COLLOQUIUM MATHEMATICUM

VOL. XXI 1970 FASC. 1

TRIGONOMETRIC INTERPOLATION, II

BY

R. TABERSKI (POZNAŃ)

1. Introduction. Denote by $\omega_m(s)$ the step function equal to $2\pi k/(2m+1)$ for $s \in \langle s_{k-1}, s_k)$ and by $\omega_n(t)$ the step function equal to $2\pi l/(2n+1)$ for $t \in \langle t_{l-1}, t_l)$, where

$$s_k = s_k^{(m)} = \frac{2\pi k}{2m+1}, \quad t_l = t_l^{(n)} = \frac{2\pi l}{2n+1}$$

($k, l = 0, \pm 1, \pm 2, \ldots$; m and n mean non-negative integers).

The numbers s_k and t_l are called the *fundamental points* of interpolation.

Given a rectangle $R = [a, b; c, d]$, let

$$s_{\alpha-1}^{(m)} < a \leqslant s_\alpha^{(m)} < s_{\alpha+1}^{(m)} < \ldots < s_\beta^{(m)} < b \leqslant s_{\beta+1}^{(m)},$$

$$t_{\gamma-1}^{(n)} < c \leqslant t_\gamma^{(n)} < t_{\gamma+1}^{(n)} < \ldots < t_\delta^{(n)} < d \leqslant t_{\delta+1}^{(n)}.$$

Write

$$\int_a^b \int_c^d \varphi(s, t)\, d\omega_m(s)\, d\omega_n(t) = \frac{4\pi^2}{(2m+1)(2n+1)} \sum_{k=\alpha}^{\beta} \sum_{l=\gamma}^{\delta} \varphi(s_k^{(m)}, t_l^{(n)})$$

for any function $\varphi(s, t)$ defined in R. Retain the symbol $\int_a^b \varphi(s)\, d\omega_m(s)$ used in [4].

The main theorems of this paper, presented in section 5, concern convergence of trigonometric polynomials

$$I_{\mu,\nu}^{m,n}(x, y; f) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t)$$

($0 \leqslant \mu \leqslant m$, $0 \leqslant \nu \leqslant n$) connected with a function $f(s, t)$ which is Riemann-integrable over the square $Q = [-\pi, \pi; -\pi, \pi]$; here, as in [4],

$$D_\nu(z) = \frac{1}{2} + \sum_{r=1}^{\nu} \cos rz = \frac{\sin(\nu+\frac{1}{2})z}{2 \sin \frac{1}{2} z}.$$

In sections 2, 3, and 4 we introduce suitable notions and give some auxiliary results.

2. Inequality of Young's type. Let

$$A = (a_{i,k}),\ B = (b_{i,k}) \quad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$$

be two matrices of real or complex numbers, and let

$$A' = (a'_{i,l}),\ B' = (b'_{i,l}) \quad (i = 1, 2, \ldots, m;\ 1 = 1, 2, \ldots, \nu)$$

$$[A^- = (a^-_{j,k}),\ B^- = (b^-_{j,k}) \quad (j = 1, 2, \ldots, \mu;\ k = 1, 2, \ldots, n)]$$

denote the matrices in which every column [row] is the sum of a number ($\geqslant 1$) of different neighbouring columns [rows] in A or B, respectively. Write, for $p, q > 0$,

$$S'_{p,q}(A, B) = \max_{A'B'} \Big[\sum_{l=1}^{\nu} \Big(\sum_{i=1}^{m} |a'_{i,l}|\Big)^p\Big]^{1/p} \Big[\sum_{l=1}^{\nu} \Big(\max_{1\leqslant k\leqslant m} \Big|\sum_{i=1}^{k} b'_{i,l}\Big|\Big)^q\Big]^{1/q},$$

$$S^-_{p,q}(A, B) = \max_{A^-,B^-} \Big[\sum_{j=1}^{\mu} \Big(\sum_{k=1}^{n} |a^-_{j,k}|\Big)^p\Big]^{1/p} \Big[\sum_{j=1}^{\mu} \Big(\max_{1\leqslant l\leqslant n} \Big|\sum_{k=1}^{l} b^-_{j,k}\Big|\Big)^q\Big]^{1/q},$$

$$S_{p,q}(A, B) = \min\{S'_{p,q}(A, B),\ S^-_{p,q}(A, B)\}.$$

We shall prove an analogue of inequality (5.1), announced in [5].

2.1. THEOREM. *Given positive p and q such that $1/p+1/q > 1$, we have*

$$(1) \qquad \Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(a_{i,j}\sum_{r=1}^{i}\sum_{s=1}^{j} b_{r,s}\Big)\Big| \leqslant C_{p,q}S_{p,q}(A, B),$$

where

$$C_{p,q} = 1+\zeta\left(\frac{1}{p}+\frac{1}{q}\right) = 1+\sum_{k=1}^{\infty} k^{-(1/p+1/q)}.$$

Proof. Let, for $i = 1, 2, \ldots, m$,

$$a'_{i,l} = \begin{cases} a_{i,l} & \text{if } l < k \leqslant n-1, \\ a_{i,k}+a_{i,k+1} & \text{if } l = k \leqslant n-1, \\ a_{i,l+1} & \text{if } k < l \leqslant n-1, \end{cases}$$

$$b'_{i,l} = \begin{cases} b_{i,l} & \text{if } l < k \leqslant n-1, \\ b_{i,k}+b_{i,k+1} & \text{if } l = k \leqslant n-1, \\ b_{i,l+1} & \text{if } k < l \leqslant n-1. \end{cases}$$

Then an easy calculation shows that

$$\sum_{i=1}^{m}\sum_{l=1}^{n-1}\Big(a'_{i,l}\sum_{r=1}^{i}\sum_{s=1}^{l} b'_{r,s}\Big) = \sum_{i=1}^{m}\Big(a_{i,k}\sum_{r=1}^{i} b_{r,k+1}\Big)+\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(a_{i,j}\sum_{r=1}^{i}\sum_{s=1}^{j} b_{r,s}\Big).$$

Putting

$$Z_k = \sum_{i=1}^{m} \Big(a_{i,k} \sum_{r=1}^{i} b_{r,k+1}\Big), \qquad M_s = \max_{1\leqslant i\leqslant m} \Big|\sum_{r=1}^{i} b_{r,s}\Big|,$$

we have

$$|Z_k| \leqslant \left[\frac{1}{n-1}\sum_{l=1}^{n-1}\Big(\sum_{i=1}^{m}|a_{i,l}|\Big)^p\right]^{1/p}\left[\frac{1}{n-1}\sum_{l=1}^{n-1}|M_{l+1}|^q\right]^{1/q}$$

for a certain $k \leqslant n-1$ (see [5], § 2). Consequently, with this k,

$$\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(a_{i,j}\sum_{r=1}^{i}\sum_{s=1}^{j} b_{r,s}\Big)\Big|$$

$$\leqslant \Big|\sum_{i=1}^{m}\sum_{l=1}^{n-1}\Big(a'_{i,l}\sum_{r=1}^{i}\sum_{s=1}^{l} b'_{r,s}\Big)\Big| + \frac{1}{(n-1)^{1/p+1/q}}\Big[\sum_{l=1}^{n}\Big(\sum_{i=1}^{m}|a_{i,l}|\Big)^p\Big]^{1/p} \times$$

$$\times\Big[\sum_{l=1}^{n}\Big(\max_{1\leqslant i\leqslant m}\Big|\sum_{r=1}^{i} b_{r,l}\Big|\Big)^q\Big]^{1/q}.$$

The first sum on the right can be similarly estimated, and so on. Hence inequality (1) in which $S_{p,q}(A,B)$ is replaced by $S'_{p,q}(A,B)$ follows.

Starting with $a^-_{j,k}, b^-_{j,k}$, we get (1) with $S^-_{p,q}(A,B)$ instead of $S_{p,q}(A,B)$. Thus, the proof is completed.

Note that the right-hand side of (1) does not exceed

$$C_{p,q}\min\Big\{\max_{A'}\Big[\sum_{l=1}^{\nu}\Big(\sum_{i=1}^{m}|a'_{i,l}|\Big)^p\Big]^{1/p}, \max_{A^-}\Big[\sum_{j=1}^{\mu}\Big(\sum_{k=1}^{n}|a^-_{j,k}|\Big)^p\Big]^{1/p}\Big\}\times$$

$$\times\max\Big\{\max_{B'}\Big[\sum_{l=1}^{\nu}\Big(\max_{1\leqslant k\leqslant m}\Big|\sum_{i=1}^{k} b'_{i,l}\Big|\Big)^q\Big]^{1/q}, \max_{B^-}\Big[\sum_{j=1}^{\mu}\Big(\max_{1\leqslant l\leqslant n}\Big|\sum_{j=1}^{l} b^-_{j,k}\Big|\Big)^q\Big]^{1/q}\Big\}.$$

3. Variations and pseudovariation. Let $\Pi^{(1)}$ and $\Pi^{(2)}$ be partitions of the intervals $I_1 = \langle a, b\rangle$ and $I_2 = \langle c, d\rangle$, generated by the points

$$a = x_1 < x_2 < \ldots < x_i < \ldots < x_m < x_{m+1} = b,$$

$$c = y_1 < y_2 < \ldots < y_j < \ldots < y_n < y_{n+1} = d,$$

respectively. Denote by Π the partition of the rectangle $R = [a, b; c, d]$, formed by the lines parallel to the axes and passing through all division points of $\Pi^{(1)}$ and $\Pi^{(2)}$.

Suppose that the function $f(s,t)$ is defined in R, and set

$$\Delta f(x_i, y_j) = f(x_i, y_j) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1}).$$

Write, for $p > 0$,

$$W'_p(f; R) = \sup_{\Pi} \{ \sum_{j=1}^{n} [\sum_{i=1}^{m} |\Delta f(x_i, y_j)|]^p \}^{1/p},$$

$$W''_p(f; R) = \sup_{\Pi} \{ \sum_{i=1}^{m} [\sum_{j=1}^{n} |\Delta f(x_i, y_j)|]^p \}^{1/p},$$

and

$$W^*_p(f; R) = \min\{W'_p(f; R), W''_p(f; R)\},$$

$$W_p(f; R) = \max\{W'_p(f; R), W''_p(f; R)\}.$$

The last quantity is called the *p-th pseudovariation* of f over the rectangle R.

The p-th partial variations of $f(s, t)$ with respect to the successive variables are defined by the following formulae:

$$V_p\big(f(\cdot, t); I_1\big) = \sup_{\Pi^{(1)}} \{ \sum_{i=1}^{m} |f(x_{i+1}, t) - f(x_i, t)|^p \}^{1/p} \qquad (t \in I_2),$$

$$V_p\big(f(s, \cdot); I_2\big) = \sup_{\Pi^{(2)}} \{ \sum_{j=1}^{n} |f(s, y_{j+1}) - f(s, y_j)|^p \}^{1/p} \qquad (s \in I_1).$$

We define the p-th variation $V_p(f; R)$ of f over the rectangle R as in [3], § 1.

An argument similar to that of [1], p. 38-39, leads to

3.1. LEMMA. *Suppose that*

$$W_p(f; R) < \infty \tag{2}$$

for a fixed $p > 0$. Then, given any point (x, y) in the interior of R and any positive ε, there are two positive numbers σ_1 and σ_2 such that, for each positive $\delta_1 < \sigma_1$ and $\delta_2 < \sigma_2$, we have

$$W'_p(f; R^{\pm}_{\delta_2}) < \varepsilon \quad \textit{and} \quad W''_p(f; \tilde{R}^{\pm}_{\delta_1}) < \varepsilon,$$

where

$$R^+_\delta = [x-\sigma_1, x+\sigma_1; y+\delta, y+\sigma_2], \qquad R^-_\delta = [x-\sigma_1, x+\sigma_1; y-\sigma_2, y-\delta],$$

$$\tilde{R}^+_\delta = [x+\delta, x+\sigma_1; y-\sigma_2, y+\sigma_2], \qquad \tilde{R}^-_\delta = [x-\sigma_1, x-\delta; y-\sigma_2, y+\sigma_2].$$

Now we shall present three theorems.

3.2. THEOREM. *Let (x, y) be an interior point of the rectangle R, and let condition* (2) *holds for a certain $p \geq 1$. Write*

$$Q_\sigma = Q_\sigma(x, y) = [x-\sigma, x+\sigma; y-\sigma, y+\sigma], \qquad \sigma > 0.$$

(i) *If*

(3) $$\lim_{t\to y} f(s,t) = f(s,y)$$

for any $s \in I_1$, *then*

$$\lim_{\sigma\to 0+} W_p'(f; Q_\sigma) = 0.$$

(ii) *If*

(4) $$\lim_{s\to x} f(s,t) = f(x,t)$$

for any $t \in I_2$, *then*

$$\lim_{\sigma\to 0+} W_p''(f; Q_\sigma) = 0.$$

Proof of (i). According to 3.1 choose, for an arbitrary $\varepsilon > 0$, a positive σ such that

$$W_p'(f; \hat{R}_\delta^+) < \varepsilon \quad \text{and} \quad W_p'(f; \hat{R}_\delta^-) < \varepsilon$$

if $0 < \delta < \sigma$, where

$$\hat{R}_\delta^+ = [x-\sigma, x+\sigma; y+\delta, y+\sigma] \quad \text{and} \quad \hat{R}_\delta^- = [x-\sigma, x+\sigma; y-\sigma, y-\delta).$$

Write

$$Q_\sigma^+ = [x-\sigma, x+\sigma; y, y+\sigma] \quad \text{and} \quad Q_\sigma^- = [x-\sigma, x+\sigma; y-\sigma, y].$$

Evidently, there is a partition of the rectangle Q_σ^+, connected with the division points

$$x-\sigma = x_1 < x_2 < \ldots < x_i < \ldots < x_m < x_{m+1} = x+\sigma,$$

$$y = y_1 < y_2 < \ldots < y_j < \ldots < y_n < y_{n+1} = y+\sigma$$

on the s- and t-axes, for which

$$W_p'(f; Q_\sigma^+) \leqslant \{\sum_{j=1}^{n} [\sum_{i=1}^{m} |\Delta f(x_i, y_j)|]^p\}^{1/p} + \varepsilon.$$

By the continuity of f, we can find a number y_1' $(y_1 < y_1' < y_2)$ such that

$$|f(x_i, y_1) - f(x_i, y_1')| < \frac{\varepsilon}{m+1} \quad \text{for } i = 1, 2, \ldots, m+1.$$

Then if $y_2' = y_2$, $y_3' = y_3, \ldots, y_{n+1}' = y_{n+1}$ and $\delta = y_1' - y_1$, we have

$$\{\sum_{j=1}^{n} [\sum_{i=1}^{m} |\Delta f(x_i, y_j)|]^p\}^{1/p}$$

$$\leqslant \sum_{i=1}^{m} |f(x_i, y_1) - f(x_{i+1}, y_1) - f(x_i, y_1') + f(x_{i+1}, y_1')| + \{\sum_{j=1}^{n} [\sum_{i=1}^{m} |\Delta f(x_i, y_j')|]^p\}^{1/p}$$

$$\leqslant 2\varepsilon + W_p'(f; \hat{R}_\delta^+) < 3\varepsilon.$$

Consequently, $W'_p(f; Q_\sigma^+) < 4\varepsilon$ for sufficiently small σ, and, by symmetry, the symbol Q_σ^+ can be replaced by Q_σ^-. Hence

$$W'_p(f; Q_\sigma) \leqslant W'_p(f; Q_\sigma^+) + W'_p(f; Q_\sigma^-) < 8\varepsilon,$$

which completes the proof.

3.3. THEOREM. *Let* (2) *be fulfilled with a* $p \geqslant 1$ *and let* $R_0 = [a_0, b_0; c_0, d_0]$ *be a rectangle interior to* R. *Suppose that the function* $f(s, t)$ *is continuous in each variable, separately, at every point of* R_0, *being continuous in* s [*resp.* t] *at* $s = a_0$, $s = b_0$ *for every* $t \in I_2$ [*at* $t = c_0$, $t = d_0$ *for every* $s \in I_1$]. *Then,*

$$\lim_{\sigma\to 0+} W_p^*\big(f; Q_\sigma(x, y)\big) = 0$$

uniformly in $(x, y) \in R_0$.

Proof. Supposing the contrary, we could find an $\varepsilon > 0$ and a sequence of squares $q_n = Q_{1/n}(x_n, y_n)$ with $(x_n, y_n) \in R_0$ such that

$$\min\{W'_p(f; q_n),\ W''_p(f; q_n)\} \geqslant \varepsilon.$$

Let (ξ, η) be an accumulation point of (x_n, y_n). Then

$$W'_p\big(f; Q_\delta(\xi, \eta)\big) \geqslant \varepsilon \quad \text{and} \quad W''_p\big(f; Q_\delta(\xi, \eta)\big) \geqslant \varepsilon$$

for every $\delta > 0$. On the other hand, by 3.2,

$$W'_p\big(f; Q_\delta(\xi, \eta)\big) < \varepsilon \quad \text{or} \quad W''_p\big(f; Q_\delta(\xi, \eta)\big) < \varepsilon$$

whenever δ is small enough.

Thus, the proof is completed (cf. [1], p. 39-40).

3.4. THEOREM. *Suppose that*

$$\sup_{t\in I_2} V_{p_1}\big(f(\cdot, t); I_1\big) < \infty \quad \textit{and} \quad \sup_{s\in I_1} V_{p_2}\big(f(s, \cdot); I_2\big) < \infty \tag{5}$$

for some positive p_1 *and* p_2. *Choose a pair of numbers* $p'_1 > p_1$ *and* $p'_2 > p_2$; *set* $I_\sigma(u) = \langle u-\sigma, u+\sigma\rangle$ *and retain the symbol* R_0 *used in* 3.3. *Then*

(i) *if* $f(s, t)$ *is continuous at an interior point* (x, y) *of the rectangle* R, *then*

$$\lim_{\substack{\sigma\to 0+\\ k\to 0}} V_p\big(f(\cdot, y+k); I_\sigma(x)\big) = 0 = \lim_{\substack{\sigma\to 0+\\ h\to 0}} V_p\big(f(x+h, \cdot); I_\sigma(y)\big); \tag{6}$$

(ii) *if*

$$\lim_{h,k\to 0} f(x+h,\ y+k) = f(x, y) \tag{7}$$

uniformly in $(x, y) \in R_0$, *then relation* (6) *holds uniformly in* $(x, y) \in R_0$, *too.*

Proof of (i). By (8.2a) of [5],

$$V_{p_1'}\big(f(\cdot, y+k); I_\sigma(x)\big) \leqslant \{V_{p_1}\big(f(\cdot, y+k); I_\sigma(x)\big)\}^{p_1/p_1'}\{\operatorname*{Osc}_{x-\sigma\leqslant s\leqslant x+\sigma} f(s, y+k)\}^{(p_1'-p_1)/p_1'}.$$

Observing that

$$V_{p_1}\big(f(\cdot, y+k); I_\sigma(x)\big) \leqslant \sup_{t\in I_2} V_{p_1}\big(f(\cdot, t); I_1\big) < \infty$$

and

$$\lim_{\substack{\sigma\to 0+\\ k\to 0}} \operatorname*{Osc}_{x-\sigma\leqslant s\leqslant x+\sigma} f(s, y+k) = 0,$$

we get the first part of (i).

In case of (ii) the last relation holds uniformly in $(x, y)\in R_0$; whence the desired result follows.

4. Analogues of the Riemann-Lebesgue theorem. In the sequel it will be assumed that $0 \leqslant \mu \leqslant m$ and $0 \leqslant \nu \leqslant n$.

By the analysis of the proof of Lemma 2.3 in [4], we obtain

4.1. THEOREM. *Consider a function $f(s, t)$ defined and bounded in the rectangle $[-\pi, \pi; -\pi, \pi]$. Suppose that, for any positive λ, there is a partition*

$$-\pi = x_1 < x_2 < \ldots < x_r < x_{r+1} = \pi$$

such that

$$\max_{1\leqslant k\leqslant r}(x_{k+1}-x_k) + \sum_{k=1}^{r}(x_{k+1}-x_k) \operatorname*{Osc}_{x_k\leqslant s\leqslant x_{k+1}} f(s, t) < \lambda$$

for all $t\in \langle c, d\rangle$. Then, if $0 < \delta < \pi$, we have

$$\lim_{\mu\to\infty} \int_{-\pi}^{x-\delta} f(s, t) D_\mu(s-x)\, d\omega_m(s) = 0 = \lim_{\mu\to\infty} \int_{x+\delta}^{\pi} f(s, t) D_\mu(s-x)\, d\omega_m(s)$$

uniformly in $(x, t)\in[-\pi+\delta, \pi-\delta; c, d]$.

A similar calculation leads to

4.2. THEOREM. *Let $f(s, t)$ be Riemann integrable over the square $[-\pi, \pi; -\pi, \pi]$, and let $0 < \delta, \eta < \pi$. Then*

$$\lim_{\mu,\nu\to\infty} \int_{-\pi}^{x-\delta} \int_{y+\eta}^{\pi} f(s, t) D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t) = 0$$

uniformly in $(x, y)\in[-\pi+\delta, \pi-\delta; -\pi+\eta, \pi-\eta]$. The assertion remains valid for the integrals

$$\int_{x+\delta}^{\pi}\int_{-\pi}^{y-\eta}, \quad \int_{x+\delta}^{\pi}\int_{y+\eta}^{\pi}, \quad \int_{-\pi}^{x-\delta}\int_{-\pi}^{y-\eta}.$$

Now, we shall give the following

4.3. LEMMA. *Suppose that the functions*

$$F_1(s),\ F_2(s),\ \ldots,\ F_n(s),\ \ldots$$

are Riemann integrable and uniformly bounded over an interval $\langle a, b\rangle$. *Then if*

$$\lim_{n\to\infty} F_n(s) = 0 \quad \textit{for } s\in\langle a, b\rangle,$$

we have

$$\lim_{n\to\infty}\Big\{\lim_{m\to\infty}\int_a^b F_n(s)\,d\omega_m(s)\Big\} = 0.$$

Proof. By the Riemann integrability of $F_n(s)$,

$$\lim_{m\to\infty}\int_a^b F_n(s)\,d\omega_m(s) = \int_a^b F_n(s)\,ds \qquad (n = 1, 2, \ldots).$$

On the other hand,

$$\lim_{n\to\infty}\int_a^b F_n(s)\,ds = 0.$$

Applying this Lemma we shall prove two further results.

4.4. THEOREM. *Let the assumptions of* 4.2 *be satisfied with* $\delta\leqslant\eta$, *and let* $R = [a, b; c, d]$ *be a rectangle such that* $-\pi+\eta\leqslant a<b\leqslant\pi-\eta$ *and* $-\pi+\eta\leqslant c<d\leqslant\pi-\eta$. *Write, as previously,* $I_1 = \langle a, b\rangle$, $I_2 = \langle c, d\rangle$, *and* $I_\sigma(z) = \langle z-\sigma, z+\sigma\rangle$.

(i) *Given* $(x, y)\in R$, *let* $f(s, t)$ *be Riemann integrable in* s [*resp. in* t] *over the interval* $\langle -\pi, \pi\rangle$ *for* $t = y$ [$s = x$]. *Suppose that, for some positive* p_1 *and* p_2,

$$V_{p_1}\big(f(\cdot, t);\ I_\sigma(x)\big)\leqslant \Phi_x(t;\sigma) \quad \textit{and} \quad V_{p_2}\big(f(s, \cdot);\ I_\sigma(y)\big)\leqslant \Psi_y(s;\sigma) \tag{8}$$

when $-\pi\leqslant t,\ s\leqslant\pi$ *and* $0<\sigma\leqslant\eta$, *where the functions* $\Phi_x(t;\sigma)$ *and* $\Psi_y(s;\sigma)$ *are uniformly bounded, Riemann integrable in* t *and* s *over* $\langle -\pi, \pi\rangle$ *and*

$$\lim_{\sigma\to0+}\Phi_x(t;\sigma) = 0 = \lim_{\sigma\to0+}\Psi_y(s;\sigma).$$

Then

$$\lim_{\mu,\nu\to\infty}\int_{-\pi}^{x+\delta}\int_{y-\delta}^{y+\delta} f(s, t)D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t) = 0, \tag{9}$$

and the last integral can be replaced, successively, by

$$\int_{x+\delta}^{\pi}\int_{y-\delta}^{y-\delta},\quad \int_{-\pi}^{y-\delta}\int_{x-\delta}^{x+\delta},\quad \int_{y+\delta}^{\pi}\int_{x-\delta}^{x+\delta}. \tag{10}$$

(ii) *Suppose that there exist, for a given $\lambda > 0$, two partitions*

$$-\pi = x_1 < x_2 < \ldots < x_{r+1} = \pi, \qquad -\pi = y_1 < y_2 < \ldots < y_{\varrho+1} = \pi$$

such that

$$\max_{1\leqslant k\leqslant r}(x_{k+1}-x_k)+\sum_{k=1}^{r}(x_{k+1}-x_k)\operatorname*{Osc}_{x_k\leqslant s\leqslant x_{k+1}} f(s,t) < \lambda \quad \textit{if } t\in I_2, \tag{11}$$

$$\max_{1\leqslant l\leqslant \varrho}(y_{l+1}-y_l)+\sum_{l=1}^{\varrho}(y_{l+1}-y_l)\operatorname*{Osc}_{y_l\leqslant t\leqslant y_{l+1}} f(s,t) < \lambda \quad \textit{if } s\in I_1. \tag{12}$$

Suppose that the conditions of (i) *are fulfilled with*

$$\Phi(t;\sigma) = \Phi_x(t;\sigma), \qquad \Psi(s;\sigma) = \Psi_y(s;\sigma)$$

independent of $(x,y)\in R$. *Then the convergence of integrals* (9) *and* (10) *is uniform in* R.

Proof of (i). Confine ourselves to the integral

$$J = \int_{x+\delta}^{\pi}\int_{y}^{y+\delta} f(s,t)D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t).$$

Given a positive ε, there is a positive $\tau < \delta$ such that

$$\int_{-\pi}^{\pi}\Psi_y(s;\tau)\,d\omega_m(s) < \varepsilon$$

if m is large enough, by 4.3. Write

$$J = \Big(\int_{x+\delta}^{\pi}\int_{y}^{y+\tau} + \int_{x+\delta}^{\pi}\int_{y+\tau}^{y+\delta}\Big)\{f(s,t)-f(s,y)\}D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t)+$$

$$+\int_{x+\delta}^{\pi}\int_{y}^{y+\delta} f(s,y)D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t) = (J_1+J_2)+J_3.$$

Let $t_k^{(n)}, t_{k+1}^{(n)}, \ldots, t_q^{(n)}$ be all fundamental points belonging to the interval $\langle y, y+\tau)$. Then,

$$|J_1| \leqslant \frac{1}{2\sin(\delta/2)}\int_{x+\delta}^{\pi}\frac{2\pi}{2n+1}\Big|\sum_{j=k}^{q}\{f(s,t_j^{(n)})-f(s,y)\}D_\nu(t_j^{(n)}-y)\Big|\,d\omega_m(s),$$

and, by the Abel transformation,

$$|J_1| \leqslant \frac{1}{2\sin(\delta/2)}\int_{x+\delta}^{\pi}\frac{2\pi}{2n+1}\Big|\sum_{j=k}^{q-1}\sum_{l=k}^{j}\{f(s,t_j^{(n)})-f(s,t_{j+1}^{(n)})\}D_\nu(t_l^{(n)}-y)+$$

$$+\{f(s,t_q^{(n)})-f(s,y)\}\sum_{l=k}^{q}D_\nu(t_l^{(n)}-y)\Big|\,d\omega_m(s).$$

Inequality (5.1) of [5] and Lemma 2.1 of [4] lead to

$$|J_1| \leqslant \frac{C_1}{2\sin(\delta/2)} \int_{x+\delta}^{\pi} \{V_{p_2}(f(s,\cdot);\langle y, y+\tau\rangle) + |f(s,t_q^{(n)}) - f(s,y)|\} d\omega_m(s)$$

$$\leqslant \frac{\pi C_1}{2\delta} \cdot 2 \int_{-\pi}^{\pi} \Psi_y(s;\tau) d\omega_m(s),$$

where C_1 is a constant. Consequently, $|J_1| < \pi C_1 \varepsilon/\delta$ for large m and n.

In view of 4.2, $|J_2| < \varepsilon$ if μ and ν are large enough. Further,

$$|J_3| \leqslant \Big|\int_{x+\delta}^{\pi} f(s,y) D_\mu(s-x) d\omega_m(s)\Big| \Big|\int_{y}^{y+\delta} D_\nu(t-y) d\omega_n(t)\Big|.$$

Applying 2.1 and 2.3 of [4], we conclude that $|J_3| < \varepsilon$, provided μ is sufficiently large. Thus the proof is completed.

In case (ii) we observe that

$$|f(s,t_q^{(n)}) - f(s,y)| \leqslant V_{p_2}(f(s,\cdot);\, I_\tau(y)) \leqslant \Psi(s;\tau),$$

and we apply 4.1 instead of 2.3 of [4].

4.5. THEOREM. *Retain the initial assumptions of* 4.4; *take* $(x,y) \in R$.

(i) *Suppose that*

$$|f(x+u,t) - f(x,t)| \leqslant \varphi_x(t;|u|), \qquad |f(s,y+v) - f(s,y)| \leqslant \psi_y(s;|v|),$$

if $-\eta \leqslant u, v \leqslant \eta$ *and* $-\pi \leqslant t, s \leqslant \pi$, *where* $\varphi_x(t;u)$ *and* $\psi_y(s;v)$ *are non-decreasing in* $u, v \geqslant 0$, *Riemann integrable in* t *and* s *over* $\langle -\pi, \pi\rangle$ *and such that the functions of* t *and of* s *defined by the Lebesgue integrals*

$$\int_0^\sigma \frac{\varphi_x(t;u)}{u} du, \quad \int_0^\sigma \frac{\psi_y(s;v)}{v} dv \qquad (0 < \sigma \leqslant \eta)$$

together with $f(x,t)$ *and* $f(s,y)$ *are also Riemann integrable over* $\langle -\pi, \pi\rangle$. *Then the assertion of* 4.4 (i) *remains valid.*

(ii) *If, moreover, the majorants*

$$\varphi(t;u) = \varphi_x(t;u), \qquad \psi(s;v) = \psi_y(s;v) \qquad (0 \leqslant u, v \leqslant \eta)$$

are independent of x, y, *then, under the integrability conditions* (11), (12), *the conclusion of* 4.4 (ii) *holds.*

Proof. Let (x,y) be fixed. Consider only the integral

$$J = \int_{x+\delta}^{\pi} \int_{y}^{y+\delta} f(s,t) D_\mu(s-x) D_\nu(t-y) d\omega_m(s) d\omega_n(t).$$

By 4.3, for any positive ε there is a positive $\delta_1 < \delta/2$ such that

$$\int_{-\pi}^{\pi} \left\{\psi_y(s;\delta_1) + \int_0^{2\delta_1} \frac{\psi_y(s;v)}{v} dv\right\} d\omega_m(s) < \varepsilon$$

if m is large enough. Write

$$J = \Big(\int_{x+\delta}^{\pi} \int_{y}^{y+\delta_1} + \int_{x+\delta}^{\pi} \int_{y+\delta_1}^{y+\delta} \Big) \{f(s,t)-f(s,y)\} D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t) +$$

$$+ \int_{x+\delta}^{\pi} \int_{y}^{y+\delta} f(s,y) D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t) = (J'+J'')+J'''.$$

Let $t_k^{(n)}, t_{k+1}^{(n)}, \ldots, t_{k+l}^{(n)}$ be the fundamental points belonging to the interval $\langle y, y+\delta_1)$. Then

$$|J'| \leqslant \pi \int_{x+\delta}^{\pi} \left\{ |f(s, t_k^{(n)})-f(s,y)| + \frac{\pi}{2n+1} \sum_{j=k+1}^{k+l} \frac{|f(s,t_j^{(n)})-f(s,y)|}{t_j^{(n)}-y} \right\} d\omega_m(s)$$

$$\leqslant \pi \int_{-\pi}^{\pi} \left\{ \psi_y(s;\delta_1) + \frac{3}{2} \int_0^{2\delta_1} \frac{\psi_y(s;v)}{v}\, dv \right\} d\omega_m(s)$$

for large n (see [7], p. 18). Consequently, $|J'| < 3\pi\varepsilon/2$ for sufficiently large m and n.

Further we argue as in the proof of 4.4.

5. Convergence criteria. Start with the following

5.1. LEMMA. *Retain the symbols R, I_1, I_2, $I_\sigma(z)$, $Q_\sigma(x,y)$, R_0 used in 4.4, 3.2 and 3.3. Consider a function $f(s,t)$ which is Riemann integrable over $Q = [-\pi,\pi; -\pi,\pi]$.*

(i) *Let $f(s,t)$ be continuous at a point $(x,y)\in R$, and let relations* (3) *and* (4) *be satisfied for $s\in I_\eta(x)$ and $t\in I_\eta(y)$, respectively. Suppose that the partial variations* (8) *with $\sigma=\eta$ remain bounded in $t\in I_\eta(y)$ and in $s\in I_\eta(x)$, and that*

$$W_p\big(f; Q_\eta(x,y)\big) < \infty$$

for a certain $p \geqslant 1$. Then, given an arbitrary $\varepsilon>0$, there is a positive $\delta\leqslant\eta$ such that

$$(13)\qquad \Big| \int_{x-\delta}^{x+\delta} \int_{y-\delta}^{y+\delta} \{f(s,t)-f(x,y)\} D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t) \Big| < \varepsilon$$

provided m and n are large enough ($0\leqslant\mu\leqslant m$, $0\leqslant\nu\leqslant n$).

(ii) *Suppose that the assumptions of 3.3 are satisfied and that relation* (7) *is fulfilled uniformly in $(x,y)\in R_0$. Let, moreover, conditions* (5) *hold with some positive p_1 and p_2. Then inequality* (13), *with δ small enough, holds uniformly in $(x,y)\in R_0$.*

Proof of (i). Consider the squares $Q_\sigma = Q_\sigma(x,y)\,(\sigma\leqslant\eta)$. Choose an arbitrary $\varepsilon>0$, and set

$$J_\sigma = \int_{x-\sigma}^{x+\sigma} \int_{y-\sigma}^{y+\sigma} \{f(s,t)-f(x,y)\} D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t).$$

Under the assumption

$$s_{\alpha-1}^{(m)} < x-\sigma \leqslant s_{\alpha}^{(m)} < s_{\alpha+1}^{(m)} < \ldots < s_{\beta}^{(m)} < x+\sigma \leqslant s_{\beta+1}^{(m)},$$

$$t_{\gamma-1}^{(n)} < y-\sigma \leqslant t_{\gamma}^{(n)} < t_{\gamma+1}^{(n)} < \ldots < t_{\lambda}^{(n)} < y+\sigma \leqslant t_{\lambda+1}^{(n)},$$

the last integral is equal to

$$\frac{4\pi^2}{(2m+1)(2n+1)} \sum_{i=\alpha}^{\beta} \sum_{j=\gamma}^{\lambda} \{f(s_i^{(m)}, t_j^{(n)})-f(x, y)\} D_\mu(s_i^{(m)}-x) D_\nu(t_j^{(n)}-y).$$

Hence, by the Abel transformation ([2], p. 16),

$$J_\sigma = \frac{4\pi^2}{(2m+1)(2n+1)} \Big\{ U_{\beta,\lambda}[f(s_\beta, t_\lambda)-f(x, y)] + \sum_{i=\alpha}^{\beta-1} U_{i,\lambda}[f(s_i, t_\lambda)-f(s_{i+1}, t_\lambda)]$$

$$+ \sum_{j=\gamma}^{\lambda-1} U_{\beta,j}[f(s_\beta, t_j)-f(s_\beta, t_{j+1})] +$$

$$+ \sum_{i=\alpha}^{\beta-1} \sum_{j=\gamma}^{\lambda-1} U_{i,j}[f(s_i, t_j)-f(s_i, t_{j+1})-f(s_{i+1}, t_j)+f(s_{i+1}, t_{j+1})] \Big\}$$

$$= J_\sigma^{(1)}+J_\sigma^{(2)}+J_\sigma^{(3)}+J_\sigma^{(4)},$$

where

$$U_{i,j} = \sum_{k=\alpha}^{i} \sum_{l=\gamma}^{j} D_\mu(s_k-x) D_\nu(t_l-y), \qquad s_k = s_k^{(m)},\ t_l = t_l^{(n)}.$$

The continuity of f at (x, y) and the estimate

$$(14) \qquad \frac{2\pi}{2n+1} \Big| \sum_{l=\gamma}^{\lambda} D_\nu(t_l-y) \Big| < K \quad (K = \text{const}),$$

proved in [4], imply $|J_\sigma^{(1)}| < \varepsilon/4$ for sufficiently small σ and large m and n.

Further,

$$|J_\sigma^{(2)}| \leqslant \frac{2\pi K}{2m+1} \Big| \sum_{i=\alpha}^{\beta-1} \sum_{k=\alpha}^{i} [f(s_i, t_\lambda)-f(s_{i+1}, t_\lambda)] D_\mu(s_k-x) \Big|.$$

Taking $p_1' > p_1$ and $q_1' > 1$ for which $1/p_1'+1/q_1' > 1$, and applying inequality (5.1) of [5], we obtain

$$|J_\sigma^{(2)}| \leqslant KC_{p_1',q_1'} V_{p_1'}\big(f(\cdot, t_\lambda);\ I_\sigma(x)\big) V_{q_1'}\big(G_\mu^m;\ I_\sigma(x)\big),$$

where

$$G_\mu^m(u) = \int_{-\pi}^{u} D_\mu(s-x)\, d\omega_m(s);$$

whence, by 2.1 of [4] and 3.4 (i), $|J_\sigma^{(2)}| < \varepsilon/4$ if σ, $1/m$ and $1/n$ are small enough. The inequality remains also true for $J_\sigma^{(3)}$.

Let

$$G^{m,n}_{\mu,\nu}(u,v) = \int_{-\pi}^{u}\int_{-\pi}^{v} D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t),$$

$$1/p+1/q>1, \qquad q>1.$$

In view of 2.1,

$$|J^{(4)}_\sigma| \leqslant C_{p,q} W^*_p(f;Q_\sigma)\,V_q(G^{m,n}_{\mu,\nu};Q_\sigma).$$

Since $V_q(G^{m,n}_{\mu,\nu};Q_\sigma)$ is uniformly bounded and $\lim_{\sigma\to 0+} W^*_p(f;Q_\sigma)=0$, by 2.4 of [3], 2.1 of [4] and 3.2, we have $|J^{(4)}_\sigma| < \varepsilon/4$ for small σ, $1/m$ and $1/n$.

Collecting the results we get the first part of the Lemma.

The proof of the second part runs parallely. In this case we apply 3.3 and 3.4 (ii) instead of 3.2 and 3.4 (i).

Now, an analogue of Theorem 3.1 in [4] will be given.

5.2. THEOREM. (i) *If the assumptions of* 4.4 (i) *and* 5.1 (i) *are satisfied, then*

(15) $$\lim_{\mu,\nu\to\infty} I^{m,n}_{\mu,\nu}(x,y;f) = f(x,y).$$

(ii) *Under the hypotheses of* 4.4 (ii) *and* 5.1 (ii), *the convergence* (15) *is uniform in* $(x,y)\in R_0$.

Proof of (i). Given an arbitrary $\varepsilon>0$, we choose a δ as in 5.1 (i) and we write

$$I^{m,n}_{\mu,\nu}(x,y;f)-f(x,y)$$

$$=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\{f(s,t)-f(x,y)\}D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t)$$

$$=\frac{1}{\pi^2}\int_{x-\delta}^{x+\delta}\int_{y-\delta}^{y+\delta}\{f(s,t)-f(x,y)\}D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t)+H^{m,n}_{\mu,\nu}(x,y).$$

In virtue of 4.2 and 4.4 (i),

$$\lim_{\mu,\nu\to\infty} H^{m,n}_{\mu,\nu}(x,y)=0.$$

Hence, by 5.1(i), the result follows.

The proof of (ii) is the same in principle.

Next, we shall present the

5.3. LEMMA. *Retain the notation and the initial assumption of* 5.1.

(i) *Let* $f(s,t)$ *be continuous at a point* $(x,y)\in R$. *Suppose that there are three functions* $\varphi_x(t;u)$, $\psi_y(s;v)$ *and* $\chi_{x,y}(u,v)$, *which are non-decreasing in* $u, v\geqslant 0$, *such that*

$$|f(x+u,t)-f(x,t)|\leqslant \varphi_x(t;|u|), \qquad |f(s,y+v)-f(s,y)|\leqslant \psi_y(s;|v|),$$

$$|f(x+u,y+v)-f(x+u,y)-f(x,y+v)+f(x,y)|\leqslant \chi_{x,y}(|u|,|v|)$$

if $-\eta \leqslant u,\ v \leqslant \eta,\ t \in I_\eta(y)$ *and* $s \in I_\eta(x)$. *Moreover, let*

$$\int_0^\eta \int_0^\eta \frac{\chi_{x,y}(u, v)}{uv} du dv < \infty$$

and

$$\lim_{\sigma\to 0+} \int_0^\sigma \frac{\varphi_x(t; u)}{u} du = 0, \quad \lim_{\sigma\to 0+} \int_0^\sigma \frac{\psi_y(s; v)}{v} dv = 0$$

uniformly in $t \in I_\eta(y)$ *and in* $s \in I_\eta(x)$. *Then the assertion of* 5.1 (i) *holds.*

(ii) *Let relation* (7) *be fulfilled uniformly in* $(x, y) \in R$. *Suppose that the conditions of* (i) *are satisfied for all* $(x, y) \in R$. *If, moreover, the majorants*

$$\varphi(t; u) = \varphi_x(t; u), \quad \psi(s; v) = \psi_y(s; v), \quad \chi(u, v) = \chi_{x,y}(u, v)$$

are independent of x *and* y, *the conclusion of* 5.1 (i) *holds uniformly in* R.

Proof of (i). Consider the integral J_σ defined in 5.1 and write

$$J_\sigma = \int_{x-\sigma}^{x+\sigma} \int_{y-\sigma}^{y+\sigma} \{f(x, t) - f(x, y)\} D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t) +$$

$$+ \int_{x-\sigma}^{x+\sigma} \int_{y-\sigma}^{y+\sigma} \{f(s, y) - f(x, y)\} D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t) +$$

$$+ \int_{x-\sigma}^{x+\sigma} \int_{y-\sigma}^{y+\sigma} \{f(s, t) - f(x, t) - f(s, y) + f(x, y)\} D_\mu(s-x) D_\nu(t-y)\, d\omega_m(s)\, d\omega_n(t)$$

$$= J'_\sigma + J''_\sigma + J'''_\sigma.$$

Choose an arbitrary positive ε. In view of (14),

$$|J'_\sigma| \leqslant K \Big| \int_{y-\sigma}^{y+\sigma} \{f(x, t) - f(x, y)\} D_\nu(t-y)\, d\omega_n(t) \Big|;$$

whence

$$K^{-1} |J'_\sigma| \leqslant \Big(\int_{y-\sigma}^{y} + \int_{y}^{y+\sigma} \Big) |f(x, t) - f(x, y)|\, |D_\nu(t-y)|\, d\omega_n(t).$$

Let $t_k = t_k^{(n)},\ t_{k+1} = t_{k+1}^{(n)}, \ldots, t_{k+l} = t_{k+l}^{(n)}$ be all fundamental points in the interval $\langle y, y+\sigma)$. Applying the estimates

$$(16) \qquad |D_\nu(t_j - y)| \leqslant \tfrac{1}{2} + \nu, \quad |(t_j - y) D_\nu(t_j - y)| \leqslant \frac{\pi}{2},$$

we obtain

$$\int_y^{y+\sigma} |f(x, t) - f(x, y)|\, |D_\nu(t-y)|\, d\omega_n(t)$$

$$\leqslant \pi \Big\{ |f(x, t_k) - f(x, y)| + \frac{\pi}{2n+1} \sum_{j=k+1}^{k+l} \frac{|f(x, t_j) - f(x, y)|}{t_j - y} \Big\}$$

$$\leqslant \pi \Big\{ |f(x, t_k) - f(x, y)| + \tfrac{3}{2} \int_0^{2\sigma} \frac{\psi_y(x; v)}{v} dv \Big\}$$

for sufficiently large n (cf. Th. 4.5). Consequently,

$$\int_{y}^{y+\sigma} |f(x,t)-f(x,y)|\,|D_\nu(t-y)|\,d\omega_n(t) < \frac{\varepsilon}{8K}$$

if σ is small and n is large enough. The last inequality remains also true for the integral $\int_{y-\sigma}^{y}$. Hence $|J_\sigma'| < \varepsilon/4$ and, by symmetry, $|J_\sigma''| < \varepsilon/4$ for sufficiently small σ and $1/m$.

Split J_σ''' into four integrals and consider one of them:

$$J_{\sigma,1}''' = \int_{x}^{x+\sigma}\int_{y}^{y+\sigma} \{f(s,t)-f(x,t)-f(s,y)+f(x,y)\} \times \\ \times D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t).$$

Assuming that $s_p = s_p^{(m)}$, $s_{p+1} = s_{p+1}^{(m)}, \ldots, s_{p+r} = s_{p+r}^{(m)}$ are in $\langle x, x+\sigma)$, we can write

$$I_{\sigma,1}''' = \frac{4\pi^2}{(2m+1)(2n+1)}\{f(s_p,t_k)-f(x,t_k)-f(s_p,y)+f(x,y\} \times \\ \times D_\mu(s_p-x)D_\nu(t_k-y)+$$
$$+\frac{2\pi}{2m+1}D_\mu(s_p-x)\int_{t_{k+1}}^{y+\sigma}\{f(s_p,t)-f(x,t)-f(s_p,y)+f(x,y)\}D_\nu(t-y)\,d\omega_n(t)+$$
$$+\frac{2\pi}{2n+1}D_\nu(t_k-y)\int_{s_{p+1}}^{x+\sigma}\{f(s,t_k)-f(x,t_k)-f(s,y)+f(x,y)\}D_\mu(s-x)\,d\omega_m(s)+$$
$$+\int_{s_{p+1}}^{x+\sigma}\int_{t_{k+1}}^{y+\sigma}\{f(s,t)-f(x,t)-f(s,y)+f(x,y)\}D_\mu(s-x)D_\nu(t-y)\,d\omega_m(s)\,d\omega_n(t)$$
$$= A_\sigma + B_\sigma^{(1)} + B_\sigma^{(2)} + B_\sigma^{(3)}.$$

By the first estimate (16) and the continuity of f, $|A_\sigma| < \varepsilon/32$ for σ, $1/m$ and $1/n$ small enough. Further, inequalities (16) give

$$|B_\sigma^{(1)}| \leqslant \frac{\pi^2}{2}\int_{t_{k+1}}^{y+\sigma}\frac{|f(s_p,t)-f(x,t)-f(s_p,y)+f(x,y)|}{t-y}\,d\omega_n(t)$$
$$\leqslant \frac{\pi^2}{2}\left\{\frac{2\pi}{2n+1}\sum_{j=k+1}^{k+l}\frac{\psi_y(x;t_j-y)}{t_j-y}+\frac{2\pi}{2n+1}\sum_{j=k+1}^{k+l}\frac{\psi_y(s_p;t_j-y)}{t_j-y}\right\}.$$

Hence, as before, we get $|B_\sigma^{(1)}| < \varepsilon/32$ provided σ, $1/m$ and $1/n$ are small enough. The estimate remains valid for $B_\sigma^{(2)}$, too. Finally, it can easily be observed that

$$|B_\sigma^{(3)}| \leqslant \frac{\pi^4}{(2m+1)(2n+1)}\sum_{i=p+1}^{p+r}\sum_{j=k+1}^{k+l}\frac{\chi_{x,y}(s_i-x,t_j-y)}{(s_i-x)(t_j-y)}$$
$$\leqslant \frac{9\pi^2}{4}\int_0^{2\sigma}\int_0^{2\sigma}\frac{\chi_{x,y}(u,v)}{uv}\,du\,dv$$

if m and n are large. Therefore $|J'''_{\sigma,1}| < \varepsilon/8$, and then $|J'''_{\sigma}| < \varepsilon/2$ for sufficiently small σ, $1/m$ and $1/n$.

Thus the proof of (i) is completed.

Applying the last Lemma and reasoning as in 5.2, we get the following test of Dini's type (cf. (5.5) of [6] and 3.2 of [3]):

5.4. THEOREM. (i) *Suppose that the assumptions of* 4.5 (i) *and* 5.3 (i) *are satisfied. Then relation* (15) *holds.*

(ii) *Under the hypotheses of* 4.5 (ii) *and* 5.3 (ii), *the convergence* (15) *is uniform in* $(x, y) \in R$.

REFERENCES

[1] J. Marcinkiewicz, *Collected papers*, Warszawa 1964.

[2] Ch. N. Moore, *Summable series and convergence factors*, New York 1938.

[3] R. Taberski, *On double Bessel series, I*, Bulletin de l'Académie Polonaise des Sciences, Série des sci. math., astronom. et phys., 16 (1968), p. 37–45.

[4] — *Trigonometric interpolation, I*, Colloquium Mathematicum 20 (1969), p. 287–294.

[5] L. C. Young, *An inequality of the Hölder type connected with Stieltjes integration*, Acta Mathematica (Uppsala) 67 (1936), p. 251–282.

[6] A. Zygmund, *Trigonometric series, II*, Cambridge 1959.

DEPARTMENT OF MATHEMATICS I, A. MICKIEWICZ UNIVERSITY, POZNAŃ

Reçu par la Rédaction le 5. 6. 1968