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THE EXPONENTIAL OBJECTS IN TOP
(A CLASSICAL PROOF)

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1. Preliminaries. Given arbitrary spaces Y and Z, let $C_t(Y, Z)$ denote the set $C(Y, Z)$ of continuous maps on Y to Z equipped with some topology t. The topology t is said to be *splitting* (or *proper*) on $C(Y, Z)$ if for every space X the continuity of a function $f\colon X \times Y \to Z$ implies that of its adjoint function $\hat{f}\colon X \to C_t(Y, Z)$, where $\hat{f}(x)(y) = f(x, y)$ for all x and y, i.e., if the exponential injection

$$E_{XYZ}\colon C(X \times Y, Z) \to C(X, C_t(Y, Z)), \quad \text{where } E_{XYZ}(f) = \hat{f},$$

is well defined [3].

If for every X the continuity of $\hat{f}\colon X \to C_t(Y, Z)$ implies that of $f\colon X \times Y \to Z$, then t is called *jointly continuous* on $C(Y, Z)$; equivalently, t is *jointly continuous* (or *admissible*) on $C(Y, Z)$ if the evaluation function $e\colon C_t(Y, Z) \times Y \to Z$ is continuous, where $e(g, y) = g(y)$ (see [5]).

A space Y is *exponential* in TOP if for every space Z there is a splitting-jointly continuous topology on $C(Y, Z)$.

A splitting-jointly continuous topology on $C(Y, Z)$ is both the greatest splitting topology and the coarsest jointly continuous [1]. Also it is known that any jointly continuous topology is finer than any splitting topology [1].

A space Y is called *corecompact* if for every point $y \in Y$ and every open set V containing y there is some open set W bounded in V and containing y. (W is *bounded* in V if every open cover of V contains finitely many members covering W (see [6])).

Ward [9] introduced these spaces under the name *quasi-locally compact*. Subsequently, they were investigated under various names. Day and Kelly [2] studied these spaces under the name *spaces which have the property* C. Also Hofmann and Lawson [4] studied the above-mentioned spaces under the name *corecompact*. Wyler ([10], p. 227) noted that spaces Y which have the Day–Kelly property C (i.e., corecompact spaces) are exactly the exponential objects in TOP, i.e., E_{XYZ} is a bijection for all spaces X and Z.

2. The main theorem. We will prove that if a topological space Y is exponential in TOP, then Y is corecompact. To prove this, it suffices to show that if there exists a splitting-jointly continuous topology on $C(Y, 2)$, where 2 is the Sierpiński space, then Y is corecompact. (The *Sierpiński space* 2 is the set $\{0, 1\}$ with the topology $\{\{\emptyset\}, \{1\}, \{0, 1\}\}$.) Hence

$$C(Y, 2) = \{\chi_W \in C(Y, 2):\ W \in O(Y)\}$$

with

$$\chi_W(y) = \begin{cases} 1 & \text{if } y \in W, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A \subset Y$. We denote by $(A, \{1\})$ the set

$$\{\chi_W \in C(Y, 2):\ \chi_W(A) = 1\} = \{\chi_W \in C(Y, 2):\ W \supset A\}.$$

The topology r^* defined below is used in the proof of Theorem 4.6 of [1]. Let Y be an arbitrary topological space, and $\{A_i:\ i \in I\}$ a family of open subsets of Y. Setting

$$V = \bigcup_{i \in I} A_i$$

we define the *topology* r^* as follows:

The *r^*-open sets* are defined to be the sets of the form $\{\chi_W\}$, $W \neq V$, and the *r^*-open neighborhoods* of χ_V the sets of the form

$$(A^*_{i_1} \cup A^*_{i_2} \cup \ldots \cup A^*_{i_k}, \{1\}),$$

where $\{i_1, i_2, \ldots, i_k\}$ is a finite subset of I, $A^*_{i_j} \subset A_{i_j}$, and $A^*_{i_j}$ $(j = 1, 2, \ldots, k)$ are open in Y.

LEMMA 2.1 ([1]). *The topology r^* is jointly continuous on $C(Y, 2)$.*

THEOREM 2.2. *If there exists a splitting-jointly continuous topology t on $C(Y, 2)$, then Y is corecompact.*

Proof. The evaluation map e: $C_t(Y, 2) \times Y \to 2$ is continuous. Let V be an open neighborhood of an arbitrary point $y \in Y$. Then $\chi_V(y) = 1$ and the continuity of e implies the existence of an open t-neighborhood N^t of χ_V and an open neighborhood V_y of y such that

$$(*) \qquad N^t \times V_y \subset e^{-1}(\{1\}).$$

For every $z \in V_y$ the last inclusion implies

$$\chi_V(z) = e(\chi_V, z) = 1,$$

so $z \in V$, and hence $V_y \subset V$.

We observe that the set $M^t = \{\chi_U:\ U \supset V_y\}$ is a t-neighborhood of χ_V

since $N^t \subset M^t$. This inclusion holds, because if an arbitrary $\chi_U \in N^t$, then, by (*),

$$e(\chi_U, V_y) = \chi_U(V_y) = 1,$$

which implies $V_y \subset U$, and thus $U \in M^t$.

We prove now that V_y is bounded in V. Let $\{A_i\colon i \in I\}$ be a family of open subsets of V such that

$$\bigcup_{i \in I} A_i = V.$$

We consider the topology r^* defined above. Then, by Lemma 2.1, we have $t \subset r^*$. Since M^t is a t-neighborhood of χ_V, there exists an r^*-neighborhood of χ_V, say

$$(A_{i_1}^* \cup A_{i_2}^* \cup \ldots \cup A_{i_k}^*, \{1\}),$$

such that

$$\chi_V \in (A_{i_1}^* \cup A_{i_2}^* \cup \ldots \cup A_{i_k}^*, \{1\}) \subset M^t.$$

Setting

$$G = \bigcup_{j=1}^{k} A_{i_j} \quad \text{and} \quad G^* = \bigcup_{j=1}^{k} A_{i_j}^*,$$

we observe that $\chi_{G^*} \in (G^*, \{1\}) \subset M^t$, and thus $\chi_{G^*} \in M^t$, which implies

$$G^* = \bigcup_{j=1}^{k} A_{i_j}^* \supset V_y.$$

So

$$G = \bigcup_{j=1}^{k} A_{i_j} \supset G^* \supset V_y,$$

and hence Y is corecompact.

Remark 2.3. If a topological space Y is corecompact, then for every space Z there exists a splitting-jointly continuous topology on the set $C(Y, Z)$, namely the Isbell topology. Hence Y is exponential in TOP (see [7] and [8]).

REFERENCES

[1] R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific J. Math. 1 (1951), pp. 5–31.

[2] B. Day and G. M. Kelly, *On topological quotient maps preserved by pullbacks or products*, Proc. Camb. Phil. Soc. 67 (1970), pp. 553–558.

[3] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.

[4] K. H. Hofmann and J. D. Lawson, *The spectral theory of distributive continuous lattices*, Trans. Amer. Math. Soc. 246 (1978), pp. 285–310.

[5] J. L. Kelley, *General Topology*, Springer-Verlag, Berlin–Heidelberg–New York 1955.
[6] P. Th. Lambrinos, *A topological notion of boundedness*, Manuscripta Math. 10 (1973), pp. 289–296.
[7] – and B. Papadopoulos, *The (strong) Isbell topology and (weakly) continuous lattices*, Continuous Lattices and their Applications (Bremen, 1982), pp. 191–211; Lecture Notes in Pure and Appl. Math. 101, Dekker, New York 1985.
[8] F. Schwarz and S. Weck, *Scott topology, Isbell topology and continuous convergence*, Continuous Lattices and their Applications (Bremen, 1982), pp. 251–271; Lecture Notes in Pure and Appl. Math., 101, Dekker, New York 1985.
[9] A. J. Ward, *Problem in Proceedings of the International Symposium on Topology and its Applications*, Herceg-Noví 1968; D. R. Kurepa (ed.), Savez Matematicara, Beograd 1969, pp. 351–352.
[10] O. Wyler, *Convenient categories for topology*, General Topology and Appl. 3 (1973), pp. 225–242.

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