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ON CONFORMALLY SYMMETRIC RICCI-RECURRENT SPACES

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1. Introduction. An n-dimensional Riemannian space is said to be *of recurrent curvature* (briefly, a *recurrent space*) if its curvature tensor satisfies the condition (see [8] and [9])

$$R_{hijk,l} = c_l R_{hijk} \neq 0 \tag{1}$$

for some vector c_j, where the comma indicates covariant differentiation with respect to the metric.

Contracting (1) with g^{hk} we see that, for a recurrent space,

$$R_{ij,l} = c_l R_{ij}. \tag{2}$$

Spaces whose Ricci tensor satisfies (2) for some vector c_j, where $R_{ij} \neq 0 \neq c_j$, are called *Ricci-recurrent* [3].

Thus, every recurrent space with $R_{ij} \neq 0$ is Ricci-recurrent, but the converse, as we shall show, is, in general, not true.

According to Chaki and Gupta [2], an n-dimensional ($n > 3$) Riemannian space is said to be *conformally symmetric* if its Weyl's conformal tensor

$$C^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-2}(g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \\ + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{3}$$

satisfies

$$C^h{}_{ijk,l} = 0. \tag{4}$$

It follows easily from (3) and (4) that every conformally flat ($n > 3$) as well as every symmetric (in the sense of E. Cartan) Riemannian n-space ($n > 3$) is necessarily conformally symmetric.

Since a general canonical form for the metric of a conformally symmetric space is so far unknown, an interesting question arises of the existence of conformally symmetric spaces which are neither conformally flat nor symmetric in the sense of E. Cartan. Spaces of such type will be called *essentially conformally symmetric*.

Adati and Miyazawa [1] investigated conformally symmetric spaces which are simultaneously Ricci-recurrent. Their main results are the following:

(A) In a conformally symmetric Ricci-recurrent space the recurrence vector c_j is a gradient.

(B) If the Ricci tensor of a conformally symmetric space satisfies equation (2) for some non-zero vector c_j, then the following cases occur:

(a) the space is conformally flat and recurrent;

(b) the space is symmetric in the sense of E. Cartan and $R_{ij} = 0$;

(c) the scalar curvature vanishes and the recurrence vector is null.

Main aim of the present paper is a determination of a canonical form for the metric of a conformally symmetric Ricci-recurrent space. With help of this metric we shall also prove the existence of essentially conformally symmetric spaces.

2. Preliminary results. We start with a canonical form for the curvature tensor of a conformally symmetric Ricci-recurrent space. To that end we need several lemmas.

LEMMA 1. *If e_j and T_{ilm} are numbers satisfying*

$$e_i T_{jlm} + e_j T_{ilm} = 0, \tag{5}$$

then either each e_j is zero or each T_{ilm} is zero.

Proof. Suppose that one of the e's, say e_q, is not zero. Then (5) with $i = j = q$ gives $2e_q T_{qlm} = 0$, and, therefore, $T_{qlm} = 0$ for all l and m Putting $i = q$ in (5), we have $e_q T_{jlm} = 0$, whence $T_{jlm} = 0$ for all j, . and m.

LEMMA 2. *If e_j and P_{lhmjk} are numbers satisfying*

$$P_{lhmjk} = -P_{lhmkj}, \qquad 2e_i P_{lhmjk} + e_j P_{lhmik} + e_k P_{lhmji} = 0, \tag{6}$$

then either each e_j is zero or each P_{hijkl} is zero.

Proof. Suppose that one of the e's, say e_q, is not zero. Then (6) with $i = j = q$ gives $3e_q P_{lhmqk} = 0$ since $P_{lhmqq} = 0$, and, therefore, $P_{lhmqk} = 0$ for all l, h, m and k. Putting $i = q$ in the second equation of (6) and applying the first equation of (6), we have $2e_q P_{lhmjk} = 0$. Hence $P_{lhmjk} = 0$ for all l, h, m, j, k.

It has been proved ([4], Lemma 2) that, for a Ricci-recurrent space whose recurrence vector is a gradient, the following relations are satisfied:

$$R_{ij,lm} - R_{ij,ml} = 0, \tag{7}$$

$$R_{ri} R^r{}_{jlm} + R_{rj} R^r{}_{ilm} = 0, \tag{8}$$

$$R_{ri} R^r{}_j = \tfrac{1}{2} R R_{ij}, \tag{9}$$

$$R^{rs} R_{rs} = \tfrac{1}{2} R^2. \tag{10}$$

Moreover, since, for a general Ricci-recurrent space,

$$c_r R^r{}_j = \tfrac{1}{2} R c_j, \tag{11}$$

$$R_{,j} = R c_j, \tag{12}$$

(see [4], equation (13)), by Theorem (A) of Adati and Miyazawa, we have the following corollary:

COROLLARY 1. *In a conformally symmetric Ricci-recurrent space, relations* (7)-(12) *are satisfied.*

With help of Corollary 1 we can now give a short proof of the following theorem ([5], Theorem 3):

THEOREM 1. *The scalar curvature of a conformally symmetric Ricci--recurrent space vanishes and the recurrence vector is null.*

Proof. Differentiating (3) covariantly, summing over h and l, and taking into account (4) and the well-known formulas

$$R^r{}_{ijk,r} = R_{ij,k} - R_{ik,j} \quad \text{and} \quad R^r{}_{j,r} = \tfrac{1}{2} R_{,j},$$

we obtain

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)} (R_{,k} g_{ij} - R_{,j} g_{ik}),$$

which, in view of (2) and (12), yields

$$c_k R_{ij} - c_j R_{ik} = \frac{1}{2(n-1)} R (c_k g_{ij} - c_j g_{ik}). \tag{13}$$

Transvecting now (13) with R^{ik} and using (9), (10) and (11), we have

$$\frac{1}{4} R^2 c_j - \frac{1}{2} R^2 c_j = \frac{1}{2(n-1)} R \left(\frac{1}{2} R c_j - R c_j\right),$$

whence

$$(n-2) R^2 c_j = 0.$$

The last equation proves the first part of our theorem.

Substituting now $R = 0$ into (13), we obtain

$$c_k R_{ij} = c_j R_{ik}, \tag{14}$$

which, in view of (11), implies

$$c^r c_r R_{ij} = c_r R^r{}_i c_j = \tfrac{1}{2} R c_i c_j = 0.$$

This equation completes the proof.

LEMMA 3. *The Ricci tensor of a conformally symmetric Ricci-recurrent space is of the form*

$$R_{ij} = a c_i c_j, \quad \textit{where } a \neq 0, \tag{15}$$

and satisfies the condition

$$R_{ri}R^{r}{}_{jlm} = 0. \tag{16}$$

Proof. Since $c_j \neq 0$, we can find b^j such that $b^r c_r = 1$. Putting $E_i = b^r R_{ir}$ and making use of (14), we have

$$R_{ij} = E_i c_j, \tag{17}$$

which, in view of $R_{ij} = R_{ji}$, gives

$$E_i = ac_i, \quad \text{where } a = b^r b^s R_{rs} \neq 0.$$

The last equation, together with (17), leads immediately to (15).

Substituting now (15) into (8), we easily obtain

$$c_j c_r R^{r}{}_{ilm} + c_i c_r R^{r}{}_{jlm} = 0. \tag{18}$$

Writting $T_{ilm} = c_r R^{r}{}_{ilm}$, we can see that (18) is of the form (5). Hence, in virtue of Lemma 1, there is $c_r R^{r}{}_{jlm} = 0$, which together with (15) yield (16). Thus the proof of our lemma is complete.

LEMMA 4. *The curvature tensor of a conformally symmetric Ricci-recurrent space satisfies the condition*

$$c_l R_{jkhm} + c_h R_{jkml} + c_m R_{jklh} = 0. \tag{19}$$

Proof. Differentiating (3) covariantly and using (4), we have

$$R_{hijk,l} = \frac{1}{n-2}(g_{ij}R_{hk,l} - g_{ik}R_{hj,l} + g_{hk}R_{ij,l} - g_{hj}R_{ik,l}) - \\ - \frac{R_{,l}}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}), \tag{20}$$

which, in view of (7), yields $R_{hijk,lm} - R_{hijk,ml} = 0$.

By making use of the Ricci identity, this equation gives

$$R_{rijk}R^{r}{}_{hlm} + R_{hrjk}R^{r}{}_{ilm} + R_{hirk}R^{r}{}_{jlm} + R_{hijr}R^{r}{}_{klm} = 0. \tag{21}$$

As an immediate consequence of (20) and (2), with $R = 0$, we have

$$R^{h}{}_{ijk,l} = \frac{1}{n-2}c_l(g_{ij}R^{h}{}_{k} - g_{ik}R^{h}{}_{j} + \delta^h_k R_{ij} - \delta^h_j R_{ik}), \tag{22}$$

$$R_{hijk,l} = \frac{1}{n-2}c_l(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}). \tag{23}$$

Differentiating now (21) covariantly and taking into account formulae (22), (23) and (16), we get

$$R_{hl}R_{mijk} - R_{hm}R_{lijk} + R_{il}R_{hmjk} - R_{im}R_{hljk} + R_{jl}R_{himk} - \\ - R_{jm}R_{hilk} + R_{kl}R_{hijm} - R_{km}R_{hijl} = 0. \tag{24}$$

A cyclic permutation of h, l and m gives

$$(25)\qquad R_{lm}R_{hijk}-R_{lh}R_{mijk}+R_{im}R_{lhjk}-R_{ih}R_{lmjk}+R_{jm}R_{lihk}-$$
$$-R_{jh}R_{limk}+R_{km}R_{lijh}-R_{kh}R_{lijm}=0$$

and, furthermore,

$$(26)\qquad R_{mh}R_{lijk}-R_{ml}R_{hijk}+R_{ih}R_{mljk}-R_{il}R_{mhjk}+R_{jh}R_{milk}-$$
$$-R_{jl}R_{mihk}+R_{kh}R_{mijl}-R_{kl}R_{mijh}=0.$$

Adding (24), (25) and (26), we obtain

$$(27)\qquad 2(R_{il}R_{hmjk}+R_{im}R_{lhjk}+R_{ih}R_{mljk})+R_{jm}(R_{lihk}-R_{hilk})+$$
$$+R_{jl}(R_{himk}-R_{mihk})+R_{hj}(R_{milk}-R_{limk})+R_{kl}(R_{hijm}-R_{mijh})+$$
$$+R_{km}(R_{lijh}-R_{hijl})+R_{kh}(R_{mijl}-R_{lijm})=0.$$

Since

$$R_{lihk}-R_{hilk}=R_{lhik},\qquad R_{himk}-R_{mihk}=R_{hmik},$$
$$R_{milk}-R_{limk}=R_{mlik},\qquad R_{hijm}-R_{mijh}=R_{hmji},$$
$$R_{lijh}-R_{hijl}=R_{lhji},\qquad R_{mijl}-R_{lijm}=R_{mlji},$$

relation (27) can be written in the form

$$2(R_{il}R_{hmjk}+R_{im}R_{lhjk}+R_{ih}R_{mljk})+R_{jm}R_{lhik}+R_{jl}R_{hmik}+$$
$$+R_{hj}R_{mlik}+R_{kl}R_{hmji}+R_{km}R_{lhji}+R_{kh}R_{mlji}=0,$$

which, in virtue of (15), yields

$$(28)\qquad 2c_i(c_lR_{hmjk}+c_mR_{lhjk}+c_hR_{mljk})+c_j(c_lR_{hmik}+c_mR_{lhik}+c_hR_{mlik})+$$
$$+c_k(c_lR_{hmji}+c_mR_{lhji}+c_hR_{mlji})=0.$$

Putting

$$P_{lhmjk}=c_lR_{hmjk}+c_mR_{lhjk}+c_hR_{mljk},$$

we see that (28) is of the form (6). Since $c_j\neq 0$, Lemma 2 gives $P_{lhmjk}=0$, which completes our proof.

Now, with help of Lemma 4, we can follow step by step a proof of Walker (see [9], p. 45, and [8], p. 155) to show the following lemma:

LEMMA 5. *The curvature tensor of a conformally symmetric Ricci-recurrent space is of the form*

$$(29)\qquad R_{jkhm}=c_hc_kS_{mj}-c_hc_jS_{mk}+c_mc_jS_{hk}-c_mc_kS_{hj},$$

where $S_{ij}=S_{ji}=b^rb^sR_{rijs}$ *and* $b^rc_r=1$.

THEOREM 2. *Every conformally symmetric Ricci-recurrent space admits a null parallel vector field co-directional with the recurrence vector.*

Proof. Differentiating (15) covariantly, using (2) and (15) again, we get the equation

$$(ac_{i,k}+\tfrac{1}{2}c_i a_{,k}-\tfrac{1}{2}ac_i c_k)c_j+(ac_{j,k}+\tfrac{1}{2}c_j a_{,k}-\tfrac{1}{2}ac_j c_k)c_i = 0$$

which is, evidently, of the form (5). Hence, in virtue of Lemma 1,

$$ac_{j,k} = \tfrac{1}{2}ac_j c_k-\tfrac{1}{2}a_{,k}c_j. \tag{30}$$

Since $c_{j,k} = c_{k,j}$, equation (30) gives $a_{,k} = b^r a_{,r}c_k$, whence, again by (30) and by $a \neq 0$, we receive

$$c_{j,k} = Ac_j c_k \quad \text{for some } A. \tag{31}$$

Differentiating now covariantly (31) and making use of the Ricci identity and (31), we have

$$c_r R^r{}_{jkp} = (A_{,p}c_k-A_{,k}c_p)c_j. \tag{32}$$

Hence, making use of (16) and (15), and transvecting (32) by b^p, we obtain $A_{,k} = dc_k$, $d = b^r A_{,r}$. Since there is a function c such that $c_j = c_{,j}$, it follows that A is a function of c alone. Putting now

$$c_j = QB_j, \quad \text{where } Q = \exp\left(\int Adc\right), \tag{33}$$

we find easily that $c_{j,k} = AQc_{,k}B_j+QB_{j,k}$, whence, because of (31) and (33), we have $QB_{j,k} = 0$. In view of Theorem 1, $Q^2B^rB_r = c^r c_r = 0$, which completes the proof.

3. Main results. Now we proceed to main results of this paper. In this section each Latin index runs over $1,2,\ldots,n$, and each Greek index — over $2,3,\ldots,n-1$.

THEOREM 3. *In a conformally symmetric Ricci-recurrent space, coordinates can be chosen so that the metric takes the form*

$$ds^2 = \psi(dx^1)^2+k_{\lambda\mu}dx^\lambda dx^\mu+2\,dx^1dx^n, \tag{34}$$

$$\psi = \frac{1}{2(n-2)}\,C\exp\left(\int Qdx^1\right)k_{\lambda\omega}x^\lambda x^\omega+a_{\lambda\omega}x^\lambda x^\omega,$$

where $(k_{\lambda\mu})$ is a symmetric and non-singular matrix consisting of constants, $(a_{\lambda\mu})$ is a symmetric matrix of constants satisfying $k^{\lambda\omega}a_{\lambda\omega} = 0$ with $(k^{\lambda\omega}) = (k_{\lambda\omega})^{-1}$, $Q(x^1) \neq 0$ is a function of x^1 only, and $C \neq 0$ is a constant. Every space of dimension $n > 3$ and with metric of the form (34) is conformally symmetric and Ricci-recurrent.

Proof. Let be given a conformally symmetric Ricci-recurrent space. Walker ([8], p. 176-179, and [9], p. 51-54) proved that if a Riemannian space with the curvature tensor of the form (29) admits a null parallel vector field B^i satisfying (33), then one can choose coordinates so that the metric of the space can be written as

$$ds^2 = \varphi(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n, \tag{35}$$

where $k_{\lambda\mu}$ $(k_{\lambda\mu} = k_{\mu\lambda})$ are constants, $|(k_{\lambda\mu})| \neq 0$, and φ is independent of x^n.

In such a coordinate system the null parallel vector field is of the form $B^i = \delta^i_n$, whence, in view of (33), it follows that $c_j = Qg_{ij}B^i = Qg_{jn} = Q\delta^1_j$. The recurrence vector c_j is now a gradient of some function $C(x^1)$, and so Q is a function of x^1 only.

As one can easily verify,

$$(g^{ij}) = \begin{pmatrix} 0 & 0 \dots 0 & 1 \\ 0 & & 0 \\ \vdots & k^{\lambda\mu} & \vdots \\ 0 & & 0 \\ 1 & 0 \dots 0 & -\varphi \end{pmatrix},$$

and, in the metric (35), the only Christoffel symbols not identically zero are

$$\left\{ {\lambda \atop 11} \right\} = -\frac{1}{2}k^{\lambda\omega}\varphi_{.\omega}, \quad \left\{ {n \atop 11} \right\} = \frac{1}{2}\varphi_{.1}, \quad \left\{ {n \atop 1\gamma} \right\} = \frac{1}{2}\varphi_{.\gamma},$$

where the dot denotes partial differentiation with respect to coordinates.

Moreover, in view of the formula

$$R_{hijk} = \frac{1}{2}(g_{hk.ij} + g_{ij.hk} - g_{hj.ik} - g_{ik.hj}) + g_{pq}\left\{ {p \atop hk} \right\}\left\{ {q \atop ij} \right\} - g_{pq}\left\{ {p \atop hj} \right\}\left\{ {q \atop ik} \right\},$$

it follows ([8], p. 179) that the only components R_{hijk} not identically zero are those related to $R_{1\mu\lambda 1} = \frac{1}{2}\varphi_{.\mu\lambda}$.

It can be also found that $R_{11} = \frac{1}{2}k^{\beta\omega}\varphi_{.\beta\omega}$ and that all other components are identically zero.

Similarly, by an elementary but somewhat lengthy calculation, we can easily show that the only components of C_{hijk}, $R_{ij,k}$, $R_{hijk,l}$ and $C_{hijk,l}$ not identically zero are those related to

$$C_{1\lambda\mu 1} = \frac{1}{2}\varphi_{.\lambda\mu} - \frac{1}{2(n-2)}k_{\lambda\mu}(k^{\beta\omega}\varphi_{.\beta\omega}),$$

$$R_{11,1} = \frac{1}{2}k^{\beta\omega}\varphi_{.\beta\omega 1}, \quad R_{11,\lambda} = \frac{1}{2}k^{\beta\omega}\varphi_{.\beta\omega\lambda},$$

$$R_{1\gamma\lambda 1,1} = \frac{1}{2}\varphi_{.\gamma\lambda 1}, \quad R_{1\gamma\lambda 1,\mu} = \frac{1}{2}\varphi_{.\gamma\lambda\mu},$$

$$C_{1\lambda\mu 1,1} = \frac{1}{2}\varphi_{.\lambda\mu 1} - \frac{1}{2(n-2)} k_{\lambda\mu}(k^{\beta\omega}\varphi_{.\beta\omega 1}),$$

$$C_{1\lambda\mu 1,\gamma} = \frac{1}{2}\varphi_{.\lambda\mu\gamma} - \frac{1}{2(n-2)} k_{\lambda\mu}(k^{\beta\omega}\varphi_{.\beta\omega\gamma}).$$

Since our space is, by the assumption, conformally symmetric Ricci--recurrent, we have

$$(k^{\beta\omega}\varphi_{.\beta\omega})_{.p} = Q\delta_p^1(k^{\beta\omega}\varphi_{.\beta\omega}), \tag{36}$$

$$\varphi_{.\lambda\mu p} = \frac{1}{n-2} k_{\lambda\mu}(k^{\beta\omega}\varphi_{.\beta\omega})_{.p}. \tag{37}$$

As one can verify, (36) and (37) are satisfied if and only if

$$\varphi_{.\lambda\mu} = \frac{1}{n-2} C k_{\lambda\mu} \exp\left(\int Q dx^1\right) + 2a_{\lambda\mu}, \tag{38}$$

where $C \neq 0$ and $a_{\lambda\mu}$ $(a_{\lambda\mu} = a_{\mu\lambda})$ are constants such that $k^{\beta\omega}a_{\beta\omega} = 0$.
Hence

$$\varphi = \frac{1}{2(n-2)} C \exp\left(\int Q\, dx^1\right) k_{\lambda\mu}x^\lambda x^\mu + a_{\lambda\mu}x^\lambda x^\mu + \varkappa_\lambda x^\lambda + \xi, \tag{39}$$

$\varkappa_\lambda$ and ξ being functions of x^1 only.

Consider now a transformation ([8], p. 178) of the form

$$x'^\lambda = x^\lambda - k^{\lambda\mu}\sigma_\mu, \qquad x'^n = x^n + \varrho_\lambda x^\lambda + \eta$$

from $x^2, x^3, \ldots, x^n$ to new coordinates $x'^2, x'^3, \ldots, x'^n$, where ϱ_λ, σ_λ and η are functions of x^1 satisfying

$$\varrho_\lambda = \frac{1}{2}\int \varkappa_\lambda dx^1, \qquad \sigma_\lambda = \int \varrho_\lambda dx^1, \qquad \eta = \frac{1}{2}\int (\xi + k^{\beta\omega}\varrho_\beta\varrho_\omega) dx^1.$$

Transforming (35) and (39) and omitting the primes, we obtain (34) for the metric of a conformally symmetric Ricci-recurrent space.

Obviously, every Riemannian n-space $(n > 3)$ with a metric of the form (34) is conformally symmetric and Ricci-recurrent. The last remark completes the proof.

LEMMA 6. *A Riemannian space with metric* (34) *is of recurrent curvature if and only if all* $a_{\lambda\mu}$ *in* (34) *are zero.*

Proof. If all $a_{\lambda\mu}$ are zero, then (34) yields

$$\varphi_{.\lambda\mu j} = Q\delta_j^1\varphi_{.\lambda\mu} = c_j\varphi_{.\lambda\mu},$$

whence, in virtue of $R_{1\lambda\mu 1,j} = \frac{1}{2}\varphi_{.\lambda\mu j}$ and $R_{1\lambda\mu 1} = \frac{1}{2}\varphi_{.\lambda\mu}$ we infer immediately that $R_{1\lambda\mu 1,j} = c_j R_{1\lambda\mu 1}$. Hence the space is of recurrent curvature.

Conversely, the recurrence condition (1) gives

$$\tfrac{1}{2}\psi_{.\lambda\mu j} = R_{1\lambda\mu 1,j} = b_j R_{1\lambda\mu 1} = \tfrac{1}{2} b_j \psi_{.\lambda\mu},$$

whence

$$\psi_{.\lambda\mu j} = b_j \psi_{.\lambda\mu} \tag{40}$$

for some non-zero vector b_j. But we infer from (34) that

$$\psi_{.\lambda\mu j} = \frac{1}{n-2} CQ\delta_j^1 k_{\lambda\mu} \exp\left(\int Qdx^1\right),$$

which, together with (40), implies

$$\frac{1}{n-2} CQ\delta_j^1 k_{\lambda\mu} \exp\left(\int Qdx^1\right) = \frac{1}{n-2} Cb_j k_{\lambda\mu} \exp\left(\int Qdx^1\right) + 2b_j a_{\lambda\mu}. \tag{41}$$

Transvecting now (41) with $k^{\lambda\mu}$ we see that $b_j = Q\delta_j^1$. The last equation, in view of (41), yields finally $b_j a_{\lambda\mu} = 0$, which completes the proof of our lemma.

Theorem 3 and Lemma 6 yield

COROLLARY 2. *For each $n > 3$, there exist n-dimensional conformally symmetric Ricci-recurrent spaces which are not of recurrent curvature.*

It can be easily verified that every conformally flat Ricci-recurrent space is of recurrent curvature. Since a Ricci-recurrent space cannot be symmetric in the sense of E. Cartan, we have, in view of Corollary 2,

COROLLARY 3. *There exist essentially conformally symmetric spaces.*

Remark. As a generalization of the concept of a Ricci-recurrent space, Roy Chowdhury [7] investigated n-dimensional Riemannian spaces whose Ricci-tensors satisfy the relation

$$R_{ij,lm} = \bar{a}_{lm} R_{ij} \neq 0$$

for some tensor $\bar{a}_{ij}$. Spaces of this kind are called *second-order Ricci-recurrent* or, briefly, *2-Ricci-recurrent*.

The present author established [6] that every conformally symmetric 2-Ricci-recurrent space is a Ricci-recurrent space with the necessarily vanishing scalar curvature.

Since, in view of (34),

$$\bar{a}_{ij} = c_{i,j} + c_i c_j = Q_{.j}\delta_i^1 + Q^2 \delta_i^1 \delta_j^1$$

is zero if and only if $Q = 1/(C^* + x^1)$ (C^* is a constant), we have (34) with $Q \neq 1/(C^* + x^1)$ for the metric of a conformally symmetric 2-Ricci-recurrent space.

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