together with (2.21) imply (2.20). Therefore we have proved (2.20) and completed the proof of Theorem 4.

Remark 2. In the recent book of Hartman [3] an exercise is proposed (namely Exercise 6.5 on p. 33) which consists of the negation of Theorem 4. This exercise appeared there due to an uncorrect quoting of a result from the paper [6] of the first of the authors (cf. [3], notes on p. 44). In the same book an analogue of Theorem B is stated but instead of Λ_f^*, only the class Λ_f is assumed to be not empty. Since in the proof of this theorem Hartman makes use of the mentioned exercise, he proves in fact nothing more than Theorem B of this paper. Thus the question whether or not his Theorem 9.1 ([3], p. 41) is true is open. Our Theorem 4 shows that if it is true, then a proof of it cannot be reduced to a proof of Theorem B.

3. Proof of Theorem 1. The proof will be carried on in a few steps; that is we are going first to state and prove three lemmas and then we prove Theorem 1.

LEMMA 1. *Assume* $\omega \in K$ *and let* $[u_0, u_1] \subset [0, a]$, *where*

$$\text{(3.1)} \qquad \textit{either } \omega(u_0) = C \quad \textit{or} \quad \omega(u_1) = C.$$

Put

$$\text{(3.2)} \qquad B = \{u\colon u_0 \leqslant u \leqslant u_1,\ \omega(u) \leqslant C/4\}.$$

Then we have

$$\text{(3.3)} \qquad m(B) \leqslant \tfrac{1}{2}(u_1 - u_0),$$

where m stands for the measure.

Proof. Put $p = u(5C/8)$; that is

$$\text{(3.4)} \qquad \omega(p) = 5C/8 \quad \text{and} \quad \omega(u) < 5C/8 \text{ if } u < p.$$

We claim that

$$\text{(3.5)} \qquad \text{either } \inf_{b \in B}(b - u_0) \geqslant p \text{ if } \omega(u_0) = C \text{ or } \inf_{b \in B}(u_1 - b) \geqslant p \text{ if } \omega(u_1) = C.$$

We restrict ourselves to the first case of (3.5), since the other case can be handled in the same way. Let us fix $b \in B$. By (3.2) and (1.2) we have the inequality $5C/8 < 3C/4 \leqslant \omega(u_0) - \omega(b) \leqslant \omega(b-u_0)$, whence, by (3.4), $(b-u_0) > p$ for each $b \in B$. Therefore we have proved (3.5). Next we claim that if $b \in B$, then $b-p \notin B$. Indeed, by (3.2), (3.4) and (1.2) we have the inequality $C/4 < 5C/8 - C/4 \leqslant \omega(p) - \omega(b) \leqslant \omega(b-p)$, which by (3.2) shows that $b-p \notin B$. Denote now by $B_{-p} = \{u\colon u+p \in B\}$. We proved that $B_{-p} \cap B = 0$ and, by (3.5), that $B_{-p} \subset [u_0, u_1]$, which completes the proof of Lemma 1.

LEMMA 1'. *Denote by* $A = \{u\colon \omega(u) \leqslant C,\ u_0 \leqslant u \leqslant u_*\}$, *where* $\omega \in K$, $\omega(u_0) = C$ *and* u_* *is arbitrary. Put* $B_n = \{u\colon u \in A,\ \omega(u) \leqslant 4^{-n}C\}$. *Then*

$$m(B_n) \leqslant \frac{1}{2^n} m(A). \tag{3.6}$$

Proof. Since ω is continuous, the set A can be decomposed into two parts A' and A'', where A' is the union of at most denumerable number of closed intervals and $\omega(u) = C$ if $u \in A''$. The same remark applies to each B_n. Applying now Lemma 1 to each interval composing A' and taking into account that $m(A') \leqslant m(A)$ and $B_1 \subset A'$ we get that $m(B_1) \leqslant m(A)/2$ and an easy induction argument leads to (3.6).

LEMMA 2. *Let* $\omega \in K$ *and assume*

$$\omega(u) \geqslant \gamma u \quad \textit{for} \quad 0 \leqslant u \leqslant C/\gamma = \Delta_0, \tag{3.7}$$

$$\omega(u_0) = C, \quad u_0 < \Delta_0. \tag{3.8}$$

Then

$$\int_{u_0}^{\Delta_0} du/\omega(u) \leqslant M/\gamma, \tag{3.9}$$

where the constant M *depends neither on* ω *nor on* C.

Proof. Let us set $\Delta_i = 4^{-i}C/\gamma$ and $\delta_i = \{u\colon u_0 \leqslant u \leqslant \Delta_0,\ 4^{-i}C \leqslant \omega(u) < 4^{-i+1}C\}$, $i = 1, 2, \ldots$, $\delta_0 = \{u\colon u_0 \leqslant u \leqslant \Delta_0,\ \omega(u) \geqslant C\}$.

Then

$$\int_{u_0}^{\Delta_0} du/\omega(u) \leqslant \sum_{i=0}^{\infty} m(\delta_i) 4^i/C. \tag{3.10}$$

By (3.7), $\delta_i \subset [0, \Delta_{i-1}]$, $i = 1, 2, \ldots$, and, therefore, $\delta_i \subset B_{i-1} = \{u\colon u_0 \leqslant u \leqslant \Delta_{i-1},\ \omega(u) \leqslant 4^{-i+1}C\}$. Thus by Lemma 1, $m(\delta_i) \leqslant m(B_{i-1}) \leqslant 2^{-i+1}\Delta_{i-1}$, whence

$$m(\delta_i)4^i/C \leqslant 2^{-i+1}\Delta_{i-1}4^i/C = 2^{-i+1}4/\gamma, \quad i = 1, 2, \ldots, \tag{3.11}$$

and

$$m(\delta_0)/C \leqslant \Delta_0/C = 1/\gamma. \tag{3.12}$$

From (3.10), (3.11) and (3.12) we get the inequality

$$\int_{u_0}^{\Delta_0} du/\omega(u) \leqslant 1/\gamma + (4/\gamma)\sum_{i=1}^{\infty} 2^{-i+1} = 9/\gamma.$$

Therefore we have (3.9) with $M = 9$ in it, which completes the proof of Lemma 2.

LEMMA 3. *Let $\omega \in K$ and let ω^* be given by* (1.3). *Assume C is in the range of ω and put $p_i = u(3^{-i}C)$, $i = 0, 1, 2, \ldots$, where the function u is defined by* (1.4).

Then the inequality

$$\int_{p_{i+2}}^{p_{i+1}} du/\omega(u) \leqslant M_0 \int_{p_{i+1}}^{p_i} du/\omega^*(u), \qquad i = 0, 1, 2, \ldots, \tag{3.13}$$

holds true, where M_0 depends neither on ω nor on C.

Proof. Note that

$$p_i \leqslant p_{i-1}/3, \qquad i = 1, 2, \ldots \tag{3.14}$$

Indeed, by (1.2) we have $\omega(p_{i-1}) \leqslant 3\omega(p_{i-1}/3)$, but by the definition of p_i we have $\omega(p_{i-1}) = 3^{-i+1}C$. Therefore $\omega(p_{i-1}/3) \geqslant 3^{-i}C$, which shows (3.14). Put $\gamma = 3^{-i-1}C/p_i$. We claim that

$$\omega(u) \geqslant \gamma u \quad \text{if} \quad u \leqslant p_{i+1}. \tag{3.15}$$

Suppose the contrary, that is that there exists a $u_0 \leqslant p_{i+1}$ such that $\omega(u_0) < \gamma u_0$. Then by (1.2) there would exist a $u_1 (= ku_0$, k an integer) such that $p_i - u_1 \leqslant p_{i+1}$ and $\omega(u_1) < \gamma u_1 \leqslant \gamma p_i = 3^{-i-1}C$. But then we had, using (1.2) again, that $3^{-i-1}2C \leqslant \omega(p_i) - \omega(u_1) \leqslant \omega(p_i - u_1) \leqslant 3^{-i-1}C$. Hence a contradiction and thus (3.15) holds.

Put $\bar{\omega}(u) = \max\big(\omega(u), \gamma u\big)$. By Proposition 1 we have $\bar{\omega} \in K$. We shall now apply Lemma 2 to the function $\bar{\omega}$, replacing C, γ and u_0 in this lemma by $3^{-i-2}C$, $3^{-i-1}C/p_i$, p_{i+2}, respectively. Assumption (3.7) of Lemma 2 is manifestly satisfied and (3.8) follows from (3.14). By Lemma 2 we have then

$$\int_{p_{i+2}}^{\Delta_0} du/\bar{\omega}(u) \leqslant M/\gamma, \quad \text{where} \quad \Delta_0 = (3^{-i-2}C)(3^{i+1}p_i/C) = p_i/3.$$

But, by (3.14), $\Delta_0 \geqslant p_{i+1}$. Therefore, by (3.15), we get

$$\int_{p_{i+2}}^{p_{i+1}} du/\omega(u) = \int_{p_{i+2}}^{p_{i+1}} du/\bar{\omega}(u) \leqslant \int_{p_{i+2}}^{\Delta_0} du/\bar{\omega}(u) \leqslant M/\gamma. \tag{3.16}$$

On the other hand, $\omega^*(u) \leqslant 3^{-i}C$ if $u \leqslant p_i$. Therefore

$$\int_{p_{i+1}}^{p_i} du/\omega^*(u) \geqslant 3^i p_i/C = 3^{-1}/\gamma. \tag{3.17}$$

Hence, putting $M_0 = 3M$, (3.13) follows from (3.16) and (3.17). Since the constant M in Lemma 2 depends neither on ω nor on C, so does M_0. Hence the proof of Lemma 3 is complete.

Now to prove Theorem 1 we have to show the inequality

$$\int_0^{p_1} du/\omega(u) \leqslant M \int_0^{p_0} du/\omega^*(u), \tag{3.18}$$

where $p_i = u(3^{-i}C)$, $i = 0, 1$, the function u is given by (1.4) and C is the range of $\omega \in K$. But (3.18) easily follows from (3.13). Theorem 1 follows now immediately from Lemma 3. The proof of Theorem 1 is thus complete.

4. Proof of Theorem 2. We are going to define the function ω, the existence of which is asserted by Theorem 2, on the set

$$P_0 = \{(t, u): \ 0 \leqslant t \leqslant 1, \ 0 \leqslant u \leqslant 1\}.$$

Before, we introduce an auxiliary function to be used in the construction, and we prove a proposition we will need later.

Let k, a, and b be positive reals such that $k \geqslant 1$, $0 < a \leqslant 1$, $0 < b \leqslant 1$. Let us set

$$\mu(u) = \mu(u; k, a, b) = \min\big(ku, a, (b-u)k\big) \quad \text{if} \quad 0 \leqslant u \leqslant b \tag{4.1}$$

and

$$\mu(u) = \mu(u; k, a, b) = \mu(u-mb; k, a, b) \quad \text{if} \quad mb \leqslant u \leqslant (m+1)b, \tag{4.2}$$

where $m = 1, 2, \ldots$

By (4.1), $\mu(b) = 0$. Therefore the function μ defined by (4.1) and (4.2) is continuous on $[0, \infty)$ and periodic with period b.

We have manifestly

$$0 \leqslant \mu(u; k, a, b) \leqslant a, \tag{4.3}$$

and if

$$kb \geqslant 2a, \tag{4.4}$$

then μ assumes the value a at some u.

PROPOSITION 2. *The function* $\mu = \mu(u; k, a, b)$ *defined by* (4.1) *and* (4.2) *belongs to the class* K *for each* $k \geqslant 1$ *and* $a, b \in (0, 1]$ (cf. Section 1).

Proof. Condition (1.1) is manifestly satisfied. To prove condition (1.2) consider the function $\nu(u) = \mu(u+v) - \mu(u)$, where v is a fixed positive number. It is easy to see from (4.1) that the right-hand derivative μ'_+ of μ exists and is equal either to k or to 0 or to $-k$. Hence also ν'_+ exists and is piecewise constant. Farther, by (4.1) we have $\mu(b-u) = \mu(u)$ if $0 \leqslant u \leqslant b$, which together with the periodicity of μ implies that $\mu(mb-u) = \mu(u)$ for each integer m and $0 \leqslant u \leqslant mb$. The latter equation has as a consequence that $\nu(u) = -\nu(mb-u-v)$ if $mb-u-v \geqslant 0$, which in turn implies that $\max_{u\geqslant 0}\nu(u) = -\min_{u\geqslant 0}\nu(u)$.

Therefore to complete the proof it is enough to prove that $\max_{u\geqslant 0} v(u) = \nu(0) = \mu(v)$. Suppose the maximum is attained for $u = u_0$. By periodicity of ν we may assume that $0 \leqslant u_0 \leqslant b$. Since $\nu(u_0) \geqslant 0$, we may assume also that either $\mu(u_0) = ku_0 < a$ or $\mu(u_0) = k(b-u_0) < a$. In the first case, $\mu'_+(u) = k$ if $0 \leqslant u \leqslant u_0$. Therefore $\nu'_+(u) = \mu'_+(u+v)-k \leqslant 0$, if $0 \leqslant u \leqslant u_0$, whence $\nu(0) \geqslant \nu(u_0)$. In the second case, $\nu'_+(u) = \mu'_+(u+v)+ +k \geqslant 0$ if $u_0 \leqslant u \leqslant b$. Therefore again $\nu(0) = \nu(b) \geqslant \nu(u_0)$. But $\nu(u_0) = \max_{0\leqslant u\leqslant b} \nu(u)$ thus the latter inequality yields the equation $\nu(0) = \max_{u\geqslant 0}\nu(u)$, which completes the proof.

In the sequel, we shall consider the function μ restricted to $[0, 1]$.

The function ω in question will be obtained as the limit

$$\omega(t, u) = \lim_n \omega_n(t, u), \qquad (t, u) \in P_0, \tag{4.5}$$

where the functions ω_n are to be defined now.

Suppose k_n, $n = 0, 1, 2, \ldots$, is a sequence of positive reals such that

$$k_0 = 1, \qquad k_n > 2k_{n-1}^2 e^{k_{n-1}} \quad \text{if} \quad n \geqslant 1, \tag{4.6}$$

where e is the base of natural logarithms. Put

$$u_n(t) = k_n^{-2} e^{k_n(t-1)}, \qquad 0 \leqslant t \leqslant 1, \ n = 0, 1, \ldots \tag{4.7}$$

Let us set, for each $(t, u) \in P_0$,

$$\omega_0(t, u) \equiv u \tag{4.8}$$

and, inductively,

$$\omega_n(t, u) = \max\big(\omega_{n-1}(t, u),\ \mu(u; k_n, k_{n-1}^{-1}, u_{n-1}(t))\big) \quad \text{if} \quad n \geqslant 1. \tag{4.9}$$

We will prove now that for each $n = 0, 1, 2, \ldots$ the following conditions hold:

(i) $$\omega_n(t, u) = k_n u \quad \text{if} \quad 0 \leqslant u \leqslant k_n^{-2},$$

(ii) $$\omega_n^*(t, u) = \max_{0\leqslant v\leqslant u} \omega_n(t, v) \geqslant \sqrt{u} \quad \text{if} \quad u \geqslant k_n^{-2},$$

(iii) $$|\omega_n(t, u_1) - \omega_n(t, u_2)| \leqslant \omega_n(t, |u_1-u_2|) \leqslant k_n|u_1-u_2|$$

for each (t, u_1), $(t, u_2) \in P_0$,

(iv) $$du_i(t)/dt = \omega_n\big(t, u_i(t)\big) \quad \text{for} \quad i = 0, 1, \ldots, n, \ 0 \leqslant t \leqslant 1.$$

To prove (i)-(iv) we apply the induction argument. It is clear by (4.6), (4.7) and (4.8) that (i)-(iv) hold true if $n = 0$. So assume that (i)-(iv) hold for $n = N$. Then by (4.9)

$$(4.9') \qquad \omega_{N+1}(t, u) = \max\big(\omega_N(t, u),\ \mu(u; k_{N+1}, k_N^{-1}, u_N(t))\big).$$

By (4.6) we have $k_{N+1} > 2k_N^2 e^{k_N}$, therefore in particular $k_{N+1}^{-1} < k_N^{-1}$, and by (4.1) and (i) we get for $n = N$ that

$$\mu\big(u; k_{N+1}, k_N^{-1}, u_N(t)\big) = k_{N+1}u > k_N u = \omega_N(t, u) \quad \text{if} \quad u \leqslant k_{N+1}^{-1}k_N^{-1}.$$

(Note that by (4.6) and (4.7) we have $k_{N+1}u_N(t) \geqslant k_{N+1}k_N^{-2}e^{-k_N} > 2k_N^{-1}$, hence, by (4.4), $\mu\big(u; k_{N+1}, k_N^{-1}, u_N(t)\big) = k_N^{-1}$ if $u = k_N^{-1}k_{N+1}^{-1}$). Therefore by (4.9')

$$(4.10) \qquad \omega_{N+1}(t, u) = k_{N+1}u \quad \text{if} \quad u \leqslant k_N^{-1}k_{N+1}^{-1}.$$

Since by (4.6) we have $k_{N+1}^{-2} \leqslant k_N^{-1}k_{N+1}^{-1}$, (4.10) implies (i) for $n = N+1$. To see (ii) for $n = N+1$ note that $\omega_{N+1}(t, u) = \sqrt{u}$ if $u = k_{N+1}^{-2}$. Therefore, by (4.10), $\omega_{N+1}(t, u) \geqslant \sqrt{u}$ if $k_{N+1}^{-2} \leqslant u \leqslant k_{N+1}^{-1}k_N^{-1}$. But $\omega_{N+1}(t, k_{N+1}^{-1}k_N^{-1}) = k_N^{-1}$, and so

$$(4.11) \qquad \omega_{N+1}^*(t, u) \geqslant \sqrt{u} \quad \text{if} \quad k_{N+1}^{-2} \leqslant u \leqslant k_N^{-2}.$$

By (4.9') we have $\omega_{N+1}(t, u) \geqslant \omega_N(t, u)$, whence also $\omega_{N+1}^*(t, u) \geqslant \omega_N^*(t, u)$. Therefore by (ii) for $n = N$ and (4.11) we have (ii) for $n = N+1$.

Condition (iii) means that, for each fixed t, $\omega_n(t, u)$ as a function of u belongs to the class K. Since both functions in the right-hand side of (4.9') have this property, we have, by Proposition 1 of Section 3, the inequality

$$(4.12) \qquad |\omega_{N+1}(t, u_1) - \omega_{N+1}(t, u_2)| \leqslant \omega_{N+1}(t, |u_1 - u_2|)$$

for each $(t, u_1), (t, u_2) \in P_0$. By (iii), for $n = N$, $\omega_N(t, u)$ satisfies the Lipschitz condition with respect to u with the constant k_N, and by (4.1) and (4.2) the function $\mu\big(u; k_{N+1}, k_N^{-1}, u_N(t)\big)$ also satisfies the Lipschitz condition with the constant k_{N+1}. Thus by (4.9') we have the inequality $\omega_{N+1}(t, u) \leqslant k_{N+1}u$, which together with (4.12) proves (iii) for $n = N+1$.

Finally, condition (iv) for $n = N+1$ and $i = N+1$ follows from (i) for $n = N+1$ and (4.7); for $n = N+1$ and $i = N$ it follows from (iv) for $n = N$, (4.9') and the fact that $\mu\big(u; k_{N+1}, k_N^{-1}, u_N(t)\big) = 0$ if $u = u_N(t)$. Suppose now that $i \leqslant N-1$. By (iv) for $n = N$ we have $\omega_N\big(t, u_i(t)\big) = k_i u_i(t)$. Thus using (4.6) and (4.7) we get

$$(4.13) \qquad \omega_N\big(t, u_i(t)\big) \geqslant k_i^{-1}e^{-k_i} > k_{i+1}^{-1} \geqslant k_N^{-1} \quad \text{if} \quad i \leqslant N-1,$$

whence (cf. (4.3)) we get $\omega_N\big(t, u_i(t)\big) > \mu\big(u_i(t); k_{N+1}, k_N^{-1}, u_N(t)\big)$ if $i \leqslant N-1$. Therefore, by (4.9'), $\omega_N\big(t, u_i(t)\big) = \omega_{N+1}\big(t, u_i(t)\big)$ if $i \leqslant N-1$, which, using (iv) for $n = N$, completes the proof of (iv) for $n = N+1$.

Note that $k_n^{-1} \to 0$ as $n \to \infty$, therefore by (4.7)

$$u_n(t) \to 0 \text{ uniformly in } [0, 1]. \tag{4.14}$$

Now by (4.9) we have

$$\omega_n(t, u) \geqslant \omega_{n-1}(t, u) \geqslant \omega_0(t, u) = u, \qquad n = 1, 2, \ldots \tag{4.15}$$

Therefore again by (4.9) and (4.3) it is easy to see that

$$\omega_n(t, u) = \omega_{n-1}(t, u) \quad \text{if} \quad u \geqslant k_{n-1}^{-1} \tag{4.16}$$

and, taking into account (i), that

$$\omega_n(t, u) \leqslant k_n^{-1} \quad \text{if} \quad u \leqslant k_n^{-2}. \tag{4.17}$$

We shall prove now that the function ω given by (4.5) has all properties stated by Theorem 2. It follows from (4.16), (4.17) and (4.5) that

$$\omega(t, u) = \omega_n(t, u) = \omega_{n+p}(t, u) \quad \text{if} \quad u \geqslant k_n^{-1} \text{ and } p \geqslant 1 \tag{4.18}$$

and

$$\omega_n(t, u) \to 0 = \omega(t, 0) \quad \text{if} \quad u \to 0 \text{ uniformly in } t. \tag{4.19}$$

Now by (4.18), (4.5) and (iii) we have

$$|\omega(t, u_1) - \omega(t, u_2)| \leqslant \omega(t, |u_1 - u_2|) \quad \text{for any } (t, u_1), \tag{4.20}$$

$(t, u_2) \epsilon P_0$, which, together with (4.18) and (4.19), implies that ω is continuous on P_0 and satisfies conditions (1.9) and (1.10). By (4.18) and (iv) we get that $u = u_n(t)$ is a solution of

$$u' = \omega(t, u), \qquad u(0) = u_n(0) \tag{4.21}$$

for each $n = 0, 1, \ldots$, and by (iii) and (4.18) it is unique. The latter implies that the unique solution of (1.12) is $u(t) \equiv 0$.

Finally, by (4.18) and (ii) it follows that $\omega^*(t, u) = \omega_n^*(t, u) \geqslant \sqrt{u}$ if $u \geqslant k_{n-1}^{-1}$, whence

$$\omega^*(t, u) \geqslant \sqrt{u} \quad \text{if} \quad u \geqslant 0. \tag{4.22}$$

As a consequence of (4.22) we can have the differential inequality

$$\varphi'(t) \leqslant \omega^*\big(t, \varphi(t)\big), \quad \text{where} \quad \varphi(t) = t^2/4. \tag{4.23}$$

Now in view of the theory of differential inequalities it follows from (4.23) that $\varphi(t)$ is less than or equal to the maximum solution of (1.13) if $t \geqslant 0$, which means that (1.13) admits a solution which is positive for $t > 0$. Therefore we have proved that $\omega(t, u)$ given by (4.5) satisfies (1.9) and (1.10), the unique solution of (1.12) is $u(t) = 0$, while (1.13) admits a positive solution. Hence the proof of Theorem 2 is complete.

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