COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 2

ON SOME ALGEBRAS CONNECTED WITH THE THEORY OF HARMONY

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1. Introduction. Some sequences of natural numbers can be used to describe a harmonical structure of musical chords as it was done in [8]. One can improve the harmonical analysis by defining some operations on these sequences and using its properties. One obtains algebras (denoted by $\mathfrak{S}_n$) which are finite Boolean algebras with the new greatest element added and with three new unary operations. Fundamental properties of these algebras were investigated in [5]. The aim of the present note is to give some further information about these algebras and about equational classes generated by $\mathfrak{S}_n$.

All notions from universal algebra may be found in [2], and from lattice theory — in [1].

1. Preliminaries. Let T be a non-void set and let $p\colon T \to R^+$ be an injective function from T into the set of positive real numbers. In the theory of harmony, we will interpret T as the set of tones, finite subsets of T as chords, and $p(t)$ as a pitch of the tone $t \in T$. The pair (T, p) is called a *tone system*. Consider the set T with elements indexed by the integers. In this case, we identify the set T with the set Z of all integers. Therefore, let $T = Z$. Denote by T_n the tone system (Z, p) if $p(z) = 2^{z/n}$ for $z \in Z$ and for a fixed natural number n. The tone system T_n is called an *equal temperament n-tone system*. For any $z_1, z_2 \in Z$ in the tone system T_n, let $z_1 r z_2$ iff there is $z \in Z$ such that $p(z_2)/p(z_1) = 2^z$, i.e. $z_2 - z_1 = nz$. Evidently, r is an equivalence relation on the set Z. Now consider the set $C'(Z)$ of all pairs (C, z) such that C is a finite subset of Z and $z \in C$ (¹). Let $(C_1, z_1) R (C_2, z_2)$ iff $z_1 r z_2$ and for every $z \in C_1$ there is $t \in C_2$ such that zrt and, conversely, for every $u \in C_2$ there is $v \in C_1$ with urv. Evidently, R is an equivalence relation on $C'(Z)$. For each $c \in C'(Z)/R$ there is a representation $(\{z_1, \ldots, z_k\}, z_1)$ of c by elements of $C'(Z)$ such that $p(z_i)$

(¹) We can interpret the pair (C, z) as an n-tone chord with the lowest tone z.

$< p(z_{i+1})$, where $i=1,\ldots,k-1$ and $p(z_k)/p(z_1) < 2$. This representation will be called *canonical*. Let h be the mapping on $C'(Z)/R$ such that

$$h(c) = \big(z_2-z_1, z_3-z_2, \ldots, z_k-z_{k-1}, n-(z_k-z_1)\big) \quad \text{for any } c \in C'(Z)/R,$$

where $(\{z_1,\ldots,z_k\}, z_1)$ is the canonical representation of c. It is easy to see that $h(c)$ is a sequence of natural numbers less than or equal to n and the sum of all elements of $h(c)$ is equal to n. Let

$$S'_n = \Big\{(s_1,\ldots,s_k)\colon s_i \in \{1,\ldots,n\},\, k \leqslant n,\, \sum_{i=1}^{k} s_i = n\Big\}$$

and

$$S_n = S'_n \cup \{(\ )\}.$$

Let

$$C(Z) = C'(Z) \cup \{(\emptyset,\ )\}, \qquad h\big((\emptyset,\ )\big) = (\ )$$

and let $\varphi\colon C(Z) \to C(Z)/R$ [2] be the natural mapping of $C(Z)$ onto $C(Z)/R$. The composition $h\circ\varphi$ maps $C(Z)$ onto the set S_n of sequences of natural numbers which describe a harmonical structure of chords in the tone system T_n. The interpretation of the described notions and operations on S_n (defined in the sequel) in the theory of harmony can be found in [5], [7] and [8].

Let n be a fixed natural number and assume that $a = (a_1,\ldots,a_k)$ and $b = (b_1,\ldots,b_r)$ belong to S'_n, $k, r \leqslant n$. Let $a \leqslant b$ iff there exists a sequence $(v_i)_{i=0,1,\ldots,r}$ such that $0 = v_0 < v_1 < \ldots < v_r = k$ and, for $j = 1,\ldots,r$,

$$b_j = \sum_{i=v_{j-1}+1}^{v_j} a_i.$$

Moreover, for every $a \in S'_n$ we assume that $a \leqslant (\ )$.

It is easy to check (see, for example, [5]) that the relation $\leqslant$ is a partial order on S_n. The poset $\langle S'_n, \leqslant\rangle$ is a 2^{n-1}-element Boolean algebra with the greatest element (n) and the least element $0 = (1,\ldots,1)$. The greatest element 1 of S_n is equal to $(\ )$.

Note that $\langle S_n, \leqslant\rangle$ is a pseudocomplemented lattice (i.e. S_n is a lattice and, for every $a \in S_n$, there exists an element a^* such that $a \wedge x = 0$ iff $x \leqslant a^*$). One can define the operation * as follows:

$$x^* = \begin{cases} 1 & \text{if } x = 0,\\ x' & \text{if } x \in S'_n,\ x \neq 0,\\ 0 & \text{if } x = 1.\end{cases}$$

In the set S_n we define three new unary operations which play an important role in the theory of harmony. For $a = (a_1,\ldots,a_k) \in S'_n$, let

$$-a = (a_k, a_{k-1},\ldots,a_1), \qquad a^p = (a_2, a_3,\ldots,a_k, a_1),$$

$$a^\perp = (a'_2,\ldots,a'_{n-k}, a'_{n-k+1}+a'_1) \qquad \text{for } a \neq 0,$$

(2) We assume that $[(\emptyset,\ )]R = \{(\emptyset,\ )\}$.

where $(a_1', \ldots, a_{n-k+1}')$ is the pseudocomplement of a. Moreover, $-1 = 1$, $1^p = 1$ and $1^\perp = 0$, $0^\perp = 1$.

Let $\mathfrak{S}_n$ denote the algebra $\langle S_n, \vee, \wedge, -, {}^p, {}^\perp, 0, 1\rangle$.

2. Some properties of the algebras $\mathfrak{S}_n$. Fundamental properties of $\mathfrak{S}_n$ can be found in [5]. We give some further information about these algebras. It is easy to see that every algebra $\mathfrak{S}_n$ is a dual pseudocomplemented lattice and a double Heyting algebra (i.e. for all $a, b \in S_n$, there exists an element $a_* b$ such that $a \wedge x \leqslant b$ iff $x \leqslant a_* b$, and there exists an element $a_+ b$ such that $a \vee x \geqslant b$ iff $x \geqslant a_+ b$). Moreover, these three operations can be defined by means of the operations ${}^*, {}^\perp, \vee, \wedge$. If we denote by a^+ the dual pseudocomplement of $a = (a_1, \ldots, a_k) \in S_n$, then

$$a^+ = a^{(n-k)(*\perp)},$$

where $a^{1(*\perp)} = a^{*\perp}$, $a^{i(*\perp)} = a^{(i-1)(*\perp)}$ for $i > 1$.

Varlet [6] has shown that a double pseudocomplemented lattice is regular, i.e. two congruence relations having a congruence class in common coincide iff $a^* = b^*$ and $a^+ = b^+$ imply $a = b$. We can see that all $\mathfrak{S}_n$ are regular. Hence (see [4])

$$a_* b = (a^* \vee b^{**})^{**} \wedge [(a \vee a^*)^* \vee a^* \vee b \vee b^*],$$

$$a_+ b = (a^+ \wedge b^{++})^{++} \vee [(a \wedge a^+)^+ \wedge a^+ \wedge b \wedge b^+].$$

Let $\mathfrak{S}_n' = \langle S_n, \vee, \wedge, {}^*, 0, 1\rangle$.

Lemma 1. *There exists no non-trivial congruence of the algebra $\mathfrak{S}_n'$, which is a congruence with respect to the operation ${}^\perp$.*

Proof. Let φ be a non-trivial congruence of the algebra $\mathfrak{S}_n'$ which is at the same time a congruence with respect to the operation ${}^\perp$. It follows from Theorem 16.7 in [1] that $(n) \equiv 1(\theta)$ for any congruence $\theta \neq \omega$ of the algebra $\mathfrak{S}_n'$ (ω is the least element of the congruence lattice of $\mathfrak{S}_n'$). Thus also

$$(n) \equiv 1 \ (\varphi), \quad 1^\perp = 0 \quad \text{and} \quad (n)^\perp = (1, \ldots, 1, 2).$$

Hence $(1, \ldots, 1, 2) \equiv 0 \ (\varphi)$. Obviously, $0^* = 1$ and $1^\perp = 0$. Therefore

$$\begin{aligned}
(1, \ldots, 1, 2)^* &= (n-1, 1) \equiv 1 \ (\varphi),\\
(n-1, 1)^\perp &= (1, \ldots, 1, 3) \equiv 0 \ (\varphi),\\
(1, \ldots, 1, 3)^* &= (n-2, 1, 1) \equiv 1 \ (\varphi),\\
(n-2, 1, 1)^\perp &= (1, \ldots, 1, 4) \equiv 0 \ (\varphi),\\
&\ldots\ldots\ldots\ldots\ldots\\
(1, n-1)^* &= (2, 1, \ldots, 1) \equiv 1 \ (\varphi),\\
(2, 1, \ldots, 1)^\perp &= (n) \equiv 0 \ (\varphi),\\
(n)^* &= 0 \equiv 1 \ (\varphi).
\end{aligned}$$

This is a contradiction, since the congruence φ is non-trivial.

COROLLARY 1. *All algebras $\mathfrak{S}_n$ are simple (i.e. they have no non-trivial congruences).*

3. Equational classes generated by the algebras $\mathfrak{S}_n$. For $i = 1, 2, \ldots,$ we put

$$a^{ip} = (a_i, \ldots, a_k, a_1, \ldots, a_{i-1}),$$

where $a = (a_1, \ldots, a_k) \in S_n$ and $a^{i(*\perp)} = \big(\ldots\big((a^{*\perp})^{*\perp}\big)^{*\perp}\ldots\big)^{*\perp}$, the operation ${}^{*\perp}$ being repeated i times.

LEMMA 2. *Every algebra $\mathfrak{S}_n$ has exactly one proper subalgebra.*

Proof. It is evident that the set $\{0, 1\}$ is closed under all operations of $\mathfrak{S}_n$. Hence it forms the subalgebra of $\mathfrak{S}_n$ isomorphic to $\mathfrak{S}_1$. Let $a = (a_1, \ldots, a_k) \in S_n$, $a \neq 0, 1$. By the definitions of operations $*$ and $^{\perp}$ we have

$$a^{*\perp} = (a_2, \ldots, a_{k-1}, a_k + a_1).$$

Therefore, $a^{(k-1)(*\perp)} = (a_1 + \ldots + a_k) = (n)$. Hence all elements of the form $[(n)^{\perp}]^{ip}$ with $i = 1, \ldots, n-1$ are atoms of $\mathfrak{S}_n$. Therefore, the algebra $\mathfrak{S}_n$ is generated by every element $a \neq 0, 1$.

Let S denote the class of all algebras isomorphic to $\mathfrak{S}_n$. Since all algebras $\mathfrak{S}_n$ are finite, the class S is not equational. Let $K = HSP(S)$. (For a class of algebras R, the classes $P(R), H(R), S(R)$ consist of all direct products, homomorphic images and subalgebras of members of R, respectively.) Let K_n be the smallest equational class containing an algebra isomorphic to $\mathfrak{S}_n$.

LEMMA 3. *Let $\mathfrak{A} \in K$. A congruence lattice $C(\mathfrak{A})$ of all congruence relations over $\mathfrak{A}$ is distributive.*

Indeed, $C(\mathfrak{A})$ is a distributive lattice as a sublattice of the distributive congruence lattice $C(\langle A, \vee, \wedge\rangle)$.

Therefore, we can apply to the class K the following well-known Jónsson's lemma [3].

LEMMA 4. *If $HSP(R)$ is an equational class of algebras with distributive congruences, then every subdirectly irreducible algebra from $HSP(R)$ belongs to $HSP_u(R)$, where $P_u(R)$ denotes the class of all ultraproducts of members of the class R.*

The following corollary follows from Corollary 1 and Lemmas 2 and 4.

COROLLARY 2. *The algebras isomorphic to $\mathfrak{S}_n$ and $\mathfrak{S}_1$ are the only subdirectly irreducible algebras in the class K_n.*

The next corollary follows from Lemma 4 and from the fact that the property "an algebra $\mathfrak{A}$ is isomorphic to a Boolean algebra with the new greatest element added" can be expressed in the first order language.

COROLLARY 3. *Every subdirectly irreducible algebra in the class K is a Boolean algebra with the new greatest element added.*

Evidently, all known corollaries to Jónsson's lemma hold for K. Particularly, the lattice of all equational subclasses of K is distributive and $\mathfrak{S}_1, \mathfrak{S}_{n_1}, \ldots, \mathfrak{S}_{n_k} \in S$ are all subdirectly irreducible in the class $HSP(\{\mathfrak{S}_{n_1}, \ldots, \mathfrak{S}_{n_k}\})$. Therefore, the algebras $\mathfrak{S}_n$ are all finite subdirectly irreducible in the class K. It would be interesting to find an equational base for K. The classes K_n have finite equational bases, but it is unknown if K has also a finite base.

Note that for every $a \in \mathfrak{A} \in K_1$ we have $-a = a$, $a^p = a$, $a^{\perp} = a$. Hence K_1 is the class of all Boolean algebras. For every $a \in \mathfrak{A} \in K_2$,

$$-a = a, \quad a^p = a, \quad a^{\perp} = (a \wedge a^+) \vee a^*.$$

Hence K_2 is the class of all regular double Stone algebras (for the definition see [4]). All algebras from the class K are regular and all are double Heyting algebras.

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Reçu par la Rédaction le 15. 11. 1974