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EXTREMALLY DISCONNECTED RESOLUTIONS OF T_0-SPACES

BY

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The concept of an extremally disconnected resolution in the compact case is due to Gleason [5] (see also Rainwater [12] and Hager [6]). For an arbitrary Hausdorff space a construction of the extremally disconnected resolution was given by Iliadis [7] and modified by Mioduszewski and Rudolf [11], where a more general definition of extremally disconnected resolutions was also provided. As shown by Mioduszewski [10], there exist many extremally disconnected resolutions in their sense for a given space.

In the present paper we define an extremally disconnected resolution (shortly, e.d. resolution) for an arbitrary T_0-space and prove a new existence theorem which depends on the factorization lemma from [1]. Considering the set of non-equivalent representatives of all e.d. resolutions of a given space, we show that it is partially ordered by dense embeddings, and that each e.d. resolution can be obtained as a selector from the greatest one. It is also shown that an e.d. resolution $a\colon aX \to X$ is the greatest one if the map a is perfect.

A similarly general approach to e.d. resolutions is also possible by means of some generalizations of uniformities, as indicated by Kulpa [9].

All maps in this paper are assumed to be continuous.

1. Preliminaries. A map $f\colon X\to Y$ is said to be *skeletal* provided the preimage under f of each open and dense subset of Y is dense in X or, equivalently, if

$$(1)\qquad \mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) = \mathrm{Int}_X f^{-1}(\mathrm{cl}_Y U)$$

for each U being open in Y.

A map

$$h\colon X \xrightarrow{\text{onto}} Y$$

is said to be *irreducible* ([11], p. 26) if

$$(2)\qquad \mathrm{cl}_Y h(F) \neq Y$$

whenever F, $F \subset X$ and $F \neq X$, is regularly closed; *regularly closed*, shortly, *r.c.*, means that $F = \mathrm{cl}_X \mathrm{Int}_X F$.

It is known ([11], p. 27) that each irreducible map h is skeletal and that

(3) for each U open in X, there exists a V open in Y such that $\mathrm{cl}_X U = \mathrm{cl}_X h^{-1}(V)$.

The converse is also true:

THEOREM 1. *If a skeletal map $h\colon X \xrightarrow{\text{onto}} Y$ satisfies* (3), *then it is irreducible.*

Proof. Let $F = \mathrm{cl}_X U$ for some U open in X. There exists, by (3), a set V open in Y for which $\mathrm{cl}_X h^{-1}(V) = F$. Thus $h(F) \subset \mathrm{cl}_Y V$. Suppose that $h(F)$ is dense in Y. Hence V is dense in Y. Therefore, $F = X$ since h is skeletal.

A set is *regularly open*, shortly, *r.o.*, if its complement is r.c. A map $h\colon X \to Y$ is called *r.o.-minimal* ([11], p. 30) if the family $\{G \cap h^{-1}(U)\colon G$ is regularly open in X and U is open in $Y\}$ is a base in X.

LEMMA 1. *A map*

$$f\colon X \xrightarrow{\text{onto}} Y$$

is irreducible and r.o.-minimal iff the family

$$\{f^{-1}(U) \cap \mathrm{Int}_X \mathrm{cl}_X f^{-1}(V)\colon U \text{ and } V \text{ are open in } Y\}$$

is a base in X.

Proof. Let

$$\mathfrak{B} = \{f^{-1}(U) \cap \mathrm{Int}_X \mathrm{cl}_X f^{-1}(V)\colon U \text{ and } V \text{ are open in } Y\}$$

be a base in X. At the first we shall show that f is skeletal. To do this let us suppose that

$$f^{-1}(G) \cap \mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$

where G is dense and open in Y and $\mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) \cap f^{-1}(V)$ is a non-empty set from the base $\mathfrak{B}$. Then $G \cap U \cap V = \emptyset$ and, in consequence, $U \cap V = \emptyset$, G being dense. Hence

$$\mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$

a contradiction.

The r.o.-minimality is obvious.

It remains to show that f is irreducible. Let H be open in X. Hence $H = \bigcup\{W\colon W \in \mathscr{U}\}$, where $\mathscr{U} \subset \mathfrak{B}$. Put

$$G = \bigcup \{U \cap V\colon \mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) \cap f^{-1}(V) \in \mathscr{U}\}.$$

It is easy to check that $\mathrm{cl}_X H = \mathrm{cl}_X f^{-1}(G)$. Now, the irreducibility follows from Theorem 1.

The converse implication is obvious.

Recall that a space is *extremally disconnected* (shortly, *e.d.*) if the closure of each open subset of it is open (see Stone [13]).

The following version of the factorization lemma (Lemma 2) from [1] will be used in the sequel:

FACTORIZATION LEMMA. *If $f\colon X \xrightarrow{\text{onto}} Y$ is skeletal, X is e.d., and X and Y are T_0-spaces, then there exists a unique (up to homeomorphism) factorization $X \xrightarrow{g} Z \xrightarrow{h} Y$ of f such that $h\colon Z \to Y$ is irreducible and r.o.-minimal, Z is T_0 and e.d., and g is a skeletal map onto Z.*

THEOREM 2. *If a map $f\colon X \xrightarrow{\text{onto}} Y$, where X is a T_0-space and Y is e.d., is irreducible and r.o.-minimal, then it is a homeomorphism.*

Proof. Since Y is e.d. and f is skeletal, we have, by (1),

$$\mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) = \mathrm{Int}_X f^{-1}(\mathrm{cl}_Y U) = f^{-1}(\mathrm{cl}_Y U) \quad \text{for each } U \text{ open in } Y.$$

Thus, according to Lemma 1, the family

$$\mathfrak{B} = \{f^{-1}(\mathrm{cl}_Y U \cap V)\colon\ U \text{ and } V \text{ are open in } Y\}$$

is a base in X. Hence f is an open map. For every two different points from X, there exists an element from $\mathfrak{B}$ which contains exactly one of these points. Thus f is one-to-one. Therefore, it is a homeomorphism.

2. The set of e.d. resolutions of a given T_0-space. An *e.d. resolution* is an irreducible r.o.-minimal map of an e.d. T_0-space onto a given one. We say that e.d. resolutions $\mu\colon \mu X \to X$ and $\nu\colon \nu X \to X$ are *equivalent* if there exists a homeomorphism $\varphi\colon \mu X \to \nu X$ such that $\nu\circ\varphi = \mu$.

LEMMA 2. *If $h\colon X \xrightarrow{\text{onto}} Y$ is irreducible and r.o.-minimal, and X is a T_0-space, then $\mathrm{card}\, X \leqslant 2^{\mathrm{card}\mathcal{T}}$, where $\mathcal{T}$ is the topology on Y.*

Proof. By Lemma 1,

$$\mathfrak{B} = \{\mathrm{Int}_X \mathrm{cl}_X h^{-1}(U) \cap h^{-1}(V)\colon\ U \text{ and } V \text{ are open in } Y\}$$

is a base of open sets in X. Since X is T_0,

$$\mathrm{card}\, X \leqslant 2^{\mathrm{card}\mathfrak{B}} \leqslant 2^{\mathrm{card}\mathcal{T}}.$$

From this lemma it follows immediately that

THEOREM 3. *For each T_0-space X there exists the set* $\mathrm{Res}\, X$ *consisting of non-equivalent e.d. resolutions such that each e.d. resolution of X is equivalent to one from* $\mathrm{Res}\, X$.

In the sequel we restrict our considerations to the fixed set $\mathrm{Res}\, X$ of e.d. resolutions of a given T_0-space X.

We say that an e.d. resolution $\mu\colon \mu X \to X$ *majorizes* an e.d. resolution $\nu\colon \nu X \to X$ (shortly, $\mu \geqslant \nu$) if there exists $\varphi\colon \nu X \to \mu X$ such that $\mu\circ\varphi = \nu$.

LEMMA 3. *If the map $g\colon Y \xrightarrow{\text{onto}} Z$ is irreducible and r.o.-minimal, and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is skeletal, then so is f.*

Proof. Let G be dense and open in Y. Since g is irreducible and r.o.-minimal, $\mathrm{cl}_Z g(Y\setminus G)$ is nowhere dense in Z (see [11], p. 30). We have

$$(g\circ f)^{-1}\big(Z\setminus \mathrm{cl}_Z g(Y\setminus G)\big) \subset f^{-1}(G).$$

But $(g\circ f)^{-1}\big(Z\setminus \mathrm{cl}_Z g(Y\setminus G)\big)$ is dense since $g\circ f$ is skeletal. Thus $f^{-1}(G)$ is dense in X.

By this lemma we infer that the map φ, realizing the inequality $\mu \geqslant \nu$, is skeletal.

LEMMA 4. *If $\mu\colon \mu X \to X$ is an e.d. resulution, then*

$$\mu\circ f = \mu\circ g \Rightarrow f = g \tag{4}$$

for arbitrary f and g being skeletal maps from a T_0-space Y into μX.

Proof. Suppose that $\mu\circ f = \mu\circ g$ and $f(x) \neq g(x)$ for some x, $x\in Y$. Since Y is a T_0-space, hence, by Lemma 1, there exist U and V open in Z such that the set

$$W = \mathrm{Int}_{\mu X}\mathrm{cl}_{\mu X}\mu^{-1}(U)\cap \mu^{-1}(V)$$

contains exactly one of the points $f(x)$ and $g(x)$. But μX is e.d., and f is skeletal; then, by (1),

$$\begin{aligned} f^{-1}(W) &= f^{-1}\big(\mathrm{Int}_{\mu X}\mathrm{cl}_{\mu X}\mu^{-1}(U)\big)\cap f^{-1}\big(\mu^{-1}(V)\big) \\ &= f^{-1}\big(\mathrm{cl}_{\mu X}\mu^{-1}(U)\big)\cap(\mu\circ f^{-1})(V) = \mathrm{Int}_Y\mathrm{cl}_Y(\mu\circ f)^{-1}(U)\cap(\mu\circ f)^{-1}(V). \end{aligned}$$

Analogously,

$$g^{-1}(W) = \mathrm{Int}_Y\mathrm{cl}_Y(\mu\circ g)^{-1}(U)\cap(\mu\circ g)^{-1}(V).$$

Since $\mu\circ g = \mu\circ f$, we have $g^{-1}(W) = f^{-1}(W)$. Suppose that $f(x)\in W$ and $g(x)\notin W$. Then $x\in f^{-1}(W)\setminus g^{-1}(W)$, a contradiction.

By Lemma 4 we infer that the map φ, realizing the inequality $\mu \geqslant \nu$ between e.d. resolutions is unique. A standard calculation leads to the conclusion that

THEOREM 4. *The set* $\mathrm{Res}\,X$ *is partially ordered by* $\geqslant$.

We shall show that e.d. resolutions are, in fact, comparable by embeddings. This follows from Lemmas 5 and 6.

LEMMA 5. *If in the decomposition $X \xrightarrow{f} Y \xrightarrow{g} Z$ the map f is skeletal, g is irreducible and $(g\circ f)(X)$ is dense in Z, then $f(X)$ is dense in Y.*

Proof. Suppose that there exists a non-empty open set U such that $U \subset Y \setminus f(X)$. Then $f^{-1}(\mathrm{cl}_Y U) = f^{-1}(\mathrm{cl}_Y U \setminus U)$ is nowhere dense in X, f being skeletal. Since g is irreducible, there exists, by (3), a non-empty open set V, $V \subset Z$, such that $\mathrm{cl}_Y U = \mathrm{cl}_Y g^{-1}(V)$. Hence $f^{-1}(\mathrm{cl}_Y g^{-1}(V))$ is nowhere dense in X. On the other hand, $(g \circ f)^{-1}(V)$ is open and non-empty, since $(g\circ f)(X)$ is dense in Z. But $(g\circ f)^{-1}(V) \subset f^{-1}(\mathrm{cl}_Y g^{-1}(V))$, a contradiction.

LEMMA 6. *If a decomposition $X \xrightarrow{f} Y \xrightarrow{g} Z$ is irreducible and r.o.-minimal, then f, regarded as the map $\bar{f}$: $X \to f(X)$, is irreducible and r.o.-minimal.*

Proof. By Lemma 1, the family

$$\mathfrak{B} = \{\mathrm{Int}_X \mathrm{cl}_X (g\circ f)^{-1}(U) \cap (g\circ f)^{-1}(V)\colon\ U \text{ and } V \text{ are open in } Z\}$$

is a base of open sets in X. Hence

$$\mathfrak{B}' = \{\mathrm{Int}_X \mathrm{cl}_X f^{-1}(U') \cap f^{-1}(V')\colon\ U' \text{ and } V' \text{ are open in } f(X)\}$$

is also a base in X. Thus, by Lemma 1, $\bar{f}$ is irreducible and r.o.-minimal.

THEOREM 5. *If μ: $\mu X \to X$ and ν: $\nu X \to X$ are e.d. resolutions and $\mu \geqslant \nu$, then the map φ, realizing this inequality, is a dense embedding.*

Proof. By Lemma 5, the map φ, regarded as $\bar{\varphi}$: $\nu X \to \varphi(\nu X)$, is irreducible and r.o.-minimal and, by Lemma 4, $\varphi(\nu X)$ is dense in μX; thus $\varphi(\nu X)$ is e.d. Then, by Theorem 2, $\bar{\varphi}$ is a homeomorphism.

3. The existence of e.d. resolutions. Iliadis [7] has constructed, using ultrafilters, for each Hausdorff space X, an irreducible but only θ-continuous map ω^X: $\omega X \to X$ from an e.d. space ωX onto X (see also Iliadis and Fomin [8]). Mioduszewski and Rudolf [11] have modified this construction so that the map ω^X becomes continuous. Moreover, it was proved in [10] that the modified Iliadis e.d. resolution is the greatest one.

We propose a way which leads to all e.d. resolutions for an arbitrary T_0-space. The first step in the construction is the following

LEMMA 7. *For every space $(X, \mathscr{T})$, there exists an e.d. topology $\mathscr{T}'$ on X such that the identity $(X, \mathscr{T}') \to (X, \mathscr{T})$ is skeletal.*

Proof. Let $\mathscr{T}$ be the topology on X. Consider the set S of topologies on X such that

(I) if $\mathscr{T}' \in S$, then $\mathscr{T} \subset \mathscr{T}'$;

(II) if $\mathscr{T}' \in S$, then the identity map i: $(X, \mathscr{T}') \to (X, \mathscr{T})$ is skeletal.

We prove that there exists a maximal member in S.

Let L be a chain in S, and let us examine the topology $\mathscr{T}^*$ generated by $\bigcup L$. Since L is a chain, the family $\bigcup L$ is a base for $\mathscr{T}^*$. This implies $\mathscr{T}^* \in S$. Hence it is a bound of L.

Let $\tilde{\mathscr{T}}$ be a maximal element in S. It remains to show that $(X, \tilde{\mathscr{T}})$ is e.d. Suppose that there exists $U \in \tilde{\mathscr{T}}$ such that $\mathrm{cl}_{\tilde{\mathscr{T}}} U \notin \tilde{\mathscr{T}}$. Then the topology $\mathscr{P}$, generated by $\tilde{\mathscr{T}} \cup \{\mathrm{cl}_{\tilde{\mathscr{T}}} U\}$, belongs to S and is greater than $\tilde{\mathscr{T}}$; a contradiction.

Composing Lemma 7, Factorization Lemma and Theorem 5, we get

THEOREM 6. *For each T_0-space, there exist e.d. resolutions; moreover, there exist one-to-one e.d. resolutions which are minimal.*

This theorem asserts that the set $\mathrm{Res}\,X$ is non-empty. Let Y be the disjoint union of all domains of maps from $\mathrm{Res}\,X$. Clearly, Y is e.d. Using once again Factorization Lemma for the map

$$f\colon Y \xrightarrow{\text{onto}} X$$

induced by maps from $\mathrm{Res}\,X$, we obtain the greatest e.d. resolution $a\colon aX \to X$ (cf. [1], where the construction was performed in the Hausdorff case).

The present construction falls under a general categorial scheme given by Freyd [4], dual to the construction of the Čech-Stone compactification.

4. Properties of the greatest e.d. resolution. In this section we show that each e.d. resolution can be obtained by a selection from the greatest e.d. resolution. The following lemma will be used:

LEMMA 8. *If a map*

$$f\colon X \xrightarrow{\text{onto}} Y$$

is irreducible, r.o.-minimal and $f(F) = Y$, then F is dense in X.

Proof. Suppose that F is not dense in X. There exist, by Lemma 1, open sets U and V of Y such that

$$F \cap \mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) \cap f^{-1}(V) = \emptyset \quad \text{and} \quad \mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) \cap f^{-1}(V) \neq \emptyset.$$

Hence $F \cap f^{-1}(U \cap V) = \emptyset$ and $U \cap V \neq \emptyset$. Thus $f(F) \cap U \cap V = \emptyset$, a contradiction.

THEOREM 7. *If $F \subset aX$ and $a(F) = X$, then $a|F\colon F \to X$ is an e.d. resolution. Moreover, if $K \subset aX$, $F \neq K$ and $a(K) = a(F) = X$, then $a|F$ and $a|K$ are non-equivalent e.d. resolutions.*

Proof. To show the first thesis, let us note, by Lemma 8, that F is dense in aX. Hence it is e.d. By Lemma 1, it is easy to see that the restriction $a|F\colon F \to X$ is irreducible and r.o.-minimal.

Suppose that $a|F\colon F \to X$ and $a|K\colon K \to X$ are equivalent e.d. resolutions. Then there exists a homeomorphism $\varphi\colon F \to K$ such that $(a|K)\circ\varphi = a|F$. Let us consider the diagram

$$\begin{tikzcd}
\alpha X \arrow[d, "\alpha"'] & & \\
X & K \arrow[ul, "i"'] \arrow[l, "\alpha|K"'] & \\
 & & F \arrow[ul, "\varphi"'] \arrow[ull, "\alpha|F"] \arrow[uull, bend right, "j"']
\end{tikzcd}$$

where i and j are inclusions. We have $a \circ j = a \circ i \circ \varphi$. According to Lemma 4, we get $i \circ \varphi = j$. Thus $F = K$, a contradiction.

Example. Let N be the set of all natural numbers, and let $\omega N = N \cup \{\Omega\}$ be the Alexandroff one-point compactification of N. There exists a function

$$a\colon \beta N \xrightarrow{\text{onto}} \omega N$$

mapping the Čech-Stone compactification βN onto ωN and being the identity on N. It is easy to see that a is irreducible. Since βN is regular, it is also r.o.-minimal. By Theorem 5, $a\colon \beta N \to \omega N$ is the greatest e.d. resolution of ωN. Since the compact e.d. resolution is unique, there exist many non-compact e.d. resolutions of ωN. There exist exactly 2^{2^c} non-equivalent e.d. resolutions of ωN. Indeed, by Theorem 7, it suffices to note that $a^{-1}(\Omega) = \beta N \setminus N$ and $\operatorname{card}(\beta N \setminus N) = 2^c$ (see Engelking [2]).

From Theorems 7 and 5 we get

THEOREM 8. *For each set $S \subset \operatorname{Res} X$, there exists the supremum of S in* $\operatorname{Res} X$.

THEOREM 9. *The supremum of the set of all minimal e.d. resolutions is equal to the greatest e.d. resolution.*

Proof. It suffices to note that, according to Theorem 7, each minimal e.d. resolution can be obtained by choosing one point from $a^{-1}(x)$ for all $x \in X$.

5. The greatest e.d. resolutions are perfect.

LEMMA 9. *If $a\colon aX \to X$ is the greatest e.d. resolution of X, X being a T_0-space, and x is an arbitrary point of X, then there does not exist a filter $\mathscr{F}$ of closed-open sets of aX containing the family*

$$\{\operatorname{cl}_{aX} a^{-1}(U)\colon\ U \text{ is open in } X \text{ and } x \in U\}$$

and such that $\bigcap \{a^{-1}(x) \cap G\colon\ G \in \mathscr{F}\} = \emptyset$.

Proof. Consider the filter ξ being maximal in the family of filters of closed-open sets containing $\mathscr{F}$. Put $\tilde{X} = aX \cup \{\xi\}$, where

$$\mathfrak{B}_\xi = \{\{\xi\} \cup a^{-1}(U) \cap G\colon\ U \text{ is open in } X,\ x \in U \text{ and } G \in \xi\}$$

is a base of neighbourhoods of ξ and the neighbourhoods of points of aX are such as in aX. We show that

(5) if G is a closed-open set in aX, then $G \in \xi \Leftrightarrow \xi \in \operatorname{cl}_{\tilde{X}} G$.

Indeed, if $\xi \in \mathrm{cl}_{\tilde{X}} G$, then $G \cap H \neq \emptyset$ for each $H \in \xi$. Thus $G \in \xi$, since ξ is maximal.

Suppose that $\xi \notin \mathrm{cl}_{\tilde{X}} G$. Then there exists a W being open in $\tilde{X}$ such that $\xi \in W$ and $W \cap G = \emptyset$. Hence $G \cap \mathrm{cl}_{aX}(W \cap aX) = \emptyset$. But we have $\mathrm{cl}_{aX}(W \cap aX) \in \xi$. Thus $G \notin \xi$.

Clearly, for each closed set F, $F \subset aX$, we have $F = aX \cap \mathrm{cl}_{\tilde{X}} F$. Therefore, by (5), we get

(6) for each closed-open set G, $G \subset aX$, we have $\mathrm{cl}_{\tilde{X}} G = G \Leftrightarrow G \notin \xi$, and $\mathrm{cl}_{\tilde{X}} G = G \cup \{\xi\} \Leftrightarrow G \in \xi$.

Let $\mu \colon \tilde{X} \to X$ be the map defined by the formula

$$\mu(\xi) = x, \quad \mu(y) = a(y) \quad \text{for each } y \in aX. \tag{7}$$

Let us consider, on the set $\tilde{X}$, the topology generated by the family

$$\mathscr{P} = \{\mu^{-1}(U) \cap \mathrm{cl}_{\tilde{X}} G \colon U \text{ is open in } X \text{ and } G \text{ is closed-open in } aX\}$$

and let us denote such a space by μX. By (6), $\mathscr{P}$ is a base in μX, closed with respect to finite intersections. Clearly, $\mu \colon \mu X \to X$ is continuous.

We show that μX is a T_0-space. Note that the topology induced in aX from that of μX is the given one. Since aX is a T_0-space, points of μX lying in aX are T_0-separated. To see that ξ and y, where $y \in aX$, are T_0-separated, consider two cases.

Case 1. Let $y \notin a^{-1}(x)$; then there exists a set U open in X, which contains exactly one of the points x and $\mu(y)$. If $x \in U$ and $\mu(y) \notin U$, then $\mu^{-1}(U)$ is an open neighbourhood of ξ which does not contain y. If $\mu(y) \in U$ and $x \notin U$, then $\mu^{-1}(U)$ is an open neighbourhood of y and $\xi \notin \mu^{-1}(U)$.

Case 2. Let $y \in a^{-1}(x)$; then, by the assumption, there exists a G such that $y \notin G$. Hence, by (6), $\mathrm{cl}_{\tilde{X}} G$ is an open neighbourhood of ξ in μX and $y \notin \mathrm{cl}_{\tilde{X}} G$.

Now we show that μX is e.d. If V and W are open in μX and disjoint, then $G = \mathrm{cl}_{aX}(V \cap aX)$ and $H = \mathrm{cl}_{aX}(W \cap aX)$ are closed-open and disjoint in aX, aX being e.d. On the other hand,

$$\mathrm{cl}_{\mu X} V \cap aX = \mathrm{cl}_{\mu X}(V \cap aX) \cap aX = \mathrm{cl}_{aX}(V \cap aX) = G,$$

since aX is dense in μX. Analogously, $\mathrm{cl}_{\mu X} W \cap aX = H$. Suppose that $\mathrm{cl}_{\mu X} V \cap \mathrm{cl}_{\mu X} W \neq \emptyset$. Then $\xi \in \mathrm{cl}_{\mu X} V \cap \mathrm{cl}_{\mu X} W$ and, in consequence, $G \cap H \in \xi$. But $G \cap H = \emptyset$, a contradiction.

It remains to show that $\mu \colon \mu X \to X$ is irreducible and r.o.-minimal. To do this we show that, for each closed-open set G, $G \subset aX$, the set $\mathrm{cl}_{\tilde{X}} G$ is closed-open in μX. In fact, if $G \in \xi$, then, by (6), $\mathrm{cl}_X G = G \cup \{\xi\}$.

Hence we obtain $\mu X\setminus \mathrm{cl}_{\tilde{X}} G = aX\setminus G$. But $aX\setminus G\notin \xi$, thus, by (6), $aX\setminus G = \mathrm{cl}_{\tilde{X}}(aX\setminus G)$. Therefore, $\mu X\setminus \mathrm{cl}_{\tilde{X}} G$ is open in μX. We do the same in the case where $G\notin \xi$.

The above shows that μ is r.o.-minimal in view of the form of sets in the family $\mathscr{P}$ generating the topology in μX.

It remains to show that μ is irreducible. Assume that $\mu(G)$ is dense in X and G is closed-open in μX. Clearly, $\xi\in G$ since $a\colon aX\to X$ is irreducible. We show that $a(G\cap aX)$ is also dense in X. Suppose that there exists an open set U, $U\subset X$ and $U\neq \emptyset$, such that $U\cap a(G\cap aX)=\emptyset$. Since $U\cap \mu(G)\neq\emptyset$ and $\mu(G)=\{x\}\cup a(G\cap aX)$, we have $x\in U$. Thus $\mathrm{cl}_{\mu X}a^{-1}(U)\in \mathscr{F}\subset \xi$. On the other hand,

$$\mathrm{cl}_{aX}a^{-1}(U)\cap G\cap aX=\emptyset.$$

But $G\cap aX\in\xi$, a contradiction. Thus we infer that $a(G\cap aX)$ is dense in X. Since a is irreducible, $G\cap aX = aX$. Therefore, G being closed in μX, $G=\mu X$, which means that μ is irreducible.

Thus it is shown that $\mu\colon \mu X\to X$ is greater than $a\colon aX\to X$; a contradiction.

LEMMA 10. *If $\mu\colon \mu X\to X$ is an e.d. resolution, then, for each $x\in X$, $\mu^{-1}(x)$ is a Hausdorff subspace of μX.*

Proof. Indeed, if $y, z\in \mu^{-1}(x)$, then there exists a set

$$W=\mathrm{cl}_{\mu X}\mu^{-1}(U)\cap \mu^{-1}(V),$$

where U and V are open in X, which contains exactly one of the points y and z; μX being a T_0-space. Hence the set $\mathrm{cl}_{\mu X}\mu^{-1}(U)$ contains exactly one of these points. Thus the set $\mathrm{cl}_{\mu X}\mu^{-1}(U)$, being closed-open, separates y and z in the Hausdorff sense.

THEOREM 10. *An e.d. resolution $a\colon aX\to X$ is the greatest one iff a is perfect, i.e., $a(F)$ is closed whenever F is closed and $a^{-1}(x)$ is compact Hausdorff for each $x\in X$.*

Proof. I. Suppose that $a\colon aX\to X$ is the greatest e.d. resolution. At the first we show that $a^{-1}(x)$ is compact Hausdorff for each $x\in X$. By Lemma 10, $a^{-1}(x)$ is Hausdorff. It is easy to see that the family

$$\mathfrak{B}=\{a^{-1}(x)\cap G\colon G \text{ is closed-open in } aX\}$$

is a base of open sets in $a^{-1}(x)$. To prove that $a^{-1}(x)$ is compact take a filter $\mathscr{G}$, $\mathscr{G}\subset\mathfrak{B}$. Consider in aX the filter

$$\mathscr{F}=\{\mathrm{cl}_{aX}a^{-1}(U)\colon U \text{ is an open neighbourhood of } x\}\cup$$
$$\cup\{H\colon H \text{ is closed-open in } aX \text{ and } H\cap a^{-1}(x)\in\mathscr{G}\}.$$

Clearly, if $\bigcap\mathscr{G}=\emptyset$, then $\bigcap\{a^{-1}(x)\cap G\colon G\in\mathscr{F}\}=\emptyset$. The last equality contradicts Lemma 9. Thus $\bigcap\mathscr{G}\neq\emptyset$.

Now we check that $a(G)$ is closed whenever G is closed-open in aX. Suppose that $x\in \mathrm{cl}_X a(G)\setminus a(G)$. Thus the family

$$\mathscr{F} = \{G\}\cup\{\mathrm{cl}_{aX} a^{-1}(U)\colon\ U \text{ is an open neighbourhood of } x\}$$

is a filter of closed-open sets of aX. Since $G\cap a^{-1}(x) = \emptyset$, we have

$$\bigcap\{a^{-1}(x)\cap H\colon\ H\in\mathscr{F}\} = \emptyset.$$

But this contradicts Lemma 9.

Now we prove that $a(F)$ is closed for each closed F, $F\subset aX$. Let $x\in X$ be given such that $x\notin a(F)$. Hence $a^{-1}(x)\cap F = \emptyset$. Since

$$\mathfrak{B} = \{a^{-1}(U)\cap G\colon\ U \text{ is open in } X \text{ and } G \text{ is closed-open in } aX\}$$

is a base in aX, for each $y\in a^{-1}(x)$ there exists a $W\in\mathfrak{B}$ such that $y\in W$ and $W\cap F = \emptyset$. But $a^{-1}(x)$ is compact, whence there exists a family $\{W_1, \ldots, W_n\}\subset\mathfrak{B}$ such that

$$a^{-1}(x)\subset W_1\cup\ldots\cup W_n\subset aX\setminus F.$$

Put $H = X\setminus a\big(aX\setminus(W_1\cup\ldots\cup W_n)\big)$. It is easy to see that $x\in H$ and $H\cap a(F) = \emptyset$. It remains to show that H is open in X. Let W_i be of the form $W_i = a^{-1}(U_i)\cap G_i$, where U_i are open in X and G_i are closed-open in aX for $i = 1, \ldots, n$. Let us evaluate

$$\begin{aligned} aX\setminus(W_1\cup\ldots\cup W_n) &= (aX\setminus W_1)\cap\ldots\cap(aX\setminus W_n)\\ &= \bigcap_{i=1}^{n}\big[\big(aX\setminus a^{-1}(U_i)\big)\cup(aX\setminus G_i)\big] = \bigcap_{i=1}^{n}[a^{-1}(F_i)\cup E_i], \end{aligned}$$

where $E_i = aX\setminus G_i$ are closed-open in aX and $F_i = X\setminus U_i$ are closed in X. Clearly,

$$\bigcap_{i=1}^{n}[a^{-1}(F_i)\cup E_i]$$

is a union of sets of the form $a^{-1}(F)\cap E$, where F is closed in X and E is closed-open in aX. But

$$a[(a^{-1}(F)\cap E)] = F\cap a(E)$$

is closed since $a(E)$ is closed, E being closed-open in aX. Thus the set $a\big(aX\setminus(W_1\cup\ldots\cup W_n)\big)$ is closed as a finite union of closed sets.

II. Suppose that $\mu\colon \mu X\to X$ is an e.d. resolution greater than $a\colon aX\to X$. By Theorem 5, there exists a dense embedding $i\colon aX\subset\mu X$ such that $\mu\circ i = a$. Let $y\in\mu X\setminus aX$ and $x = \mu(y)$. Since $a^{-1}(x)$ is compact and, by Lemma 10, $\mu^{-1}(x)$ is Hausdorff and $a^{-1}(x)\subset\mu^{-1}(x)$, there exists a closed-open set H, $H\subset\mu X$, such that $y\in H$ and $H\cap a^{-1}(x) = \emptyset$. Clearly, $a(H\cap aX)$ is closed in X, since $H\cap aX$ is closed in aX. But $x\notin a(H\cap aX)$

and there exists an open set U, $U \subset X$, such that $x \in U$ and $U \cap a(H \cap aX)$ $= \emptyset$. Thus

$$(8) \qquad a^{-1}(U) \cap H \cap aX = \emptyset.$$

On the other hand, $\mu^{-1}(U) \cap H \neq \emptyset$ in view of $y \in H \cap \mu^{-1}(U)$. Since aX is dense in μX,

$$\mu^{-1}(U) \cap H \cap aX \neq \emptyset.$$

But $\mu^{-1}(U) \cap aX = a^{-1}(U)$. Hence $a^{-1}(U) \cap H \cap aX \neq \emptyset$, a contradiction to (8).

As a corollary we have

THEOREM 11. *If X is not e.d., then the smallest e.d. resolution does not exist.*

Proof. Suppose that X admits the smallest e.d. resolution μ: $\mu X \to X$. Hence, by Theorem 7, the greatest e.d. resolution a: $aX \to X$ is a one-to-one map. By Theorem 10, a is a closed map, hence it is a homeomorphism. Thus X is e.d.

THEOREM 12. *For $i = 1, 2, 3$, the following statements are equivalent*:

(I) *X is a T_i-space*;

(II) *aX is a T_i-space, where a is the greatest e.d. resolution*;

(III) *μX is a T_i-space for each e.d. resolution μ: $\mu X \to X$.*

Proof. 1. (I) $\Rightarrow$ (II). Let X be a T_1-space and let $x \in X$. Hence $a^{-1}(x)$ is closed in aX. By Lemma 10, for each $y \in aX$, $\{y\}$ is closed in $a^{-1}(a(y))$, and so in aX. Thus each point of aX is closed as a subset, which means that aX is a T_1-space.

If X is a Hausdorff space, then every two points y and z such that $a(y) \neq a(z)$ are separated in aX by open and disjoint sets. If $a(y) = a(z)$ and $y \neq z$, then y and z are, by Lemma 10, separated in the sense of T_2.

It is known (see Engelking [3]) that the perfect preimage of a T_3-space is a T_3-space.

2. (II) $\Rightarrow$ (III). This implication, by Theorem 5, is obvious.

3. (III) $\Rightarrow$ (I). In particular, aX is a T_1-space. We show that $a^{-1}(x)$ is closed in aX for each $x \in X$. Let y be a point of aX such that $y \notin a^{-1}(x)$. We show that $y \notin \mathrm{cl}_{aX} a^{-1}(x)$.

Case 1. There exists an open set U, $U \subset X$, such that $a(y) \in U$ and $a^{-1}(U) \cap a^{-1}(x) = \emptyset$. Since $y \in a^{-1}(U)$, $y \notin \mathrm{cl}_{aX} a^{-1}(x)$.

Case 2. For each open neighbourhood U of $a(y)$, $a^{-1}(U) \cap a^{-1}(x) \neq \emptyset$. Hence $a^{-1}(x) \subset a^{-1}(U)$ for each open neighbourhood of $a(y)$. Since aX is T_0, for each $z \in a^{-1}(x)$ there exists a closed-open neighbourhood of y which does not contain z. Suppose that each closed-open neighbourhood of y intersects $a^{-1}(x)$. Then the family

$$\mathscr{F} = \{G \cap a^{-1}(x)\colon G \text{ is a closed-open neighbourhood of } y\}$$

is a filter of closed sets in $a^{-1}(x)$. But each z from $a^{-1}(x)$ is missed by some G. Thus $\bigcap \mathscr{F} = \emptyset$, which contradicts compactness of $a^{-1}(x)$ (see Theorem 10). Hence there exists a closed-open neighbourhood G of y such that $G \cap a^{-1}(x) = \emptyset$, and so $y \notin \mathrm{cl}_{aX} a^{-1}(x)$.

Since $a^{-1}(x)$ is closed for each $x \in X$ and a is a closed map, $\{x\}$ is a closed set for each $x \in X$. Thus X is a T_1-space.

From [3] it is known that the property "to be a T_i-space" is invariant under perfect maps for $i = 2, 3$. This completes the proof.

REFERENCES

[1] A. Błaszczyk, *A factorization theorem and its application to extremally disconnected resolutions*, Colloquium Mathematicum 28 (1973), p. 33-40.

[2] R. Engelking, *Outline of general topology*, Amsterdam 1968.

[3] — *O zagadnieniach topologii ogólnej związanych z badaniem przekształceń*, Wiadomości Matematyczne 12 (1971), p. 257-284.

[4] P. Freyd, *Abelian categories*, New York, Evanston and London 1964.

[5] A. M. Gleason, *Projective topological spaces*, Illinois Journal of Mathematics 2 (1958), p. 482-489.

[6] A. W. Hager, *The projective resolution of a compact space*, Proceedings of the American Mathematical Society 28 (1971), p. 262-266.

[7] С. Илиадис, *Абсолюты хаусдорфовых пространств*, Доклады Академии наук СССР 149 (1963), p. 22-25.

[8] — и С. В. Фомин, *Метод центрированных систем в теории топологических пространств*, Успехи математических наук 21 (4) (1966), p. 47-76.

[9] W. Kulpa, *On a generalization of uniformities*, Colloquium Mathematicum 25 (1972), p. 227-240.

[10] J. Mioduszewski, *On a method which leads to extremally disconnected covers*, Proceedings of the Third Prague Topological Symposium, 1971, p. 309-311.

[11] — and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Dissertationes Mathematicae 66 (1969).

[12] J. Rainwater, *A note on projective resolutions*, Proceedings of the American Mathematical Society 10 (1959), p. 734-735.

[13] M. H. Stone, *Algebraic characterization of special Boolean rings*, Fundamenta Mathematicae 29 (1937), p. 223-302.

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