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ON AFFINE TRANSFORMATIONS OF BANACHABLE BUNDLES

BY

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1. Introduction and main results. Let $l = (E, B, p)$ be a banachable vector bundle (v.b, for short) of fibre type F, provided with a linear connection c. A v.b-automorphism (F, id_B) of l is said to be an *affine transformation* of l if it commutes with the parallel displacement T_a along any curve a of B, i.e.

$$T_a \circ F|p^{-1}(a(0)) = F \circ T_a.$$

If $L(E) = (P(E), GL(F), B, \pi)$ is the principal bundle (p.b) of frames of l, the linear connection c determines a connection ω in $L(E)$ and conversely. A $GL(F)$-B-automorphism $(f, \mathrm{id}_{GL(F)}, \mathrm{id}_B)$ of $L(E)$ is said to be *connection preserving* if $f^*\omega = \omega$.

Hence, with respect to c and ω, we have the following

THEOREM. *A $GL(F)$-B-automorphism of $L(E)$ is connection preserving if and only if it is induced by an affine transformation of l.*

We say that a diffeomorphism F of B is an *affine transformation* of B if the tangent map TF is an affine transformation in the above-defined sense. Thus, for the sake of completeness, we state the following

COROLLARY. *F is an affine transformation of B if and only if the corresponding frame bundle automorphism is connection preserving.*

The previous results remind the finite-dimensional tangent bundle analogue ([3], Proposition 1.4, p. 228), which is the motivation of the present note. However, the arbitrary v.b-framework, as well as the infinite-dimensional context, require techniques of the v.b-theory as in [1], [2] and [4]. We also note that further refinements of the theorem, involving the so-called *fundamental form* ([3], Proposition 1.3, p. 226), are not feasible here.

The proof of the theorem is based particularly on the following result which appears to be known under various forms (e.g., [3], p. 225-226) but not explicitly written down, at least in this approach.

PROPOSITION. *Let $L = (P, G, B, \pi)$ and $L' = (P', G', B', \pi')$ be two p.b's with connections ω and ω', respectively. If (f, φ, h) is a p.b-morphism of L into L', then the following conditions are equivalent:*

(i) $f^*\omega' = \tilde{\varphi}\omega$;

(ii) $f \circ \tau_a = \tau'_{a'} \circ f | \pi^{-1}(a(0))$.

Here $\tilde{\varphi}$ is the Lie algebra homomorphism induced by φ, and τ_a (respectively, $\tau'_{a'}$) is the parallel displacement along a (respectively, $a' = h \circ a$).

2. Preliminaries. Throughout this note we follow mainly the notation and terminology of [1]-[4]. Manifolds and bundles are modelled on Banach spaces and, for the sake of simplicity, differentiability is of class C^∞ (smoothness).

A *connection* in $L = (P, G, B, \pi)$ is a G-splitting of the exact sequence of v.b's,

$$0 \to P \times \mathbf{G} \to TP \xrightarrow{T\pi!} \pi^*(TB) \to 0,$$

where $\mathbf{G}$ is the Lie algebra of G, $\pi^*(TB)$ the pull-back of TB by π, and $T\pi!$ the v.b-morphism defined by the universal property of pull-backs. Hence, there exists a G-v.b-morphism $C: \pi^*(TB) \to TP$ such that $T\pi! \circ C = \mathrm{id}_{\pi^*(TB)}$ (cf. [7] and [10]). Equivalently, C can be given by a differential $\mathbf{G}$-valued 1-form ω on P, satisfying the well-known properties ([3], p. 64, and [9]).

If L and L' are p.b's with connections C and C', respectively, and (f, φ, h) is a p.b-morphism of L into L', then we have

LEMMA 1 (Vassiliou [9]). *The following conditions are equivalent:*

(i) $f^*\omega' = \tilde{\varphi}\omega$;

(ii) $C' \circ (f \times Th) = Tf \circ C$;

(iii) $Tf(u^v) = (Tf(u))^{v'}$, $u \in TP$;

(iv) $Tf(u^h) = (Tf(u))^{h'}$, $u \in TP$.

Here the exponents v and h denote vertical and horizontal parts of tangent vectors, respectively, after the decomposition $TP = HP \oplus VP$ with $HP = \mathrm{Im}(C)$.

Similarly, a *linear connection* in $l = (E, B, p)$ is a splitting of the exact sequence of v.b's,

$$0 \to VE \to TE \xrightarrow{Tp!} p^*(TB) \to 0,$$

so that the *connection map* $D: TE \to E$ is linear on the p_*-fibres (cf. [12]). We denote by c the right splitting map and we have

LEMMA 2 (Penot [5], p. 40). *If l and l' are two v.b's with linear connections c and c', respectively, and (F, h) is a v.b-morphism of l into l', then the following conditions are equivalent:*

(i) $TF\circ c = c'\circ(F\times Th)$;

(ii) $F\circ D = D'\circ TF$;

(iii) $F\circ T_a = T'_{a'}\circ F|p^{-1}(a(0))$.

3. Parallel displacement in principal bundles. Let L and ω be as above. The *horizontal lifting* a_p of $a: I\to B$ with the initial condition $p\in P$ is given by $a_p(t) = \beta(t)s(t)$, where $\beta(t)$ is an arbitrary lifting of a through p, and $s(t)$ is determined by the equation ([3], p. 68-70, [5], p. 7, 34)

$$\dot{s}(t) = -\omega(\dot{\beta}(t))s(t), \quad s(0) = e.$$

Equivalently, we write

$$\omega(\dot{\beta}(t)) = T_{s(t)}R_{s(t)^{-1}}(\dot{s}(t)) \tag{3.1}$$

with R_g denoting the right translation on G.

We are in a position to give

Proof of the Proposition. (i) $\Rightarrow$ (ii). Since $\dot{a}_p(t)$ is horizontal for every $p\in P$, Lemma 1 implies that $(f\circ a)^{\cdot}(t)$ is a horizontal vector. Hence $f\circ a_p$ is the horizontal lifting of a' through $f(p)$, and so

$$\tau'_{a'}(f(p)) = (f\circ a_p)(1) = (f\circ\tau_a)(p) \quad \text{for every } p\in P.$$

(ii) $\Rightarrow$ (i). Let p and $u\in T_pP$ be arbitrarily chosen, and let β be a smooth curve with $\beta(0) = p$ and $\dot{\beta}(0) = u$. Setting $a = \pi\circ\beta$, we have

$$(f\circ\tau_a)(p) = f(a_p(1)) = f(\beta(1))\varphi(s(t)).$$

On the other hand, the horizontal lifting $a'_{f(p)}(t)$ is given by

$$a'_{f(p)}(t) = (f\circ\beta)(t)s'(t),$$

where $s'(t)$ is determined by

$$\dot{s}'(t) = -f^*\omega'(\dot{\beta}(t))s'(t), \quad s'(0) = e'.$$

Thus $(\tau'_{a'}\circ f)(p) = (f\circ\beta)(1)s'(t)$ and, by assumption,

$$s'(0) = \varphi(s(0)), \quad s'(1) = \varphi(s(1)). \tag{3.2}$$

If now $t_0\in(0,1)$ and $\gamma = a|[0, t_0]$, then the uniqueness of horizontal liftings yields $\gamma_p(t) = a_p(t)$ for $t\in[0, t_0]$. Thus $f\circ\tau_\gamma(p) = f(\beta(t_0))s(t)$ and, similarly, for $\gamma' = h\circ\gamma$, $\tau'_{\gamma'}(f(p)) = f(\beta(t_0))s'(t_0)$. The last equalities along with the assumption yield $s'(t_0) = \varphi(s(t_0))$ and, in virtue of (3.2), we get

$$s'(t) = \varphi(s(t)), \quad t\in I. \tag{3.3}$$

Differentiating both sides of (3.3) and taking into account the routinely checked equation

$$T_{\varphi(s(t))}R'_{\varphi(s(t))^{-1}}\circ T_{s(t)}\varphi = \tilde{\varphi}\circ T_{s(t)}R_{s(t)^{-1}},$$

we conclude that $(f^*\omega')(\dot{\beta}(t)) = \tilde{\varphi}\omega(\dot{\beta}(t))$ or, for $t = 0$, $(f^*\omega')_p(u) = \tilde{\varphi}(\omega_p(u))$. Since p and u were arbitrarily chosen, the proof is complete.

Remarks. (i) Let us denote by Φ_p the holonomy group at p (cf. [3]) and assume that a is a closed curve of B. Then condition (ii) of the Proposition leads to $\varphi(\Phi_p) \subset \Phi'_{f(p)}$ for every $p \in P$. Indeed, if $g \in \Phi_p$, then there is a horizontal curve γ joining p and pg. Hence, for the closed curve $a = \pi\circ\gamma$ we have

$$f(\tau_a(p)) = f(p)\varphi(g) \quad \text{and} \quad \tau'_{a'}(f(p)) = f(p)g' \quad \text{for some } g \in \Phi'_{f(p)};$$

hence, $g' = \varphi(g)$.

(ii) Furthermore, if G is abelian and $(f, \mathrm{id}_G, \mathrm{id}_B)$ is a p.b-automorphism, then the conditions of the Proposition are equivalent to

$$\exp\Big[-\int_0^1 \omega(\dot{\beta}(\tau))\,d\tau\Big] = \exp\Big[-\int_0^1 (f^*\omega)(\dot{\beta}(\tau))\,d\tau\Big]$$

for every smooth curve β in P. Indeed, if condition (i) holds, then the equality above is clear. Conversely, for arbitrary p and $u \in T_pP$ we choose β with $\beta(0) = p$ and $\dot{\beta}(0) = u$. Then $\beta(t)s(t)$ is the horizontal lifting of $\pi\circ\beta$ with the initial condition p if $s(t)$ satisfies (3.1). If G is abelian, then, in particular,

$$s(t) = \exp\Big[-\int_0^1 \omega(\dot{\beta}(\tau))\,d\tau\Big]$$

(cf. [8]). On the other hand, $f(\beta(t))\sigma(t)$ is a horizontal lifting of $\pi\circ\beta$ if $\sigma(t)$ satisfies the analogous equation

$$\sigma(t) = \exp\Big[-\int_0^1 (f^*\omega)(\dot{\beta}(\tau))\,d\tau\Big].$$

Thus $\sigma(t) = s(t)$, which implies that $(f^*\omega)(\dot{\beta}(t)) = \omega(\dot{\beta}(t))$ or, for $t = 0$, $(f^*\omega)_p(u) = \omega_p(u)$, which completes the proof.

4. Affine transformations. Let l be a v.b of fibre type F and let $L(E) = (P(E), GL(F), B, \pi)$ be the corresponding p.b of frames. The morphism

$$\varrho: P(E)\times F \to E$$

given by $\varrho(r_x, u) := r(u)$, where $r_x = (x, r) \in \{x\} \times \mathrm{Lis}(F, E_x)$, determines on E the structure of an associated bundle. In particular, $p(\varrho(r_x, u)) = \pi(r_x) = x$ (cf. [2], p. 66 and 68, for details). Obviously, a p.b-endomorphism (f, φ, h) of $L(E)$ induces a unique v.b-morphism (F, h) of l into itself, so that for every $(r_x, u) \in P(E) \times F$ we have

$$F(\varrho(r_x, u)) = \varrho(f(r_x), u). \tag{4.1}$$

Let now ω be a connection in $L(E)$ with the corresponding splitting map C. It determines a unique linear connection, say c, in l by

$$c(e, v) = T^1_{(r_x,u)}\varrho(C(r_x, v)) \tag{4.2}$$

for every $(e, v) \in E \times_B TB$ with $e = r(u)$ for some $(r_x, u) \in P(E) \times F$ (cf. [6]). Note that the converse is true as well.

We need the following

LEMMA 3. *Let l be a v.b of fibre type F. Let also (f, φ, h) be any p.b-endomorphism of $L(E)$. If (F, h) is the induced v.b-morphism of l into itself, then the following conditions are equivalent:*

(i) $F \circ T_a = T_{a'} \circ F | p^{-1}(a(0))$;

(ii) $f \circ \tau_a = \tau_{a'} \circ f | \pi^{-1}(a(0))$.

Proof. Assume that (i) holds. Then, for every $(r_x, v) \in L(E) \times_B TB$, setting $e = r(u)$, by (4.2) we get

$$(TF \circ c)(e, v) = (T_e F \circ T^1_{(r_x,u)}\varrho \circ C)(r_x, v). \tag{4.3}$$

Similarly, since $F(e) = \varrho(f(r_x), u)$, we obtain

$$(c \circ (F \times Th))(e, v) = (T^1_{(f(r_x),u)}\varrho \circ C \circ (f \times Th))(r_x, v). \tag{4.4}$$

Differentiating (4.1) we transform (4.3) into

$$(TF \circ c)(e, v) = (T^1_{(f(r_x),u)}\varrho \circ T_{r_x} f \circ C)(r_x, v). \tag{4.5}$$

Hence, (4.4) and (4.5) along with the assumption imply

$$(T^1\varrho \circ C \circ (f \times Th))(r_x, v) = (T^1\varrho \circ Tf \circ C)(r_x, v).$$

Since $T^1\varrho$ is an injection on each horizontal subspace (cf. [11]), the last equality implies (ii) in virtue of Lemma 1 and the Proposition.

Conversely, we show that (ii) implies (i) using the same equalities in a reverse way.

We are in a position to give now

Proof of the Theorem. For simplicity, we denote by f the $GL(F)$-B-automorphism. Then f determines the v.b-automorphism $F := (F, \mathrm{id}_B)$ of l which, by Lemma 3, is an affine transformation. We induce on $L(E)$

the map F' given by

$F'(r_x) = (x, F_x \circ r)$ with $F_x := F|E_x$ and $r_x = (x, r) = \{x\} \times \mathrm{Lis}(F, E_x)$.

First, we check the smoothness of F' using the local structure of $L(E)$ (cf. [1], p. 12 and 21, for a more general setting). For a given chart (U, φ) of B we denote by (Ψ, φ, U) the corresponding v.b-chart of l with

$$\Psi : p^{-1}(U) \xrightarrow{\cong} \varphi(U) \times F,$$

and by (L_φ, φ, U) the corresponding chart of $L(E)$ with

$$L_\varphi : L(E)|U \xrightarrow{\cong} \varphi(U) \times \mathrm{Lis}(F).$$

Then the local representative of (F, id_B) is $(L_\varphi \circ F' \circ L_\varphi^{-1}, \mathrm{id}_{\varphi(U)})$, and we show it is a local v.b-automorphism. Indeed, the following diagram is commutative:

$$\begin{array}{ccc} \varphi(U) \times \mathrm{Lis}(F) & \xrightarrow{L_\varphi \circ F' \circ L_\varphi^{-1}} & \varphi(U) \times \mathrm{Lis}(F) \\ \downarrow & & \downarrow \\ \varphi(U) & \xrightarrow{\quad\quad \mathrm{id} \quad\quad} & \varphi(U) \end{array}$$

Furthermore, we should define a C^∞-morphism

$$(L_\varphi \circ F' \circ L_\varphi^{-1})^{\#} : \varphi(U) \to \mathrm{Lis}\big(\mathrm{Lis}(F), \mathrm{Lis}(F)\big)$$

so that

$$(L_\varphi \circ F' \circ L_\varphi^{-1})\big(\varphi(x), l\big) = \big(\varphi(x), (L_\varphi \circ F' \circ L_\varphi^{-1})^{\#}_{\varphi(x)}(l)\big)$$

for every $\varphi(x) \in \varphi(U)$ and $l \in \mathrm{Lis}(F)$. By a routine checking, for each $l \in \mathrm{Lis}(F)$ we have

$$(L_\varphi \circ F' \circ L_\varphi^{-1})^{\#}_{\varphi(x)}(l) = (\Psi \circ F \circ \Psi^{-1})^{\#}_{\varphi(x)} \circ l.$$

Thus the morphism

$$\varphi(U) \ni \varphi(x) \to (L_\varphi \circ F' \circ L_\varphi^{-1})^{\#}_{\varphi(x)}(l) \in \mathrm{Lis}(F)$$

is smooth. Hence, in virtue of [7], Lemme 3.1, $(L_\varphi \circ F' \circ L_\varphi^{-1})$ is C^∞, and this proves the smoothness of F'.

F' is a G-morphism, since for every $g \in GL(F)$ we have

$$F'(r_x g) := F'(x, r \circ g) = \big(x, F_x \circ (r \circ g)\big) = F'(r_x) g.$$

Finally, we check that $F' = f$ as follows: f can be written as

$$f(r_x) = (x, r').$$

Since now $\varrho(r_x, u) = \varrho((x, r), u) = r(u)$, we have

$$F(\varrho(r_x, u)) = F(r(u)) \quad \text{and} \quad \varrho(f(r_x, u)) = \varrho((x, r'), u) = r'(u).$$

Hence, in virtue of (4.1), we get $F(r(u)) = r'(u)$. Since for the pair $r_x = (r, u)$ we have $F(r(u)) = F_x(r(u))$, we conclude that $F' = f$.

Conversely, assume that f is induced by an affine transformation F. The v.b-morphism F' corresponding to f is exactly F. Thus F' is an affine transformation. In virtue of the Proposition and Lemma 1 we conclude that f is connection preserving.

The proof of the Corollary is clear.

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