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# ON A REPRESENTATION OF SEMIGROUPS BY PRODUCTS OF ALGEBRAS AND RELATIONS

BY

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We construct a representation of an arbitrary commutative semigroup $(S, +)$ by products of unary algebras or graphs. This means that we find a collection $\{A_x; x \in S\}$ of non-isomorphic algebras or graphs such that $A_{x+y}$ is always isomorphic to the Cartesian product $A_x \times A_y$. For example, graphs $A_x$ can be forests of length 3.

For a special class of $C$-embeddable semigroups $S$ (including, e. g., all Abelian groups) a stronger result is obtained: we characterize those varieties of unary algebras with one operation for which every $C$-embeddable semigroup can be represented by products. In the remaining varieties, no group can be represented.

In the interesting case of a finite cyclic group one tries to find, for any natural $p$, an algebra $A$ with $A^n$ isomorphic to $A^m$ iff $n = m \pmod p$. This was done for groups in [3], for Boolean algebras, rings and semigroups in [8], etc. The general concept of representations was defined by Trnková in [8], where she proved that a representation in $\mathfrak{A}(1, 1)$ exists for a class of semigroups and showed in [9] that this class, in fact, contains all commutative semigroups. Our representation method is a modification of hers.

A related problem, initiated by McKenzie, is the Cantor-Bernstein theorem for a class $\mathscr{C}$ of graphs. It states that two graphs $A, B \in \mathscr{C}$ are isomorphic whenever there exist graphs $X, Y \in \mathscr{C}$ with $A$ isomorphic to $B \times X$ and $B$ isomorphic to $A \times Y$. We give some examples of such a class $\mathscr{C}$.

Some results of this paper have been announced in [2].

It is our pleasant duty to express our gratitude to Věra Trnková, who initiated our work and paid a close attention to it. We are also much indebted to the referee of this paper for his valuable comments.

## I. C-EMBEDDABLE SEMIGROUPS

**I.1.** Trnková proved an interesting result concerning a universality of semigroups $\exp N^M$, where $N$ denotes the additive semigroup of non-negative integers, $M$ is a set (denoting here a product of "$M$ copies" of $N$)

and exp denotes the global. Thus, elements of $\exp N^M$ are non-void sets of functions $f\colon M\to N$ with addition defined by

$$\mathscr{A}+\mathscr{B}=\{f+g;\, f\in\mathscr{A},\, g\in\mathscr{B}\}.$$

THEOREM 1 (Trnková [9]). *Every commutative semigroup $S$ can be embedded into $\exp N^M$, where $M$ is a set of power* $\operatorname{card} S\cdot\aleph_0$.

Notice that the function const 0 (defined by $f(m)=0$ for all $m$) can be "omitted" in the above embedding. More precisely, for every commutative semigroup $S$ there exists an embedding $\varphi^*\colon S\to\exp N^{M^*}$($M^*$ is a set of power $\operatorname{card} S\cdot\aleph_0$) such that $\operatorname{const} 0\notin\varphi^*(s)$, $s\in S$. Indeed, given an arbitrary embedding $\varphi\colon S\to\exp N^M$, denote by $M^*$ the disjoint union of $M$ and $N$ and, if $\varphi(s)=\mathscr{A}$, put

$$\varphi^*(s)=\{f\colon M^*\to N;\, f/M\in\mathscr{A} \text{ and } f(n)\neq 0 \text{ for an infinite number of } n\in N\}.$$

Then $\varphi^*$ is clearly an embedding, and $\operatorname{card} M^*=\operatorname{card} M$.

**I.2.** Definition 1. We say that a commutative semigroup is *C-embeddable* if it can be embedded into $(\exp N^N)^K$ for some set $K$.

Clearly, $C$-embeddable semigroups form a quasi-variety, which contains no subdirectly irreducible semigroup of power greater than $\operatorname{card}(\exp N^N)$. On the other hand, all free commutative semigroups are $C$-embeddable (as will be seen from the next theorem); therefore, $C$-embeddable semigroups do not form a variety.

THEOREM 2. *Every subdirect product of countable commutative semigroups is C-embeddable. In particular, all Abelian groups, all regular and all countable commutative semigroups are C-embeddable.*

Proof. By Theorem 1, every countable commutative semigroup can be embedded into $\exp N^N$. Therefore, a subdirect product of countable commutative semigroups belongs to every quasi-variety containing $\exp N^N$. The $C$-embeddability of regular commutative semigroups follows from a result of Schein [7]: every regular commutative semigroup is a subdirect product of countable semigroups (Schein proves this result for inverse semigroups, but the present generalization is obvious).

**I.3.** Definition 2. Let $(M, \dashv)$ be a well-ordered set. Given $\mathscr{A}\in\exp N^M$, let $\overline{\mathscr{A}}\in\exp N^M$ denote the set of all the functions $f\colon M\to N$ with the following property: for every increasing $\omega_0$-sequence $\{m_i\}_{i=0}^{\infty}$ in $(M, \dashv)$ there exists $g\in\mathscr{A}$ such that $f(m_i)=g(m_i)$ for all $i$. We say that $\mathscr{A}$ is *closed* if $\overline{\mathscr{A}}=\mathscr{A}$.

THEOREM 3. *A commutative semigroup $S$ is C-embeddable iff there exist a well-ordered set $(M, \dashv)$ and an embedding $\varphi\colon S\to\exp N^M$ such that $\varphi(x)$ is closed for all $x\in S$. Moreover, $\varphi$ can be chosen so that no $\varphi(x)$, $x\in S$, contains the function* const 0.

Proof. Necessity. Assume, for short, that $S$ is a subsemigroup of $\exp N^M$ with $\overline{\mathscr{A}} = \mathscr{A}$ for all $\mathscr{A} \in S$. Denote by $K$ the set of all increasing $\omega_0$-sequences in $(M, \dashv)$ and define a mapping

$$\psi\colon S \to (\exp N^N)^K.$$

(Recall that elements of $(\exp N^N)^K$ are collections $\{\mathscr{A}_\mu; \mu \in K\}$ with $\mathscr{A}_\mu \subset N^N$.) Put

$$\psi(\mathscr{A}) = \{\mathscr{A}_\mu; \mu \in K\},$$

where for $\mu = \{m_i\}_{i=0}^{\infty}$ we have

$$\mathscr{A}_\mu = \{f \in N^N; f(i) = g(m_i),\ i = 0, 1, 2, \ldots \text{ for some } g \in \mathscr{A}\}.$$

It is easy to see that $\psi(\mathscr{A} + \mathscr{B}) = \psi(\mathscr{A}) + \psi(\mathscr{B})$. Moreover, since each $\mathscr{A} \in S$ is closed, $\psi$ is an embedding. Indeed, given distinct elements $\mathscr{A}$ and $\mathscr{B}$ in $S$, we can assume that, e. g., $\mathscr{A} - \mathscr{B} \neq \emptyset$. Choose $g \in \mathscr{A} - \mathscr{B}$; then there exists $\mu \in K$, $\mu = \{m_i\}$, such that for no $h \in \mathscr{B}$ we have $g(m_i) = h(m_i)$, $i = 0, 1, 2, \ldots$ Thus $\mathscr{A}_\mu - \mathscr{B}_\mu \neq \emptyset$, and so $\psi(\mathscr{A}) \neq \psi(\mathscr{B})$.

Sufficiency. Assume, for short again, that $S$ is a subsemigroup of $(\exp N^N)^K$ and let $(M, \dashv)$ be the set $M = N \times \operatorname{card} K$ with the lexicographic order ($N$ is ordered in the natural way, and $\operatorname{card} K$ is considered, as usual, to be the well-ordered set of all ordinals of type less than $\operatorname{card} K$; we also have a bijection $k \mapsto k'$ of $K$ onto $\operatorname{card} K$). Define $\varphi\colon S \to \exp N^M$ as follows:

$$\varphi(\{\mathscr{A}_k\}_{k \in K}) = \{f\colon N \times \operatorname{card} K \to N;\ f(-, k') \in \mathscr{A}_k \text{ for every } k \in K\}.$$

It is easy to see that $\varphi$ is an embedding. Let us verify that, for every $x = \{\mathscr{A}_k\}$ in $S$, $\varphi(x)$ is closed. Given $f \in \overline{\varphi(x)}$, we choose special increasing sequences $\mu_k$ in $M$, $k \in K$, such that $\mu_k = \{(i, k')\}_{i=0}^{\infty}$. For each $k$ we thus obtain $g \in \varphi(x)$ with $g(i, k') = f(i, k')$, so $f(-, k') \in \mathscr{A}_k$. Therefore $f \in \varphi(x)$.

In case where const $0 \in \varphi(s)$ for some $s \in S$ we proceed as in Section I.1, denoting by $M^*$ the ordinal sum of $(M, \dashv)$ and $N$ (naturally ordered). It is clear that if $\varphi(s)$ is closed, then so is $\varphi^*(s)$.

Note 1. We shall use Theorems 1 and 3 without mentioning: given a commutative semigroup $S$ we shall always consider it as a subsemigroup of $\exp N^M$, whose elements do not contain const 0; if, by hypothesis, $S$ is $C$-embeddable, its elements are considered to be closed with respect to a fixed well-ordering of $M$.

## II. REPRESENTATIONS OF AN ARBITRARY SEMIGROUP

**II.1.** The aim of this section is to prove that every commutative semigroup can be represented by forests of length 3 and by unary algebras with one operation $f$ satisfying the equation $f^4 = f^3$. We use I.1 and represent the semigroup $\exp N^M$ (more precisely, $\exp(N^M - \{\text{const } 0\})$), where $M$ is an arbitrary infinite set.

**II.2.** Throughout the paper we deal with graphs (binary relations) $(X, R)$, where $R \subset X \times X$, and with unary algebras $(X, f)$, where $f: X \to X$. It is convenient to consider such an algebra as a graph $f \subset X \times X$ because the basic notions that we use (product, sum and isomorphism) are the same for graphs and for these algebras. We recall that, given relations $(X_i, R_i)$, $i \in I$, their (Cartesian) product is the relation $(\bigtimes_{i \in I} X_i, R)$ with $\{x_i\} R \{y_i\}$ equivalent to $x_i R_i y_i$ for all $i$. Their sum (disjoint union) is the relation $(\bigvee_{i \in I} X_i, S)$, where $\bigvee X_i$ denotes the disjoint union of the set $X_i$ and

$$S \cap (X_i \times X_j) = \begin{cases} R_i & \text{if } i = j, \\ \emptyset & \text{if } i \neq j. \end{cases}$$

Sums and products of graphs are distributive in the following sense: for arbitrary collections of graphs $\{A_i; i \in I\}$ and $\{B_j; j \in J\}$ we have

$$\Big(\bigvee_{i \in I} A_i\Big) \times \Big(\bigvee_{j \in J} B_j\Big) \simeq \bigvee_{i \in I} \bigvee_{j \in J} (A_i \times B_j),$$

where $\simeq$ denotes isomorphism of graphs. This property is basic for the representation method of Trnková [8]; in her terminology, the category of graphs is distributive. The same holds for much more general relational structures and, in fact, many results presented in this paper can easily be generalized in this direction.

**II.3.** Every cardinal $a$ is, as usual, considered as the set of all ordinals $i$ of type less than $a$ $(i < a)$.

Recall that a *forest* is a graph $(X, R)$ which has no cycles and such that if $(y_1, x) \in R$, $(y_2, x) \in R$, then $y_1 = y_2$. The supremum length of its paths is called the *length of the forest*. For a vertex $x \in X$ we denote by $B(x)$ the set of all vertices to which a path leads from $x$ (including $x$ itself). The restricted graph $(B(x), R \cap B(x)^2)$ is called the *branch* of $x$. We put $\operatorname{st} x = \operatorname{card} B(x)$.

Given a graph $G$, denote by $G^n$, $n \in N$, either the product of $n$ copies of $G$ (for $n \neq 0$) or the single loop $G^0 = (\{q\}, \{(q, q)\})$ (for $n = 0$).

**II.4. Representation by forests of length 3.** Given a cardinal $a$, denote by $R(a) = (X, R)$ the forest (see Fig. 1), where

$$X = \{a, b, c, d, b_0\} \cup \{c_i, d_i, d_i'\}_{i \in a},$$

$$R = \{(a, b), (b, c), (c, d), (a, b_0)\} \cup \{(b_0, c_i), (c_i, d_i), (c_i, d_i')\}_{i \in a}.$$

Let $S$ be a commutative semigroup. We consider $S$ to be a subsemigroup of $\exp N^M$, where $M$ is an infinite set and no element $\mathscr{A} \in S$ $(\mathscr{A} \in N^M)$ contains const 0. Choose pairwise distinct infinite regular cardinals $a_m$,

$m \in M$, and put $R_m = R(a_m)$. Then a representation of $S$ is defined by products of forests of length 3 as follows:

$$r(\mathscr{A}) = \bigvee_{h \in \mathscr{A}} \Big(\mathop{\times}_{m \in M} R_m^{h(m)}\Big)_{\aleph}, \qquad \mathscr{A} \in S.$$

Here $\aleph = \operatorname{card} N^M$ and $G_\aleph$ denotes the sum of $\aleph$ copies of $G$.

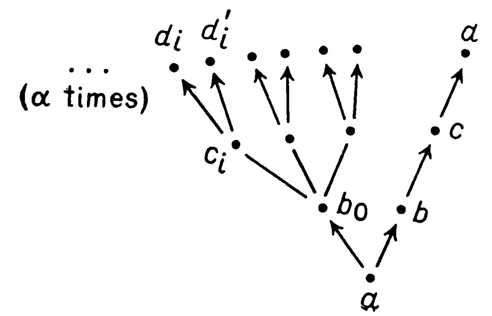


Fig. 1

Note 2. The forests $r(\mathscr{A})$, used for the representation, have the following property:

*Given $\mathscr{A}, \mathscr{B} \in S$, $\mathscr{A} \neq \mathscr{B}$, there is a tree of length 3 either isomorphic to a component of $r(\mathscr{A})$ but non-isomorphic to any component of $r(\mathscr{B})$ or, conversely, isomorphic to a component of $r(\mathscr{B})$ but not to that of $r(\mathscr{A})$.*

*Moreover, for every tree of length 3 in $r(\mathscr{A})$, $\mathscr{A} \in S$, no leaf has a distance from the root shorter than 3.*

Proof. (I) $r(\mathscr{A})$ is a forest of length 3.

Of course, $R_m$ are forests of length 3. It is easy to see that the product and sum of forests of length 3 are again forests of length 3. The only trouble is that $R_m^0$ is not a forest, but a single loop. On the other hand, for an arbitrary graph $T$, we have $T \simeq T \times R_m^0$ and, since $h \neq \text{const}\, 0$ whenever $h \in \mathscr{A}$, all $r(\mathscr{A})$ are clearly forests of length 3.

(II) $r(\mathscr{A} + \mathscr{B}) \simeq r(\mathscr{A}) \times r(\mathscr{B})$.

Due to the distributivity of products and sums we have

$$r(\mathscr{A}) \times r(\mathscr{B}) \simeq \bigvee_{h \in \mathscr{A}} \bigvee_{k \in \mathscr{B}} \Big(\big(\mathop{\times}_{m \in M} R_m^{h(m)}\big)_{\aleph} \times \big(\mathop{\times}_{m \in M} R_m^{k(m)}\big)_{\aleph}\Big) \simeq \bigvee_{(h,k) \in \mathscr{A} \times \mathscr{B}} \Big(\mathop{\times}_{m \in M} R_m^{(h+k)(m)}\Big)_{\aleph}.$$

Now, for every $f \in \mathscr{A} \dot{+} \mathscr{B}$ there exist at most $\aleph = \operatorname{card} N^M$ pairs $(h, k) \in \mathscr{A} \times \mathscr{B}$ with $f = h + k$. Since we take $\aleph$ copies of every graph, we have

$$r(\mathscr{A}) \times r(\mathscr{B}) \simeq \bigvee_{f \in \mathscr{A} + \mathscr{B}} \Big(\mathop{\times}_{m \in M} R_m^{f(m)}\Big)_{\aleph}.$$

(III) $r(\mathscr{A}) \simeq r(\mathscr{B})$ implies $\mathscr{A} = \mathscr{B}$.

Since isomorphisms of graphs preserve branches of points, the following lemma completes the proof.

LEMMA 1. *Let $f \in N^M$ and $\mathscr{A} \in S$. Then $f \in \mathscr{A}$ iff $r(\mathscr{A})$ has a tree $T$ of length 3 such that for every $m \in M$ there are just $f(m)$ points $x \in T$,* *distinct from the root, with the following property: $B(x)$ contains exactly $a_m$ P-points, i.e. points the branch of which has length* 1 *and power* 3.

Proof. Since $r(\mathscr{A})$ is a sum of graphs $\bigtimes_{m\in M} R_m^{f(m)}$, $f\in\mathscr{A}$, each with $\aleph$ copies, we can consider the vertices of $r(\mathscr{A})$ as triples $(x, f, i)$ with $f\in\mathscr{A}$, $i\in\aleph$ and $x\in \bigtimes_{m\in M} R_m^{f(m)}$. Recalling that $f\neq \text{const}\,0$, we can represent $x$ as a collection from which the points $q$ (corresponding to the graphs $R_m^0 = (\{q\}, \{(q, q)\})$) are excluded:

$$x = \{t_{m,j};\, m\in M, f(m)\neq 0, j = 0, 1, 2, \dots, f(m)-1\}, \qquad t_{m,j}\in R_m.$$

To avoid confusion, we often denote by $p^{(m)}$ the vertex $p$ in $R_m$. Consider the graph

$$\bigtimes_{m\in M} R_m^{f(m)} \qquad (f\neq \text{const}\,0)$$

which has just one component $K_0$ of length 3 (the component of $\{t_{m,j}\}$, where $t_{m,j} = a^{(m)}$ for all $(m, j)$ with $j < f(m)\neq 0$). Then, clearly, its point $\{t_{m,j}\}$ is a $P$-point iff there exists $(m_0, j_0)$ with $t_{m_0,j_0} = c_k^{(m)}$ $(k\in a_m)$ and, for $(m, j)\neq (m_0, j_0)$, $t_{m,j} = c^{(m)}$.

Necessity. For $f\in\mathscr{A}$, denote by $K_0$ the component of $\bigtimes_{m\in M} R_m^{f(m)}$ of length 3 and let $K$ be the component of $r(\mathscr{A})$ consisting of points $(\{t_{m,j}\}, f, 0)$, $\{t_{m,j}\}\in K_0$. It is clear from the above that for $x = (\{t_{m,j}\}, f, 0)$ in $K$, distinct from the root, there exist infinitely many $P$-points in $B(x)$ iff, for some $(m_0, j_0)$,

$$t_{m_0,j_0} = b_0^{(m)}, \qquad t_{m,j} = b^{(m)} \text{ whenever } (m, j)\neq (m_0, j_0).$$

Then, obviously, there exist exactly $a_{m_0}$ $P$-points in $B(x)$, viz. $(\{s_{m,j}\}, f, 0)$, where $s_{m_0,j_0} = c_k^{(m)}$ for some $k\in a_{m_0}$ and $s_{m,j} = c^{(m)}$ whenever $(m, j)\neq (m_0, j_0)$. For a fixed $m_0$ we get just $f(m_0)$ such points $x$ by varying $j_0$ from 0 to $f(m_0)-1$.

Sufficiency. Assume that, for a given $f\in N^M$, there exists a component $K$ of $r(\mathscr{A})$ with the above-mentioned properties. Obviously, $K$ is a component of a copy of $\bigtimes_{m\in M} R_m^{h(m)}$ for some $h\in\mathscr{A}$, more precisely, there are a component $K_0$ of $\bigtimes_{m\in M} R_m^{h(m)}$ and an element $i\in\aleph$ such that

$$K = \{(\{t_{m,j}\}, h, i);\, \{t_{m,j}\}\in K_0\};$$

without loss of generality we may assume that $i = 0$. By hypothesis, $K_0$ has length 3 and, for every $m$, it contains exactly $f(m)$ points $x$ with the property: "there exist just $a_m$ $P$-points in $B(x)$". But $\bigtimes_{m\in M} R_m^{h(m)}$ contains just one component of length 3, and this is $K_0$; therefore, it must be the component mentioned in the Necessity of this proof. Thus, for every $m$ there are exactly $h(m)$ points $x$ with the above-mentioned property. Therefore $h = f$, which proves that $f\in\mathscr{A}$.

**II.5.** Our next investigation concerns unary algebras with one operation $(X, f)$, where $f\colon X \to X$ is a transformation. We need a complete list of subvarieties of the variety $\mathfrak{A}(1)$ of unary algebras (see [6]):

$\mathscr{V}_{k,m}$ $(k \geqslant m;\ k, m = 0, 1, 2, \ldots)$ – the variety of algebras with $f^k(x) = f^m(x)$;

$\mathscr{W}^r$ $(r = 0, 1, 2, \ldots)$ – the variety of algebras with $f^r(x) = f^r(y)$ (i. e. $f^r$ is a constant mapping).

We denote by $f^0$ the identity of $X$. Notice that $\mathscr{V}_{k,k} = \mathfrak{A}(1)$.

**II.6. Representation in $\mathscr{V}_{4,3}$.** Every commutative semigroup has a representation by products of algebras in $\mathscr{V}_{4,3}$. The proof is quite analogous to that given in Note 2. The basic algebras are shown in Fig. 2.

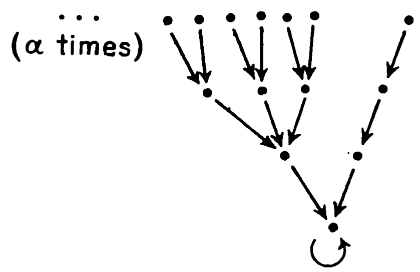


Fig. 2

**II.7. Representation in $\mathscr{W}_4$.** The representation in $\mathscr{W}_4$ is complicated by the fact that sums of $\mathscr{W}_4$-algebras are very distinct from their disjoint unions and, therefore, there is no distributivity with respect to sums and products in $\mathscr{W}_4$. We deal with a general $\mathscr{W}_n$ so as to be able to apply the auxiliary construction to $\mathscr{W}_3$-algebras later.

We use a different notion of branches here ($\mathscr{W}_n$-algebras are, roughly speaking, dual trees with a loop adjoint to the root): given $(X, f)$ in $\mathscr{W}_n$, the *branch* of a point $x$ is the algebra $(B(x), f_x)$, where

$$B(x) = \{y \in X;\ f^k(y) = x,\ k = 0, 1, 2, \ldots, n\}$$

and $f_x = f$ on $B(x) - \{x\}$, while $f_x(x) = x$. Denote by $\bar{B}(x)$ the tree $(B(x), R)$ with $aRb$ iff $f(b) = a$ and $b \neq a$.

Definition 3. Given an infinite cardinal $\beta$ and a natural number $n$, we say that a forest $R$ is *$(\beta, n)$-standard* provided that

(i) $R$ has length $n$;

(ii) in every tree $T$ of $R$ the distances of any two leaves from the root are equal; moreover, $\operatorname{card} T < \beta$;

(iii) if a tree has property (ii) and has a length smaller than $n$, then it is isomorphic to some tree of $R$.

Notation. For a cardinal $\beta$ we write

$$\exp^2\beta = \exp(\exp\beta), \quad \exp^3\beta = \exp(\exp^2\beta), \quad \ldots$$

LEMMA 2. *For every $(\beta, n)$-standard forest $R$ and for every cardinal $a > \exp^n\beta$ there exists an algebra $R^* = (X, f)$ in $\mathscr{W}_{n+1}$ with the following properties:*

(a) *for every $x, y \in X$ with $f^{-1}(x) = f^{-1}(y) = \emptyset$ the following implication holds: if $f^k(x) = f^r(y)$, then $f^k(x) = f^k(y)$;*

(b) *for every $x \in X$: either $\operatorname{card} f^{-1}(x) = a$ and then for each $y \in f^{-1}(x)$, $y \neq x$, there exist $a$ distinct points $y_i \in f^{-1}(x)$ with $B(y) \simeq B(y_i)$; or $\operatorname{card} f^{-1}(x) < a$ and then $\bar{B}(x)$ is isomorphic to a tree of $R$;*

(c) *for every algebra $A$ in $\mathscr{W}_n - \mathscr{W}_{n-1}$ with properties* (a) *and* (b) *there exist $a$ points $x \in X$, whose branches are isomorphic to $A$, such that $f^2(x) = f(x)$.*

*Moreover, the algebra $R^*$ is unique in $\mathscr{W}_{n+1}$ up to isomorphism.*

**Proof.** Let $A_i$, $i \in I$, be all representants of isomorphism types of algebras in $\mathscr{W}_n - \mathscr{W}_{n-1}$ with properties (a) and (b). There is just one way of constructing $R^*$,

$$R^* = (X_1 \cup \{a\}, f),$$

where $(X_1, R_1)$ is the disjoint union of all $A_i$, each with $a$ disjoint copies, i. e.,

$$(X_1, R_1) = \bigvee_{i \in I} (A_i)_a,$$

and $f(x) = a$ if $(x, x) \in R_1$; $f(x) = y$ if $(x, y) \in R_1$ and $x \neq y$; $f(a) = a$. Therefore, the uniqueness is clear. To prove the existence, we must show that, in the above-described algebra $R^*$, $\operatorname{card} f^{-1}(a) \leqslant a$. Then $R^*$ has properties (a), (b), and (c).

Thus, we want to verify that $\operatorname{card} I \leqslant \exp^n\beta$. Let us proceed by induction on $n$. For $n = 1$ the inequality is obvious. Given an algebra in $\mathscr{W}_n$ with properties (a) and (b), say $B = (Y, g)$ with $b \in Y$ and $g(b) = b$, either $\operatorname{card} g^{-1}(b) < a$ (but then $\operatorname{card} Y < \beta$ and the number of such non-isomorphic algebras cannot exceed $2^\beta$) or $\operatorname{card} g^{-1}(b) = a$. Then the isomorphism type of $B$ is determined by isomorphism types of branches of the points $y \in g^{-1}(b)$, since each branch occurs with exactly $a$ copies. If the number of isomorphism types of the branches is less than or equal to $\exp^{n-1}\beta$, then the number of types of algebras $B$ is less than or equal to $\exp^n\beta$.

LEMMA 3. *Let $R_1$ and $R_2$ be $(\beta, n)$-standard forests, and let $a > \exp^n\beta$. Then $R_1 \times R_2$ is a $(\beta, n)$-standard forest and*

$$(R_1 \times R_2)^* \simeq R_1^* \times R_2^*.$$

*Moreover, if $R_1$ and $R_2$ are non-isomorphic graphs, then $R_1^*$ and $R_2^*$ are non-isomorphic algebras.*

Proof. (1) $R_1 \times R_2$ is a $(\beta, n)$-standard forest.

Properties (i) and (ii) of Definition 3 are easy to verify. For (iii), consider a tree $T$ with property (ii) — we can assume that $T$ is a tree in $R_1$. It is clear that $R_2$ contains a tree $S$ which is a chain of length $n-1$. Then a component of $T \times S$ is isomorphic to $T$, so is a component of $R_1 \times R_2$.

(2) $(R_1 \times R_2)^* \simeq R_1^* \times R_2^*$.

It suffices to verify that the algebra $R_1^* \times R_2^*$ has properties (a), (b), and (c) of Lemma 2 with respect to the forest $R_1 \times R_2$. Put

$$R_1^* = (X_1, f_1), \quad R_2^* = (X_2, f_2) \quad \text{and} \quad (X_1 \times X_2, g) = R_1^* \times R_2^*.$$

Property (a) is clear.

For (b), let $(x_1, x_2) \in X_1 \times X_2$ be given.

I. $\operatorname{card} g^{-1}(x_1, x_2) = a$. Then either $\operatorname{card} f_1^{-1}(x_1) = a$ or $\operatorname{card} f_2^{-1}(x_2) = a$. Assume the former. Let $(y_1, y_2) \in g^{-1}(x_1, x_2)$ be arbitrary with $(y_1, y_2) \neq (x_1, x_2)$. If $y_1 \neq x_1$, then there exist $a$ points $y_1^i \in f_1^{-1}(x_1)$ with $B(y_1^i) \simeq B(y_1)$. Then we have $a$ points $(y_1^i, y_2)$ with $B(y_1^i, y_2) \simeq B(y_1, y_2)$. Now assume that $x_1 = y_1$; then $x_1 = y_1$ is the root of $R_1^*$ (i. e., $f_1(y_1) = y_1$). We know that $y_2 \neq x_2$ (since $(y_1, y_2) \neq (x_1, x_2)$); therefore, $y_2$ is not the root of $R_2^*$.

Denote by $A$ the subalgebra of $R_1^*$ over all $x \in X_1$ with $f^n(x) = x_1$ (i. e., $X - A$ are the leaves of $R_1^*$). Then $A$ is a $\mathscr{W}_n$-algebra with properties (a) and (b), and so there exist $a$ points $t^i \in X_1$ with $B(t^i) \simeq A$ and $f_1^2(t^i) = f_1(t^i)$; the last means that $t^i \in f_1^{-1}(x_1)$. Since $R_2$ is a standard forest and since $y_2$ is not a root in $R_2^*$, it is very easy to check that the branches of $(x_1, y_2)$ and $(t^i, y_2)$ in $R_1^* \times R_2^*$ are isomorphic. Therefore, we have again $a$ points $(t^i, y_2) \in g^{-1}(x_1, x_2)$ with branches isomorphic to the branch of $(x_1, y_2) = (y_1, y_2)$.

II. $\operatorname{card} g^{-1}(x_1, x_2) < a$. Then either

$$g^{-1}(x_1, x_2) = \emptyset$$

and, therefore, $\bar{B}(x_1, x_2)$ is a tree of $R_1 \times R_2$ or

$$\operatorname{card} f_1^{-1}(x_1) < a \quad \text{and} \quad \operatorname{card} f_2^{-1}(x_2) < a.$$

Then $\bar{B}(x_1)$ is a tree of $R_1$ and $B(x_2)$ is a tree of $\bar{R}_2$, and thus $\bar{B}(x_1, x_2)$ is a tree of $R_1 \times R_2$.

To verify (c), let $A$ be an algebra in $\mathscr{W}_n$ with properties (a) and (b). Since $R_1$ and $R_1 \times R_2$ have the same trees of lengths smaller than $n$, it is clear that

either $\operatorname{card} A < a$ and the length of $A$ is $n$

or $A$ fulfils (a) and (b) also with respect to $R_1$.

In the former case, since $R_1^* \times R_2^*$ fulfils (b), $A$ is isomorphic to the branch of some $(x_1, x_2) \in X_1 \times X_2$ with

$$g^2(x_1, x_2) = g(x_1, x_2) \neq (x_1, x_2);$$

then there exist $a$ such points.

In the latter case we have a point $x_1 \in X_1$ with $f_1^2(x_1) = f_1(x_1)$ such that $B(x_1) \simeq A$. Clearly, there exists a point $x_2 \in X_2$, with $f_2^2(x_2) = f_2(x_2) \neq x_2$, whose branch consists of $a$ chains of length $n-1$ (see Fig. 3).

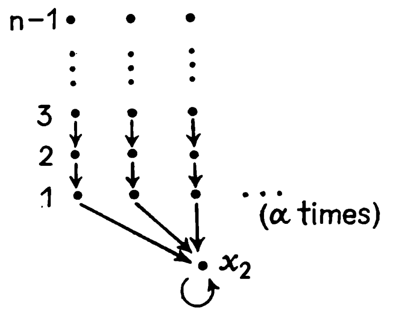


Fig. 3

It is evident that $B(x_1, x_2) \simeq B(x_1) \simeq A$, and then there are $a$ such points.

(3) $R_1 \not\simeq R_2$ implies $R_1^* \not\simeq R_2^*$.

This follows immediately from the uniqueness of the algebras $R_1^*$ and $R_2^*$.

THEOREM 4. *Every commutative semigroup can be represented by products of algebras in $\mathscr{W}_4$.*

Proof. By Lemma 3 it suffices to represent a given semigroup $S$ by products of $(\beta, 3)$-standard forests (for some cardinal $\beta$). We have a representation $\{R_s; s \in S\}$ of $S$ by products of forests of length 3 having the property in Note 2. Choose a cardinal $\beta > \operatorname{card} R_s$ for all $s$. Let $\bar{R}_s$ denote the forest which has the following trees: Its trees of length 3 are exactly those of $R_s$, and its trees of smaller lengths are just all the trees $T$ such that

(*) any two leaves have the same distance from the root and $\operatorname{card} T < \beta$,

and such a tree is contained, with exactly $\beta$ copies, in $R_s$.

It is clear that $\bar{R}_s$, $s \in S$, are pairwise non-isomorphic $(\beta, 3)$-standard forests. Let us verify that $\bar{R}_{s+s'} \simeq \bar{R}_s \times \bar{R}_{s'}$. It is clear that $\bar{R}_s \times \bar{R}_{s'}$ contains $\beta$ copies of every tree $T$ of length less than 3 and with property (*). As far as trees of $R_s \times R_{s'}$ of length 3 are concerned, recall that $R_s \times R_{s'} \simeq R_{s+s'}$. Given trees $T_1$ and $T_2$ such that the forest $T_1 \times T_2$ has length 3, the length of $T_1$ and $T_2$ must clearly be 3. Therefore, the trees of length 3 in $\bar{R}_s \times \bar{R}_{s'}$ are the same as in $R_s \times R_{s'}$. Since $R_s \times R_{s'} \simeq R_{s+s'}$, and $\bar{R}_{s+s'}$ has the same trees of length 3 as $R_{s+s'}$, the proof is completed.

## III. REPRESENTATIONS OF A C-EMBEDDABLE SEMIGROUP

**III.1.** In this section we show a representation of a $C$-embeddable semigroup by products of forests of length 2 and of unary algebras. We assume that a well-ordered set $(M, \dashv)$ is given and that $S$ is a subsemigroup of $\exp N^M$ the elements of which are closed subsets of $N^M$, not containing const 0.

**III.2. Representation by forests of length 2.** Given infinite cardinals $a$ and $\beta$, denote by $R(a, \beta) = (X, R)$ the forest (see Fig. 4), where

$$X = \{x_i\}_{i\in a} \cup \{y_i\}_{i\in\beta},$$

$$R = \{(x_0, x_1), (x_1, x_2), (x_0, x_3)\} \cup \{(y_0, y_1), (y_1, y_2), (y_0, y_3)\} \cup$$
$$\cup \{(x_3, x_i)\}_{i\in a} \cup \{(y_3, y_i)\}_{i\in\beta}.$$

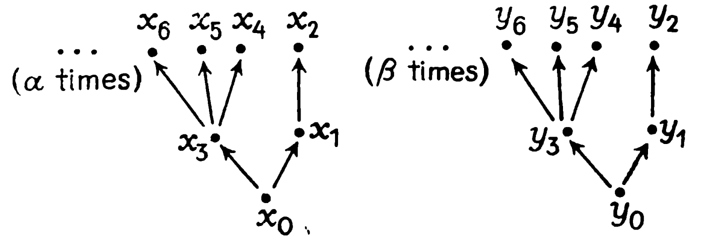


Fig. 4

Choose infinite regular cardinals $a_m$ and $\beta_m$ for $m \in M$ so that, if $m \dashv n$ in $M$, then $a_m < a_n < \beta_m < \beta_n$. Put $R_m = R(a_m, \beta_m)$. Then a representation of $S$ is defined by products of forests of length 2 analogously to the case of forests of length 3 (see II.4). Also, replacing in Note 2 the words "length 3" by "length 2" we have the property valid for the forests $r(\mathscr{A})$ with trees of length 2, used for the representation.

Indeed, the proof that all $r(\mathscr{A})$ are trees of length 2 and that

$$r(\mathscr{A} + \mathscr{B}) \simeq r(\mathscr{A}) \times r(\mathscr{B})$$

is quite analogous to the proof in II.4. To verify that $r(\mathscr{A}) \simeq r(\mathscr{B})$ implies $\mathscr{A} = \mathscr{B}$ it suffices to prove the following lemma:

LEMMA 4. *Let $f \in N^M$ and $\mathscr{A} \in S$. Then $f \in \mathscr{A}$ iff for every increasing $\omega_0$-sequence $\mu = \{m_i\}_{i=1}$ in $(M, \dashv)$ there exists a vertex in $r(\mathscr{A})$ whose branch*

(1) *has length* 2;

(2) *contains exactly*

$$2^{f(m_1)}-1 \textit{ points with } \mathrm{st} = a_{m_1},$$

$$(2^{f(m_2)}-1)\cdot 2^{f(m_1)} \textit{ points with } \mathrm{st} = a_{m_2},$$

$$\cdots\cdots\cdots\cdots$$

$$(2^{f(m_k)}-1)\cdot 2^{f(m_1)+\ldots+f(m_{k-1})} \textit{ points with } \mathrm{st} = a_{m_k};$$

(3) *does not contain any point with* st $= a_m$, $m \notin \mu$, *or* st $= \beta_m$, $m \in \mu$.

**Proof.** As in the proof of Lemma 1, the vertices of $r(\mathscr{A})$ are of the form $(x, f, i)$ with $f \in \mathscr{A}$, $i \in \aleph$ and $x \in \mathop{\times}\limits_{m \in M} R_m^{f(m)}$. Further,

$$x = \{t_{m,j};\, m \in M,\, f(m) > 0,\, j = 0, 1, 2, \ldots, f(m)-1\}, \qquad t_{m,j} \in R_m.$$

Again, we denote by $p^{(m)}$ the vertex $p$ in $R_m$.

**Necessity.** Let $f \in \mathscr{A}$. Given an increasing sequence $\mu = \{m_i\}$ in $M$, define a vertex $v(f, \mu) = (\{t_{m,j}\}, f, 0)$ in $r(\mathscr{A})$, by $t_{m,j} = x_0^{(m)}$ if $m \in \mu$, and $t_{m,j} = y_0^{(m)}$ if $m \notin \mu$. Let us verify that $v(f, \mu)$ has the required properties. The length of its branch is clearly 2; let $w = (\{z_{m,j}\}, f, 0)$ belong to this branch. We want to show that if $\operatorname{st} w = \beta_{m_0}$, then $m_0 \notin \mu$, and if $\operatorname{st} w = a_{m_0}$, then $m_0 \in \mu$, and that there are just

$$(2^{f(m_k)}-1)\cdot 2^{f(m_1)+\ldots+f(m_{k-1})}$$

such points (if $m_0 = m_k$). It is clear that if $w$ is in the branch of $v(f, \mu)$ and $\operatorname{st} w$ is infinite, then either

$$w = v(f, \mu)$$

(and, clearly, $\operatorname{st} w = \sup\{\beta_m;\, m \in M - \mu\}$) or

$$z_{m,j} = \begin{cases} x_1^{(m)} \text{ or } x_3^{(m)} & \text{if } m \in \mu, \\ y_1^{(m)} \text{ or } y_3^{(m)} & \text{if } m \notin \mu. \end{cases}$$

If $\operatorname{st} w = \beta_{m_0}$, then, obviously, $z_{m,j} = y_1^{(m)}$ for all $m \in M - \mu$ with $m > m_0$ (else, $\operatorname{st} w \geqslant \beta_m > \beta_{m_0}$). If $m_0 \in \mu$, then

$$\operatorname{st} w \leqslant \sup(\{\beta_m;\, m \notin \mu,\, m < m_0\} \cup \{a_m\}_{m \in \mu}).$$

Since $\beta_{m_0}$ is a regular cardinal, $\operatorname{st} w < \beta_{m_0}$ — a contradiction. Therefore $m_0 \notin \mu$.

If $\operatorname{st} w = a_{m_0}$, then, obviously, $z_{m,j} = y_1^{(m)}$ for all $m \in M - \mu$, and $z_{m,j} = x_1^{(m)}$ for all $m \in \mu$, $m > m_0$. Thus $m_0 \in \mu$, since, otherwise,

$$\operatorname{st} w \leqslant \sup\{a_m;\, m \in \mu,\, m < m_0\},$$

and so $\operatorname{st} w < a_{m_0}$.

If $m_0 = m_1$, then all $z_{m,j}$, $m \neq m_1$, are determined (as $x_1^{(m)}$ or $y_1^{(m)}$). We can choose $z_{m_1,j} = x_1^{(m_1)}$ or $x_3^{(m_1)}$ for $j = 0, 1, 2, \ldots, f(m_1)-1$, but at least once we must choose $x_3^{(m_1)}$. There exist just $2^{f(m_1)}-1$ such points $w$.

If $m_0 = m_2$, then all $z_{m,j}$, $m \neq m_1, m_2$, are determined, and we can choose $z_{m_2,j} = x_1^{(m_2)}$ or $x_3^{(m_2)}$, choosing at least once $x_3^{(m_2)}$. And, moreover, we choose arbitrarily $z_{m_1,j} = x_1^{(m_1)}$ or $x_3^{(m_1)}$. There exist just $(2^{f(m_2)}-1)\cdot 2^{f(m_1)}$ such points. Etc.

**Sufficiency.** Let $f \in N^M$ be such that for every increasing sequence $\mu$ there is a vertex $t(f, \mu)$ in $r(\mathscr{A})$ with the described properties. To prove that $f \in \mathscr{A}$ it suffices to find an element $h \in \mathscr{A}$ with $h = f$ on $\mu$ (since $\mathscr{A}$ is closed). We have $t(f, \mu) = (\{t_{m,j}\}, h, i)$ for some $h \in \mathscr{A}$ and $i \in \aleph$ — without loss of generality, $i = 0$. We prove that in this case $h = f$ on $\mu$. It suffices, clearly, to show that $t(f, \mu) = v(h, \mu)$. Indeed, then

$$2^{h(m_1)} - 1 = 2^{f(m_1)} - 1,$$
$$(2^{h(m_2)} - 1) \cdot 2^{h(m_1)} = (2^{f(m_2)} - 1) \cdot 2^{f(m_1)}, \ldots,$$

thus $h(m_i) = f(m_i)$.

Since $t(f, \mu)$ has the branch of length 2, each $t_{m,j}$ has, clearly, the branch of length 2 in $R_m$, i. e., $t_{m,j} = x_0^{(m)}$ or $y_0^{(m)}$. We want to show that $t_{m,j} = x_0^{(m)}$ iff $m \in \mu$.

(a) Let $t_{m_0,j_0} = x_0^{(m_0)}$. Then put $w = (\{z_{m,j}\}, h, 0)$, where $z_{m_0,j_0} = x_3^{(m_0)}$ and, for $(m, j) \neq (m_0, j_0)$, put

$$z_{m,j} = \begin{cases} x_1^{(m)} & \text{if } t_{m,j} = x_0^{(m)}, \\ y_1^{(m)} & \text{if } t_{m,j} = y_0^{(m)}. \end{cases}$$

Clearly, $w$ is in the branch of $t(f, \mu)$ and $\mathrm{st}w = \alpha_{m_0}$. Thus $m_0 \in \mu$.

(b) Analogously, if $t_{m_0,j_0} = y_0^{(m_0)}$, then we find $w$ in the branch of $t(f, \mu)$ with $\mathrm{st}w = \beta_{m_0}$ (here $z_{m_0,j_0} = y_3^{(m_0)}$). Thus $m_0 \notin \mu$.

This completes the proof.

**III.3. Representation in $\mathscr{W}_3$.**

*Every C-embeddable semigroup has a representation by products of algebras in $\mathscr{W}_3$.*

The proof is quite analogous to that of Theorem 4.

**III.4. Representation in $\mathscr{V}_{3,2}$.**

*Every C-embeddable semigroup has a representation by products of algebras in $\mathscr{V}_{3,2}$.*

The proof is quite analogous to the above one, concerning forests. The basic algebras are shown in Fig. 5.

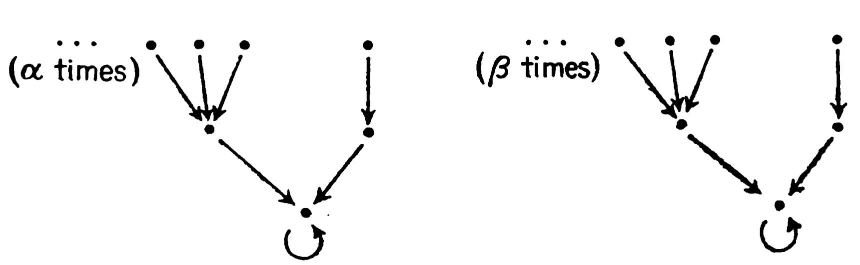


Fig. 5

**III.5. Representation by bipartite graphs.** Recall that a symmetric graph $(X, R)$ is *bipartite* if there exists a partition $X = A \cup B$ such that

$$R \subset (A \times B) \cup (B \times A).$$

By a *diameter of a graph G* we understand the least number $n$ such that any two points in a component of $G$ can be connected by a path of length at most $n$.

*Every C-embeddable semigroup can be represented by products of bipartite graphs of diameter* 3.

The construction is quite analogous to that for forests in III.2. The basic graphs are shown in Fig. 6.

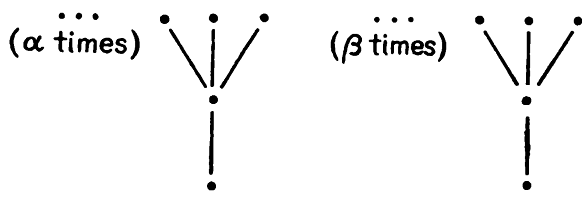


Fig. 6

## *IV. CANTOR-BERNSTEIN THEOREM*

**IV.1.** Following McKenzie [6] we say that a class $\mathscr{C}$ of graphs *fulfils the Cantor-Bernstein theorem* if, for $G_1, G_2, X, Y$ in $\mathscr{C}$,

$$G_1 \times X \simeq G_2 \text{ and } G_2 \times Y \simeq G_1 \quad \text{implies} \quad G_1 \simeq G_2.$$

In this section we show that this is true for the class $\mathscr{C}$ of all forests of length 1, all algebras in $\mathscr{W}_2$ and $\mathscr{V}_{n,1}$ (see II.5), and all bipartite graphs of diameter 2.

These are very special classes, of course, but notice that the results cannot be improved: if a class $\mathscr{C}$ fulfils the Cantor-Bernstein theorem, then no non-trivial group (in fact, no semigroup in which there exist $a, b, x, y$ with $a \neq b$, $ax = b$, $by = a$) can be represented by products of graphs in $\mathscr{C}$. Therefore, forests of length 2, algebras in $\mathscr{W}_3$ and $\mathscr{V}_{n,2}$, and bipartite graphs of diameter 3 do not fulfil the Cantor-Bernstein theorem.

**IV.2.** We show a general method for proving the Cantor-Bernstein theorem for classes of graphs. Notice that this method can be generalized to arbitrary relational structures, more generally, to categories $\mathscr{K}$ with the following properties:

(1) sums commute with finite products;

(2) each object is a sum of indecomposable objects (i. e., objects $A$ such that if $A \simeq A_1 \vee A_2$, then $A \simeq A_1$ or $A \simeq A_2$, and the injections agree).

Let $\mathscr{A}$ be a class of graphs, closed under sums and components (i. e. a graph belongs to $\mathscr{A}$ iff each of its components does). Denote by $\mathscr{A}_0$ the subclass of all connected graphs in $\mathscr{A}$. By an *order* $\leqslant$ on isomorphism types of $\mathscr{A}_0$ we mean a quasi-order on the class $\mathscr{A}_0$ with $G_1 \simeq G_2$ equivalent to $G_1 \leqslant G_2$ and $G_2 \leqslant G_1$. Recall that the *ACC* (*ascending chain condition*) on an order means that every set contains a minimal element. We use the ACC for quasi-orders in the obvious sense: the corresponding quotient-order has the ACC.

Definition 4. Let $\mathscr{A}$ be as above. An order $\leqslant$ on isomorphism types of $\mathscr{A}_0$ is said to be *productive* if it fulfils the ACC and, for $G_1, G_2, G_1'$ in $\mathscr{A}_0$,

(1) every component of $G_1 \times G_2$ is greater than or equal to both $G_1$ and $G_2$;

(2) if $G_1 \times G_2$ has a component isomorphic to $G_1$, and if $G_1' \geqslant G_1$, then also $G_1' \times G_2$ has a component isomorphic to $G_1'$.

THEOREM 5. *Let $\mathscr{A}$ be a class of graphs, closed under sums and components. If there exists a productive order on isomorphism types of $\mathscr{A}_0$, then $\mathscr{A}$ fulfils the Cantor-Bernstein theorem.*

Proof. Let $G_1, G_2, X$ and $Y$ in $\mathscr{A}$ be given with

$$G_1 \times X \simeq G_2 \quad \text{and} \quad G_2 \times Y \simeq G_1.$$

For every $K \in \mathscr{A}_0$ isomorphic to $a$ components of $G_1$ ($a > 0$), we find $a$ components of $G_2$ also isomorphic to $K$. Then, by symmetry,we see that $G_1 \simeq G_2$, since the two graphs have the same components with the same number of copies.

By the ACC there exists a minimal element $K_m$ in the set of all those components of $G_1$ which are less than or equal to $K$. Since $G_1 \simeq G_2 \times Y$, there must exist components $K_m'$ of $G_2$, and $K_m''$ of $Y$ such that $K_m$ is isomorphic to a component of $K_m' \times K_m''$. By condition (1) of Definition 4 we have $K_m \geqslant K_m'$ and $K_m \geqslant K_m''$. Analogously, since $K_m'$ is a component of $G_2 \simeq G_1 \times X$, there exist components $L$ of $G_1$, and $L'$ of $X$ such that $K_m'$ is isomorphic to a component of $L \times L'$. By condition (1) we have $K_m' \geqslant L$. But since $L$ is a component of $G_1$, and $L \leqslant K_m' \leqslant K_m \leqslant K$, we have $L \simeq K_m \simeq K_m'$ (by the minimality of $K_m$). Since $K_m$ is isomorphic to a component of $L \times L'$ and $L \simeq K_m \simeq K_m'$, $K_m$ is also isomorphic to a component of $K_m \times L'$. Now, by condition (2) of Definition 4, for each of the $a$ components $K_i$ of $G_1$ with $K_i \simeq K$, $K$ is isomorphic to a component of $K_i \times L'$. Since, moreover, for $i \neq j$ the subgraphs $K_i \times L'$ and $K_j \times L'$ are disjoint, we get $a$ distinct components of $G_2$ isomorphic to $K$. This completes the proof.

**IV.3. $\mathscr{V}_{n,1}$-algebras.** A connected algebra $A = (X, f)$ in $\mathscr{V}_{n,1}$ has a cycle $\{x_0, x_1, \ldots, x_{n-1}\}$ (not necessarily of pairwise distinct points) such that

$$X = \bigcup_{i=0}^{n-1} f^{-1}(x_i).$$

Put $a_i = \operatorname{card} f^{-1}(x_i)$; then the sequence $\{a_i\}_{i=0}^{n-1}$ is called the *type of* $A$, and the least $k > 0$ with $x_k = x_0$ is called the *order of* $A$ (it divides $n$ and fulfils $x_{k+1} = x_1, x_{k+2} = x_2, \ldots$). We also write $A[x_0, x_1, \ldots, x_{n-1}]$ to stress the cycle.

Denote by $\mathscr{A}_0$ the class of all connected algebras in the class $\mathscr{V}_{n,1}$. If $A[x_0, x_1, \ldots, x_{n-1}]$ and $B[y_0, y_1, \ldots, y_{n-1}]$ are in $\mathscr{V}_{n,1}$, then $A \times B$ has components

$$A \times B[(x_0, y_p), (x_1, y_{p+1}), \ldots, (x_n, y_{p+n})] \quad \text{for } p = 0, 1, \ldots, n-1$$

($p+i$ is modulo $n$). The type of such a component is $\{a_i \cdot b_{p+i}\}$, where $\{a_i\}$ and $\{b_i\}$ are the types of $A$ and $B$, respectively. Furthermore, the order of any component in $A \times B$ is the least common multiple of the orders of $A$ and $B$. Therefore, it is clear that the following ordering on the isomorphism types of $\mathscr{A}_0$ is productive: given $A$ and $B$ with orders $h$ and $k$ and types $\{a_i\}$ and $\{b_i\}$, respectively, put

$A \leqslant B$ iff $k$ is divided by $h$ and there exists $p = 0, 1, \ldots, n-1$ such that $a_{i+p} \leqslant b_i$, $i = 0, 1, \ldots, n-1$.

COROLLARY 1. *The class* $\mathscr{V}_{n,1}$ *fulfils the Cantor-Bernstein theorem.*

**IV.4. Forests of length 1.** Denote by $\mathscr{A}$ the class of forests of length 0 or 1. Assume that $A$ and $B$ are trees with length 1. Then $A \times B$ is a forest, all trees of which but one have length 0 and, if $T$ is a tree of $A \times B$ of length 1, then

$$\operatorname{card} T = \operatorname{card} A \cdot \operatorname{card} B.$$

Thus, the following order on isomorphism types of $\mathscr{A}_0$ is productive:

$A \leqslant B$ iff either $B$ has length 0 or $A$ and $B$ have length 1 and $\operatorname{card} A \leqslant \operatorname{card} B$.

COROLLARY 2. *Forests of length* 1 *fulfil the Cantor-Bernstein theorem.*

**IV.5. Bipartite graphs of diameter 2.** Consider the class $\mathscr{A}$ of bipartite graphs of diameter 2. Then $\mathscr{A}_0$, clearly, consists just of graphs $(X, R)$ with $R = (A \times B) \cup (B \times A)$ for a decomposition $X = A \cup B$. Denote the last graph by $[A, B]$ and put

$$[A, B] \leqslant [A', B']$$

iff

$$\max(\operatorname{card} A, \operatorname{card} B) \leqslant \max(\operatorname{card} A', \operatorname{card} B')$$

and

$$\min(\operatorname{card} A, \operatorname{card} B) \leqslant \min(\operatorname{card} A', \operatorname{card} B').$$

This is clearly an ACC order. Further notice that

$$[A, B] \times [A', B'] \simeq [A \times A', B \times B'] \vee [A \times B', A' \times B],$$

so that conditions (1) and (2) of Definition 4 are satisfied.

COROLLARY 3. *Bipartite graphs of diameter 2 fulfil the Cantor-Bernstein theorem.*

**IV.6. $\mathscr{W}_2$-algebras.** The class of $\mathscr{W}_2$-algebras also fulfils the Cantor-Bernstein theorem. But since this class is not closed under disjoint unions, we cannot use the above method and we shall proceed in a different way.

PROPOSITION. *$\mathscr{W}_2$-algebras form a class of graphs fulfilling the Cantor-Bernstein theorem.*

Proof. Let $A, B, X$, and $Y$ be $\mathscr{W}_2$-algebras with $A \times X \simeq B$ and $B \times Y \simeq A$, say $A = (Z_A, f_A)$ with $f_A^2 = \text{const}\, z_A$ (analogously $(Z_B, f_B)$, $z_B$; $(Z_X, f_X)$, $z_X$; and $(Z_Y, f_Y)$, $z_Y$). Denote by $L_A$ the set of all $x \in Z_A - \{z_A\}$ with $f_A(x) = z_A$ and $f_A^{-1}(x) \neq \emptyset$ (analogously $L_B$, $L_X$, and $L_Y$). If $L_A = \emptyset$, i. e., $A$ is a $\mathscr{W}_1$-algebra, then clearly $A \simeq B$. Assume that $L_A \neq \emptyset$ and put

$$\beta = \min\{\text{card} f_A^{-1}(x);\, x \in L_A\}.$$

We have $d \in L_A$ with $\text{card} f_A^{-1}(d) = \beta$. Since

$$A \simeq B \times Y \simeq X \times (A \times Y),$$

there exists $d' \in L_X$ with $\text{card} f_X^{-1}(d') \cdot \beta = \beta$: given an isomorphism

$$\varphi\colon A \to X \times A \times Y,$$

consider the point $\varphi(d) = (d', -, -)$ in $Z_X \times Z_A \times Z_Y$.

Now, for every $x \in L_A$, we have

$$\text{card} f_A^{-1}(x) = \text{card}(f_A \times f_X)^{-1}(x, d).$$

Since $A \times X \simeq B$, we obtain a one-one mapping $\varrho\colon L_A \to L_B$ such that

$$\text{card} f_A^{-1}(x) = \text{card} f_B^{-1}\big(\varrho(x)\big) \quad \text{for all } x \in L_A.$$

By symmetry of $A$ and $B$ we see (by the classical Cantor-Bernstein theorem) that $\varrho$ can be chosen to be bijective. Finally, $A \times X \simeq B$ and $B \times Y \simeq A$ imply that the number of points $x \in Z_A$ with $f_A^{-1}(x) = \emptyset$ and $f_A(x) = z_A$ equals the number of points $y \in Z_B$ with $f_B^{-1}(y) = \emptyset$ and $f_B(y) = z_B$. Hence $A \simeq B$ since we can extend $\varrho$ to an isomorphism.

## *V. SUMMARIZATION*

**V.1. Algebras.**

THEOREM 6. *Let $\mathscr{K}$ be a variety of unary algebras with one operation. Then one of the following cases 1° and 2° holds:*

1° *$\mathscr{K}$ contains $\mathscr{V}_{3,2}$ or $\mathscr{W}_3$, in which case every C-embeddable semigroup can be represented by products in $\mathscr{K}$;*

*2° $\mathscr{K}$ does not contain $\mathscr{V}_{3,2}$ and $\mathscr{W}_3$, in which case no non-trivial group can be so represented.*

*If $\mathscr{K}$ contains $\mathscr{V}_{4,3}$ or $\mathscr{W}_4$, then every commutative semigroup can be represented by products in $\mathscr{K}$.*

**Proof.** Case 1° holds by III.3-4 and II.6-7. Case 2° follows from the simple observation that every variety $\mathscr{K}$, containing neither $\mathscr{V}_{3,2}$ nor $\mathscr{W}_3$, must be contained in $\mathscr{W}_2$ or in $\mathscr{V}_{n,1}$ for some $n$. Then, by IV.3 and IV.6, $\mathscr{K}$ fulfils the Cantor-Bernstein theorem and, therefore, no non-trivial group has a representation by products in $\mathscr{K}$.

COROLLARY 4. *Given any type of universal algebras, every commutative semigroup has a representation by products of algebras of this type.*

**Proof.** We define an embedding $\Phi_\Delta$ of $\mathscr{W}_4$ into the class $\mathfrak{A}(\Delta)$ of algebras of type $\Delta$. Given $A = (X, f)$, where $f^4 = \text{const}\, a$, define an arbitrary $n$-ary operation $\omega\colon X^n \to X$ as follows: if $n = 0$, then $\omega = a$; if $n \neq 0$, then $\omega(x_0, x_1, \ldots, x_{n-1}) = f(x_0)$. It is easy to see that we obtain an algebra $\Phi_\Delta A$ in $\mathfrak{A}(\Delta)$ and that

$$\Phi_\Delta(A \times B) = \Phi_\Delta A \times \Phi_\Delta B.$$

Therefore, if $\{A_s; s \in S\}$ is a representation of a semigroup $S$ by products in $\mathscr{W}_4$, then $\{\Phi_\Delta A_s; s \in S\}$ is its representation in $\mathfrak{A}(\Delta)$.

**V.2. Forests.** A commutative semigroup $S$ can be represented by products of forests of length 3 — always (II.4), and of length 2 — if $S$ is $C$-embeddable (III.2).

No non-trivial group can be represented by products of forests of length 1 (IV.4).

**V.3. Bipartite graphs.** Every $C$-embeddable semigroup can be represented by products of bipartite graphs of diameter 3 (III.5). But no non-trivial group has a representation by bipartite graphs of diameter 2 (IV.5).

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