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A SINGULAR PLANE CURVE

BY

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1. The problem. The following two properties play a particular role in the topological theory of *curves* (i.e. of 1-dimensional metric continua):

(1) the absence of *indecomposable subcontinua* (i.e. of subcontinua which are not unions of two their subcontinua no one of whose contains the other); this property is called *hereditary decomposability*;

(2) the presence, for each pair of points, of a unique arc joining them; we call this property *one-arcwise connectedness*.

Properties (1) and (2) appear especially in the study of *dendroids* (i.e. of curves acyclic in the sense that for each pair of their points there exists exactly one continuum irreducible between them, which is an arc). Some information on this matter can be found, for example, in papers [3], p. 239, and [4], p. 197, by Charatonik (1).

It is known that (2) does not imply (1), at least for 2-dimensional continua in the euclidean 3-space. Such is, e.g., the cone C over the *pseudo-arc*, i.e., the hereditarily indecomposable continuum $\mathscr{K}_3$ (see my paper [6], p. 275-279).

During his stay in Wrocław in 1974, Sam B. Nadler, Jr. posed a question whether also in the plane, thus for curves, (2) does not imply (1). Each plane 2-dimensional continuum contains disks (see [11], Theorem 4, p. 81), therefore many arcs between the same pair of points, and it also contains indecomposable continua (see [10], p. 327).

The answer is affirmative. The example of a plane curve $\mathscr{S}$ that will be constructed in this paper has property (2) without having property (1), which shows in particular that, also in the plane, property (2) alone does not suffice to assure hereditary decomposability of a continuum.

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2. The union $\mathscr{F}$. The construction will heavily depend on the union $\mathscr{F}$ of a family containing $2^{\aleph_0}$ disjoint curves, each homeomorphic to the continuum defined by the conditions

$$y = \sin\frac{\pi}{x} \quad \text{for } 0 < x \leqslant 1, \tag{3}$$

$$-1 \leqslant y \leqslant 1 \quad \text{for } x = 0 \tag{4}$$

and situated in the square Q with the opposite vertices $(0, 0)$ and $(1, 1)$. This family of curves has been constructed in my paper [7] (see Fig. V, p. 286, with the under-text and three final propositions, p. 288 and 289).

Let $\mathfrak{N}$ be the Cantor discontinuum in the segment $0 \leqslant x \leqslant 1$, $y = 0$ and let I be the set composed of the points $(0, 0)$, $(1, 0)$ and of the left end-points of the intervals contiguous to $\mathfrak{N}$. Let M_t be a line homeomorphic to (3) and L_t be the vertical segment playing the role of (4) for M_t, where $L_t \cup M_t \subset \mathscr{F}$ for every $(t, 0) \in \mathfrak{N} \setminus I$. Thus every of these $L_t \cup M_t$ is a continuum irreducible between the points $(t, 0)$ and $(1, 1-t)$, and the first of these two points is the lower end-point of L_t, whereas the members of the union $\mathscr{F}$ for values t satisfying $0 \neq t \neq 1$ and $(t, 0) \in I$ are, in the construction of $\mathscr{F}$ mentioned above, broken lines, thus some arcs. Note the following easy property of $\mathscr{F}$:

$$t = \lim_{n\to\infty} t_n \text{ implies } L_t \cup M_t = \operatorname*{Lim}_{n\to\infty}(L_{t_n} \cup M_{t_n}), \tag{5}$$

which means that the decomposition of $\mathscr{F}$ into the curves $L_t \cup M_t$ is continuous (see [9], p. 61 and 62).

3. The continuum $\mathscr{B}_0$. Let $\mathscr{B}_0$ be the simplest indecomposable continuum composed of all semicircles D_t with ordinates $y \geqslant 0$, having the center $(2^{-1}, 0)$ and passing through all points $(t, 0) \in \mathfrak{N}$ and of all ones of ordinates $y \leqslant 0$ with the centers $(5\cdot 2^{-1}\cdot 3^{-n}, 0)$, where $n = 1, 2, \ldots$, passing through all the points $(t, 0) \in \mathfrak{N}$ with $2\cdot 3^{-n} \leqslant t \leqslant 3^{-n+1}$.

Put $\mathfrak{N}_n = \{(t, 0) \in \mathfrak{N}\colon\ 2\cdot 3^{-n} \leqslant t \leqslant 3^{-n+1}\}$. Thus we have

$$\mathfrak{N} = \{0\} \cup \bigcup_{n=1}^{\infty} \mathfrak{N}_n, \tag{6}$$

and putting

$$B = \bigcup_{(t,0)\in\mathfrak{N}} D_t, \quad B_n = \bigcup_{(t,0)\in\mathfrak{N}_n} D_t \tag{7}$$

we obtain

$$\mathscr{B}_0 = B \cup \bigcup_{n=1}^{\infty} B_n. \tag{8}$$

The set $\mathscr{B}_0$ is an indecomposable continuum (see [8], p. 40).

4. The union $\mathscr{F}^*$. The leading idea of the construction of the curve $\mathscr{E}$ consists in the replacing of the semicircles, the curve $\mathscr{B}_0$ is composed of, by curves homeomorphic to irreducible continua $L_t \cup M_t$ transformed in a proper way in order to obtain a new indecomposable continuum; we denote it by $\mathscr{B}_s$. But it needs some preliminary considerations.

Since the union of curves $L_t \cup M_t$ for $(t, 0) \in \mathfrak{N} \setminus I$ is not a closed set, it should be replaced by a subunion $\mathscr{F}^*$ which is closed.

To this end let J denote the set composed of points $(0, 0)$, $(0, 1)$ and of end-points of all intervals of the complement of the Cantor set $\mathfrak{N}$ to the unit interval $0 \leqslant x \leqslant 1$. The set $\mathfrak{N} \setminus J$ is obviously homeomorphic to the set C of binary irrational numbers of the interval $0 \leqslant x \leqslant 1$ under the Cantor step-function ("la fonction scalariforme de Cantor", see [2], p. 386). Let $\mathfrak{N}'$ be the homeomorphic image in $\mathfrak{N} \setminus J$ of some perfect subset contained in C. Further, the sets $\mathfrak{N}$ and $\mathfrak{N}'$ being perfect, 0-dimensional and linear, a homeomorphism

$$h' \colon \mathfrak{N} \to \mathfrak{N}' \tag{9}$$

between them can be chosen in such a way that it preserves the order $<$ between corresponding points ([1], p. 146). Denote by $\mathscr{F}^*$ a subset of the union $\mathscr{F}$ formed from the curves $L_t \cup M$ for $(t, 0) \in \mathfrak{N}'$, i.e.,

$$\mathscr{F}^* = \bigcup \big(L_t \cup M_t \colon \ (t, 0) \in \mathfrak{N}'\big).$$

Thus the set $\mathscr{F}^*$, defined in this way, is compact, it has — similarly as $\mathscr{F}$ — property (5), but — contrary to $\mathscr{F}$ — no its member is a broken line, but all are irreducible continua $L_t \cup M$ different from arcs. This property of the set $\mathscr{F}^*$ is essential for the construction of the curve $\mathscr{B}_s$.

5. The indecomposable continuum $\mathscr{B}_s$ and the curve $\mathscr{E}$. The following notation will be used in the sequel. For every t with $(t, 0) \in \mathfrak{N}'$ the member $L_t \cup M_t$ of the union $\mathscr{F}^*$ is irreducible between points $(t, 0)$ and $(1, 1-t)$. Let $\mathfrak{N}''$ be the set of these points, i.e.

$$\mathfrak{N}'' = \{(1, 1-t) \colon \ t \in \mathfrak{N}'\} \subset Q$$

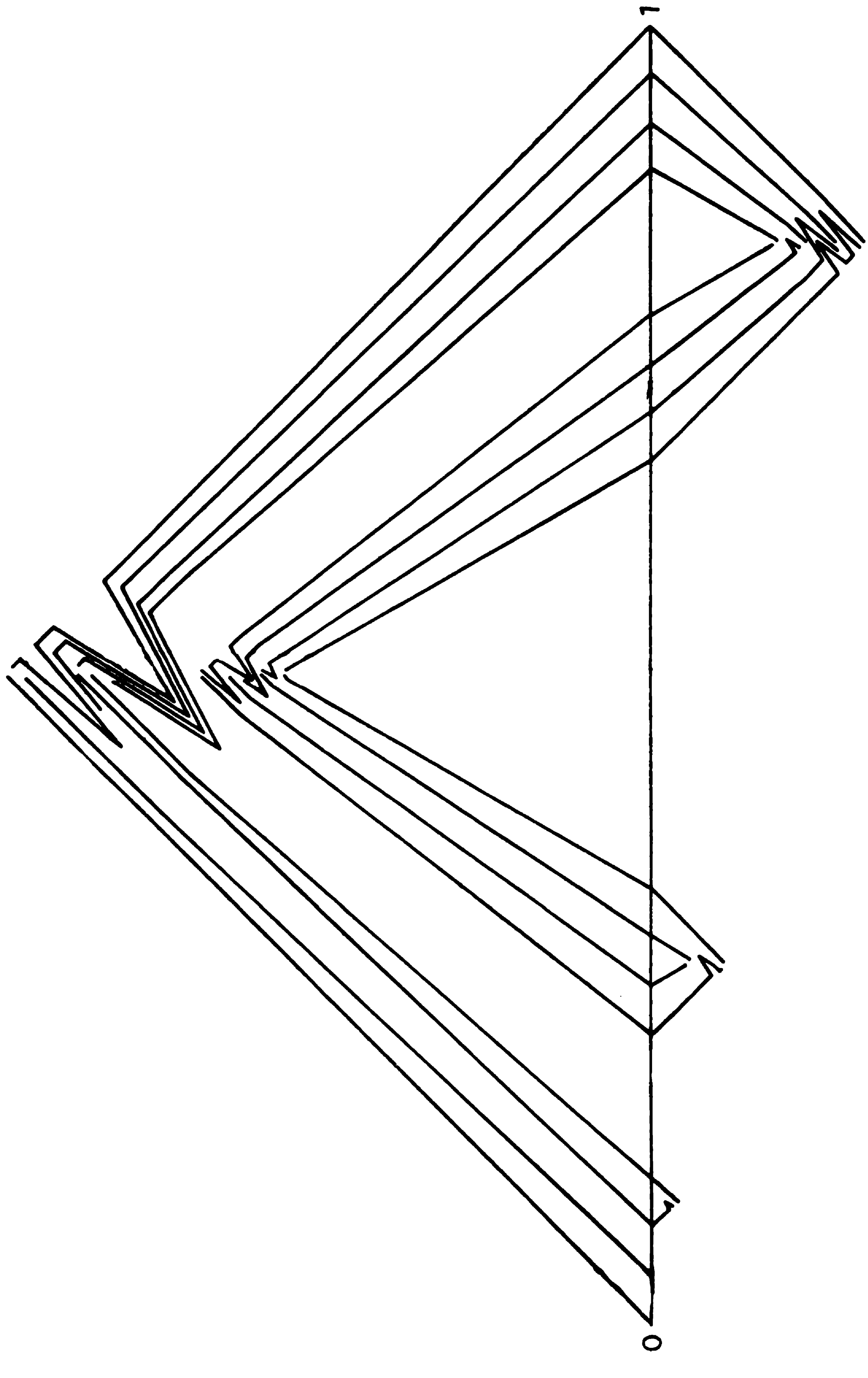
and let — similarly to (9) —

$$h'' \colon \mathfrak{N} \to \mathfrak{N}'' \tag{10}$$

be a homeomorphism preserving the order $<$ between corresponding points.

Having defined the set $\mathscr{F}^*$ which lies in the unit square Q, we are ready to describe the curve $\mathscr{B}_s$. To begin with we define a geometrical operation Γ on the compact set

$$B = \bigcup_{t \in \mathfrak{N}} D_t$$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(see (7)) which is the union of semicircles D_t (with ordinates $y \geqslant 0$). Let (see (6))

$$\mathfrak{N}_0 = \mathfrak{N} \setminus \mathfrak{N}_1. \tag{11}$$

Thus $\mathfrak{N}_0$ is the set of left end-points of semicircles D_t, and $\mathfrak{N}_1$ is the set of their right end-points. Obviously both these sets are homeomorphic with the Cantor set $\mathfrak{N}$. Let $\gamma_0\colon \mathfrak{N}_0 \to \mathfrak{N}$ and $\gamma_1\colon \mathfrak{N}_1 \to \mathfrak{N}$ be homeomorphisms preserving the order, i.e. defined by formulae

$$\gamma_0(t) = 3t \quad \text{for } t \in \mathfrak{N}_0,$$

$$\gamma_1(t) = 3t-2 \quad \text{for } t \in \mathfrak{N}_1.$$

Let us divide every of the semicircles D_t ($t \in \mathfrak{N}$), which are members of the union B, into three equal parts and remove from each of them the arc being the middle one-third of the semicircle D_t. In the free place obtained in this way in the plane we put the image $\gamma(Q)$ of the unit square Q under a similarity transformation γ so that the points

$$\left(\frac{1}{2}, \frac{1}{2}\right) = \gamma(0, 1) \quad \text{and} \quad \left(\frac{1}{2}, \frac{1}{2} - \frac{2}{9}\right) = \gamma(1, 0)$$

are opposite vertices of the square $\gamma(Q)$ and that

$$\gamma(0, 0) = \left(\frac{1}{2} - \frac{1}{9}, \frac{1}{2} - \frac{1}{9}\right).$$

Put

$$h_0 = \gamma h' \gamma_0\colon \mathfrak{N}_0 \to \gamma(\mathfrak{N}') \quad \text{and} \quad h_1 = \gamma h'' \gamma_1\colon \mathfrak{N}_1 \to \gamma(\mathfrak{N}''),$$

where h' and h'' are homeomorphisms (9) and (10).

Now let us take away the remaining left and right one-third parts of the semicircles D_t and join every point $(t, 0) \in \mathfrak{N}_0$ with its image $h_0(t, 0) \in \gamma(\mathfrak{N}') \subset \gamma(Q)$ by the straight line segment A'_t and, analogously, join every point $(t, 0) \in \mathfrak{N}_1$ with its image $h_1(t, 0) \in \gamma(\mathfrak{N}'') \subset \gamma(Q)$ by the straight line segment A''_t. Observe that for different points $(t, 0) \in \mathfrak{N}_0$ (or $(t, 0) \in \mathfrak{N}_1$, respectively) the corresponding segments A'_t (or A''_t, respectively) are disjoint.

Finally, put

$$\Gamma(B) = \bigcup \big(A'_t\colon (t, 0) \in \mathfrak{N}_0\big) \cup \gamma(\mathscr{F}^*) \cup \bigcup \big(A''_t\colon (t, 0) \in \mathfrak{N}_1\big). \tag{12}$$

In this manner every semicircle D_t, being a member of the union B, has been replaced, to form $\Gamma(B)$, by an irreducible continuum composed of straight line segments A'_{t_1} and A''_{t_2} for proper indices t_1 and t_2 and of the image $\gamma(L_u \cup M_u)$, lying between these segments, of the irreducible continuum $L_u \cup M_u \subset \mathscr{F}^*$ (for some $(u, 0) \in \mathfrak{N}'$). More exactly, if a point $(t, 0) \in \mathfrak{N}_0$ is the left end-point of the semicircle D_t being a member of the union B (see (7)), then to obtain $\Gamma(B)$, the semicircle D_t is replaced by $A'_t \cup \gamma(L_u \cup M_u) \cup A''_v$, where (see (9) and (10))

$$(13) \qquad (u, 0) = h'(t, 0)$$

and

$$(14) \qquad (v, 0) = (h'')^{-1}\big((1, 1-u)\big).$$

We obtain in this way the following equivalent description of the operation Γ:

$$(15) \qquad \Gamma(B) = \bigcup [A'_t \cup \gamma(L_u \cup M_u) \cup A''_v\colon (t, 0) \in \mathfrak{N}_0],$$

where the indices u and v are defined by equalities (13) and (14).

Observe that, according to the definition of segments A'_t and A''_v, we have

$$(16) \qquad A'_t \cap \gamma(M_u) = \big\{h_0\big((t, 0)\big)\big\}$$

and

$$(17) \qquad A''_v \cap \gamma(L_u) = \big\{h_1\big((v, 0)\big)\big\}.$$

Exactly in the same manner we define the operation Γ on compact sets B_n for $n = 1, 2, \ldots$ (see (7)), every of which is – similarly to B – the union of semicircles D_t, where $(t, 0) \in \mathfrak{N}_n$. Consequently, for every natural n and for every point $(t, 0) \in \mathfrak{N}_n$, the semicircle D_t is replaced by an irreducible continuum composed of two straight line segments joined by a homeomorphic image of a curve described by formulae (3) and (4).

Finally we define

$$(18) \qquad \mathscr{B}_s = \Gamma(B) \cup \bigcup_{n=1}^{\infty} \Gamma(B_n)$$

and

$$(19) \qquad \mathscr{E} = \mathscr{B}_s \cup \{(x, 0)\colon 0 \leqslant x \leqslant 1\}.$$

6. Proofs of properties. First we show that $\mathscr{B}_s$ is an indecomposable continuum.

Recall (see [8], p. 40 and 41) that the union of the sequence of semicircles $S_1, S_2, \ldots, S_n, \ldots$ in $\mathscr{B}_0$, where $S_1, S_2, \ldots$ have points $(0, 0)$, $(1, 0)$, $(2\cdot 3^{-1}, 0)$, $(3^{-1}, 0)$, ... as their end-points successively, and the common end-points of S_n and S_{n+1}, for every $n = 1, 2, \ldots$, pass through all points of the set J which is dense in $\mathfrak{N}$. Therefore it is easy to verify that the union

$$S = \bigcup_{n=1}^{\infty} S_n$$

(which is a connected set) is dense in $\mathscr{B}_0$ (which is thereby a continuum). Similarly, $\overline{\mathfrak{N} \setminus J} = \mathfrak{N}$, whence $\overline{\mathscr{B}_0 \setminus S} = \mathscr{B}_0$. Further, $\mathscr{B}_0$ is irreducible between every point $p \in S$ and every point $q \in \mathcal{B}_0 \setminus S$ (see [8], property (π), p. 41) which is sufficient for $\mathcal{B}_0$ to be indecomposable (see [5], Theorem IV, p. 215).

Let us come back to $\mathcal{B}_s$. Consider the sequence $D'_1, D'_2, \ldots, D'_n, \ldots$, an analogue to $S_1, S_2, \ldots, S_n, \ldots$, where every D'_n is that continuum, described in the previous construction of the set $\mathcal{B}_s$, which has replaced the semicircle S_n, i.e., D'_n is composed of two straight line segments and of a line homeomorphic to $\sin(1/x)$-curve ((3) and (4)) that joins the proper end-points of the two segments.

Moreover, as for the sequence $S_1, S_2, \ldots, S_n, \ldots$, the continuum D'_1, joins the points (0, 0) and (1, 0) and, for every $n = 1, 2, \ldots$, the common part $D'_n \cap D'_{n+1} = S_n \cap S_{n+1}$ reduces to the common end-point of both these continua.

Let

$$D = \bigcup_{n=1}^{\infty} D'_n.$$

It is easy to verify by induction, in the same way as for S, that $J \subset D$ and, since $\bar{J} = \mathfrak{N}$, we have $\bar{D} = \mathcal{B}_s$. This set is therefore a continuum. Simultaneously, $\overline{\mathfrak{N} \setminus J} = \mathfrak{N}$, whence $\overline{\mathcal{B}_s \setminus D} = \mathcal{B}_s$. Thus D is a set both dense and boundary in $\mathcal{B}_s$. The remaining part of the proof that the continuum $\mathcal{B}_s$ is indecomposable, namely that it is irreducible between (0, 0) and every point of $\mathcal{B}_s \setminus D$, is completely analogous to the corresponding part of the previous argumentation for $\mathcal{B}_0$. Therefore it is also established by (19) that $\mathcal{E}$ does not have property (1).

To prove that $\mathcal{E}$ has property (2) let us remark that, according to (15) and (18), the continuum $\mathcal{B}_s$ is decomposed into disjoint lines C_t with $(t, 0) \in \mathfrak{N}$, defined as follows. For $t = 0$, C_0 is the union $A'_0 \cup \gamma(M_{h'(0,0)}) \subset \Gamma(B)$. For $t \neq 0$, C_t is the union of two arcwise connected sets having $(t, 0)$ as the only common point: one of them is equal either to $A'_t \cup \gamma(M_u)$ (see (13) and (16)) or to $A''_t \cup \gamma(L_w)$ (see (14) and (17) with $v = t$ and $u = w$) and is contained (see (15)) in $\Gamma(B)$ $(y \geqslant 0)$; the other has a similar form and is contained in $\Gamma(B_n)$ $(y \leqslant 0)$, where n is defined by $(t, 0) \in \mathfrak{N}_n$. Therefore every C_t is an arc-component (and it is one-arcwise connected) of $\mathcal{B}_s$, i.e. it is the union of all subarcs in $\mathcal{B}_s$ which contain a given point of C_t. Thus

(20) *every arc-component C_t has — by the definition — exactly one point in common with the straight line $y = 0$, namely the point $(t, 0) \in \mathfrak{N}$.*

Consequently,

(21) *two points which belong to different arc-components C_{t_1} and C_{t_2}, in particular two points $(t_1, 0)$ and $(t_2, 0)$ of $\mathfrak{N}$ with $t_1 < t_2$, cannot be joined by an arc lying in $\mathcal{B}_s$.*

It follows immediately from (19) that every arc which joins two points $p \in C_{t_1}$ and $q \in C_{t_2}$ in $\mathscr{E}$ goes along the x-axis from $(t_1, 0)$ to $(t_2, 0)$; therefore this arc is of the form

$$pq = p(t_1, 0) \cup (t_1, 0)(t_2, 0) \cup (t_2, 0)q.$$

According to (18) and (19) such an arc there exists indeed. The question is to prove that it is unique. Suppose that there exists another one, say A. By (20) we have $A \cap \overline{\mathscr{E} \setminus \mathscr{B}_s} \neq \emptyset$. Let us order the points of A linearly starting with p, and let r be the first point of $\overline{\mathscr{E} \setminus \mathscr{B}_s}$ in A. Thus all points of A that precede r are in $\mathscr{B}_s$, which is closed by the definition, whence $r \in \mathscr{B}_s$. Let C_t be an arc-component of $\mathscr{B}_s$ such that $r \in C_t$. The arc pr joins points $p \in C_{t_1}$ and $r \in C_t$ in $\mathscr{B}_s$, whence, by (21), we have $t_1 = t$ and thus $r = (t, 0)$ by (20). Therefore the uniqueness of the initial arc $p(t_1, 0)$ and, by the symmetry of assumptions, of the final arc $(t_2, 0)q$ has been established. Finally, if there would exist another intermediate arc, say T, in $\mathscr{E}$, besides the segment $(t_1, 0)(t_2, 0)$ in the straight line $y = 0$, the union $(t_1, 0)(t_2, 0) \cup T$ would contain a simple closed curve composed of two arcs, say ab and ba, such that $ab \subset (t_1, 0)(t_2, 0)$ and $ba \subset T$. But no straight line contains two distinct arcs having the same end-points, and since the segment $(t_1, 0)(t_2, 0)$ containing ab is contained in the straight line $y = 0$, the arc ba must be contained in $\mathscr{B}_s$, which is impossible by (20) and (21). Thus the proof of the uniqueness of the arc $pq \subset \mathscr{E}$ is complete.

7. Remarks. The plane curve $\mathscr{E}$ which satisfies (2) without satisfying (1) contains the only indecomposable continuum $\mathscr{B}_s$. To construct in the plane an analogous example containing a finite or countable family of disjoint indecomposable continua is obviously sufficient to prolong the segment $0 \leqslant x \leqslant 1$ along the x-axis to a finite or infinite sequence of smaller and smaller consecutive segments (and adjoin their limit-point of course), and to repeat the construction of $\mathscr{E}$ on each of them. However, it is not known if one can go further in this direction, i.e. the following problem seems to be open: does there exist in the plane a one-arcwise connected continuum which contains uncountably many disjoint indecomposable continua? (**P 1138**)

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