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DIMENSION OF NON-ARCHIMEDEAN BANACH SPACES

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A simple category argument shows that there are no countably infinite dimensional complex Banach spaces. If the continuum hypothesis is taken as one of the set-theoretic axioms, then this proves that every infinite dimensional Banach space has dimension greater than or equal to the cardinality of the continuum. An alternate proof, in which the continuum hypothesis is not assumed, may be given to show that every infinite dimensional complex Banach space has dimension at least the cardinality of the continuum [2]. The purpose of this note is to present analogues of these results for non-Archimedean Banach spaces.

A *non-Archimedean Banach space* is a vector space X over a field with rank one valuation $| \ |$, together with a real-valued function $\| \ \|$ on X satisfying

1. $\|x\| \geqslant 0$, and $\|x\| = 0$ if and only if $x = 0$;
2. $\|ax\| = |a| \, \|x\|$;
3. $\|x+y\| \leqslant \max(\|x\|, \|y\|)$; and
4. X is complete with respect to the topology generated by the metric $\|x-y\|$;

where a is an arbitrary scalar, and x and y are arbitrary vectors.

The cardinality of a set A will be denoted by $\# A$; the smallest infinite cardinal will be denoted by $\aleph_0$; and the cardinality of the continuum will be denoted by c.

THEOREM 1. *If X is an infinite dimensional non-Archimedean Banach space over a complete non-trivially valued field, then* $\dim X > \aleph_0$.

Proof. If $\{x_i\}$, $i = 1, 2, \ldots$, is a countable basis for X and $M_n = [x_1, \ldots, x_n]$, then

$$X = \bigcup_{1}^{\infty} M_n.$$

Since each M_n is finite dimensional, and hence closed (see [1]), $\overline{M_n^0} = M_n^0$ (i.e. the interior of M_n). This implies each M_n is nowhere dense: if $\overline{M_n^0} = M_n^0 \neq \emptyset$, then M_n contains a sphere and must be all of X, which is impossible, because X is infinite dimensional. Hence X has been written as a countable union of nowhere dense sets, contradicting the Baire category theorem. Therefore, $\{x_n\}$ could not have been a basis.

LEMMA 2.1. *If a field F is complete with respect to a discrete valuation, then* $\# F = (\# \mathfrak{o}/\mathfrak{p})^{\aleph_0}$, *where* $\mathfrak{o}/\mathfrak{p}$ *is the residue class field of* F.

Consequently, any non-trivially valued locally compact field has the cardinality of the continuum.

Proof. Each element of F may be written in the form $\sum_{-N}^{\infty} a_i \pi^i$, where π is a prime element of $\mathfrak{o}$ and the a_i are unique modulo $\mathfrak{p}$. (The consequence follows because a non-trivially valued locally compact field is complete and discrete and has a finite residue class field.)

LEMMA 2.2. *Any complete non-trivially valued field has at least the cardinality of the continuum.*

Proof. Let Π be the prime field contained in F, where $(F, |\ |)$ is as in the hypothesis, and let a be any element in F such that $|a| \neq 0$ or 1. If the valuation of F restricted to Π is non-trivial, then (to within isomorphism) Π is the field Q of rationals and has a p-adic valuation. Then $F \supset \bar{Q} = Q_p$, where Q_p is the locally compact field of p-adic numbers, so that, by Lemma 2.1, $\# F \geqslant \# Q_p = c$.

If $|\ |$ is trivial on Π, then a must be transcendental over Π, since $|\ |$ is not trivial on $\Pi(a)$. But each of the non-trivial valuations on $\Pi(a)$ which is trivial on Π is isomorphic to either a $p(a)$-adic valuation or the "degree" valuation, and hence is discrete. Therefore, letting $K = \overline{\Pi(a)}$ and k equal the residue class field of K, $\# F \geqslant \# K = (\# k)^{\aleph_0} \geqslant c$.

LEMMA 2.3. *Let the field F be complete with respect to a non-trivial valuation, and let l_∞ be the Banach space of all bounded sequences with pointwise operations and* sup *norm. Then* $\dim l_\infty = \# F$.

Proof. The collection of sequences $\{\{a^n\} \mid a \in \mathfrak{o} - \{0\}\}$, where $\mathfrak{o}$ is the ring of integers of F, is linearly independent. Therefore, $\dim l_\infty \geqslant \# \mathfrak{o} = \# F$. On the other hand, $\dim l_\infty \leqslant \# l_\infty = (\# F)^{\aleph_0} = \# F$ (since $\# F \geqslant c$).

A set B of non-zero vectors from a non-Archimedean Banach space is *orthogonal* if

$$\Big\|\sum_i a_i x_i\Big\| = \max_i \|a_i x_i\|$$

for every linear combination with vectors $\{x_i\}$ from B. An orthogonal set B is an *orthogonal base* for the space X if X is the closed subspace generated by B.

A valued field F is said to be *spherically complete* if every collection of closed spheres (i.e., sets of the form $\{a \mid |a - a_0| \leqslant \varepsilon\}$, $\varepsilon \geqslant 0$) which is totally ordered by set inclusion has a non-empty intersection. Monna [3] has shown that a spherically complete field is complete and that a field complete with respect to a discrete valuation is spherically complete.

THEOREM 2. *Let X be a non-Archimedean Banach space over a complete non-trivially valued field F. If X contains an infinite orthogonal set, the dimension of X is at least $\# F$, and hence at least c.*

Proof. Let $\{x_1, x_2, \ldots\}$ be a sequence of vectors from an orthogonal set in X. By dividing the x_n by appropriately large scalars a_n if necessary, it can be assumed that $\|x_n\|$ converges to zero. Then the function

$$\Psi: l_\infty \to X, \qquad \{a_n\} \to \sum a_n x_n,$$

is a well-defined 1-1 linear transformation, so that

$$\dim X \geqslant \dim l_\infty = \# F \geqslant c.$$

(Lemma 2.3 could, of course, be bypassed by observing directly that $\{\sum a^n x_n \mid a \in \mathfrak{o} - \{0\}\}$ is a linearly independent set.)

COROLLARY 2.1. *An infinite dimensional non-Archimedean Banach space over a spherically complete non-trivially valued field F has at least dimension $\# F$, and hence at least dimension c.*

Proof. An infinite dimensional non-Archimedean Banach space over a spherically complete field has an infinite orthogonal set [4].

It is interesting to note that in the classical case Theorem 1 subsumes Theorem 2, and Theorems 1 and 2 become equivalent if one assumes the continuum hypothesis. We see that neither of these relations hold in the non-Archimedean case.

REFERENCES

[1] I. S. Cohen, *On non-Archimedean normed spaces*, Indagationes Mathematicae 10 (1948), p. 693-698.

[2] C. Goffman and C. Pedrick, *First course in functional analysis*, Englewood 1965.

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[4] — and T. A. Springer, *Sur la structure des espaces de Banach non-Archimédiens*, ibidem 27 (1965), p. 602-614.

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