COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 1

ABSOLUTE FIXED-POINT SETS IN COMPACTA

BY

JOHN R. MARTIN (SASKATOON, SASKATCHEWAN)

1. Introduction. A subset A of a topological space X is said to be a *fixed-point set* of X if there is a map $f\colon X \to X$ such that $f(x) = x$ iff $x \in A$. In [2] the author defines a space X to be an *absolute fixed-point set* relative to a class Q of topological spaces if $X \in Q$ and whenever X is embedded as a closed subset of a Q-space Z, then X is a fixed-point set of Z. It is shown in [2] that, for many classes Q, the class of absolute fixed-point sets relative to Q contains the class of absolute retracts relative to Q and lies in the class of connected, locally connected Q-spaces.

In this paper* we study the concept of an absolute fixed-point set for the class Q of compact metric spaces. In particular, it is shown that a finite-dimensional compactum X is an absolute retract iff it is an absolute fixed-point set and an absolute neighborhood retract. If, in addition, X is planar or 1-dimensional, then X is an absolute retract iff X is an absolute fixed-point set. We give an example of a contractible compactum X which is locally n-connected for all $n = 0, 1, 2, \ldots$, and which is not an absolute fixed-point set.

2. Absolute fixed-point sets. We shall confine our attention to compact metric spaces and we shall adopt the notation used in [1]. In particular, we shall let AR (ANR) denote the class of absolute retracts (absolute neighborhood retracts) relative to the class of compact metric spaces.

Definition. A compactum X is an *absolute fixed-point set* (or an AFS-space) if whenever X is embedded as a subset of a compactum Z, then X is a fixed-point set of Z.

The results of [2] yield the following theorem:

THEOREM 1. *Every AFS-space is a Peano continuum.*

We shall find the following concept useful in the sequel. Let C be a compact subset of a Hilbert space E^ω and let A be a compact segment in E^ω which is disjoint from C.

* The research of this article was supported in part by the National Research Council of Canada (Grant A8205).

Consider the disjoint union $A\cup(C\times[0,1))$ of A and the product space $C\times[0,1)$. Let h be a one-to-one mapping from $A\cup(C\times[0,1))$ onto a continuum in E^ω such that the following properties are satisfied:

(i) $h(z) = z$ if $z\in A$.

(ii) $h(z,0) = z$ for all $z\in C$.

(iii) $h|C\times[0,1)$ is a homeomorphism.

(iv) For each $z\in C$, $h(\{z\}\times[0,1))\cup A$ is homeomorphic to the closure of the curve in the plane E^2 whose equation is $y = \sin\pi/x$ for $0<x\leqslant 1$.

The image of $A\cup(C\times[0, 1))$ under h is said to be the *cap* CA of C and A. To facilitate notation, we shall view the cap CA as the set $A\cup(C\times[0,1))$ together with an assigned topology which makes the function h an isometry.

PROPOSITION 1. *If X is a locally n-connected AFS-space, then X is n-connected.*

Proof. From Theorem 1 it follows that every AFS-space is 0-connected. Hence suppose that n is some positive integer, and X is a locally n-connected AFS-space which is not n-connected. Then there is a map g from the n-dimensional sphere S^n into X such that g is not homotopic to a constant map. Let Y denote the cap of $g(S^n)$ and a segment A. Form the compact metric space Z by taking the disjoint union of X and Y, and then identifying the set $g(S^n)$ in X with the set $g(S^n)\times\{0\}$ in Y. Since X is an AFS-space, there is a map $f\colon Z\to Z$ whose fixed-point set is precisely X.

If $f(A)\cap A\neq\emptyset$, then $f(A)\subset A$ and, consequently, f has a fixed point in A. Thus we must have $f(Y)\subset Z-A$, and hence $f(Z)\subset Z-A$. Since $g(S^n)\times\{0\}$ is a retract of $Y-A$, there is a retraction $r\colon Z\to X$ mapping Z onto X.

Let $p\in A$ and let U be a neighborhood of $r(p)$ in X. Since X is locally n-connected, there is a neighborhood V of $r(p)$ contained in U such that every map $\varphi\colon S^n\to V$ is homotopic to a constant map in U^{S^n}. Let W be a neighborhood of p in Z such that $r(W)\subset V$. Then, for some t_1 in $[0,1)$,

$$g(S^n)\times\{t_1\}\subset W.$$

Hence the map $\psi\colon S^n\to V$, defined by $\psi(z) = r(g(z),t_1)$ for all $z\in S^n$, is homotopic to a constant map. This is a contradiction since the maps $\psi, g\colon S^n\to X$ clearly belong to the same homotopy class.

Suppose that X is an AFS-space and C is an n-dimensional sphere which is a retract of X. Using the notation from the proof of Proposition 1, we let $g\colon S^n\to X$ be an embedding mapping S^n onto C. Then we obtain a retraction $r\colon Y\to C\times\{0\}$ mapping the cap Y onto $C\times\{0\}$. As in the proof of Proposition 1, we obtain a contradiction. Consequently, we obtain the following theorem:

THEOREM 2. *An AFS-space cannot contain an n-dimensional sphere as a retract.*

COROLLARY 1. *The 1-dimensional AFS-spaces coincide with the 1-dimensional AR-spaces.*

COROLLARY 2. *If X is planar, then X is an AR-space iff X is an AFS-space.*

Proof of Corollary 1. Let X be a 1-dimensional AFS-space. By Theorems 1 and 2, X is a 1-dimensional locally connected continuum which does not contain a simple closed curve as a retract. It follows that X contains no simple closed curves and is therefore a dendrite. Since the 1-dimensional AR-spaces coincide with the dendrites (see [1], p. 138), Corollary 1 follows.

Proof of Corollary 2. Let X be a subset of the plane E^2 which is an AFS-space. By Theorems 1 and 2, X is a locally connected planar continuum which does not contain a simple closed curve as a retract. It follows that X cannot separate E^2. Therefore, X is an AR-space (see [1], p. 132), and Corollary 2 follows.

THEOREM 3. *A finite-dimensional compactum X is an AR-space iff X is an AFS-space and X is an ANR-space.*

Proof. Suppose that X is a finite-dimensional compactum which is an AFS-space and an ANR-space. Then X is a locally contractible finite-dimensional AFS-space and, by Proposition 1, it follows that X must be contractible, and hence an AR-space [1]. Since every AR-space is both an AFS-space and an ANR-space, Theorem 3 follows.

Example. Let a_k $(k = 1, 2, \ldots)$ denote the point in a Hilbert space E^ω given by the formula

$$a_k = \left(\frac{2k+1}{2k(k+1)}, 0, 0, 0, \ldots\right),$$

and let a_0 denote the origin of E^ω. Let S_k^n denote the n-dimensional sphere in E^ω consisting of all points $x = (x_i)$ such that

$$d(x, a_k) = \frac{1}{2k(k+1)}$$

and such that $x_i = 0$ for $i > n+1$. It is well known (see [1], p. 31) that the set

$$Y = \{a_0\} \cup \bigcup_{k=1}^{\infty} S_k^k$$

is LC^∞. Hence, if X denotes the cone over Y with vertex p, then X is a contractible compactum which is LC^∞. We now show that X is not an AFS-space by constructing a compactum Z containing X such that X is not a fixed-point set of Z.

Let A_k $(k = 1, 2, \ldots)$ denote the segment in E^ω with endpoints

$$\left(\frac{1}{k+1}, \frac{1}{k(k+1)}, 0, 0, 0, \ldots\right) \quad \text{and} \quad \left(\frac{1}{k}, \frac{1}{k(k+1)}, 0, 0, 0, \ldots\right).$$

Let C_k^n denote a cap for S_k^n and A_k such that diameter

$$\operatorname{diam} C_k^n < \frac{2}{k(k+1)}.$$

Form the compact metric space Z obtained by taking the disjoint union $X \cup \bigcup_{k=1}^{\infty} C_k^k$ and then identifying each S_k^k in Y with $S_k^k \times \{0\}$ in C_k^k $(k = 1, 2, \ldots)$.

Suppose that $f\colon Z \to Z$ is a mapping whose fixed-point set is precisely X. Let V be a neighborhood of a_0 such that $f(V) \subset Z - \{p\}$. Since V contains infinitely many of the sets of the form C_k^k, there is an integer j such that $f(C_j^j) \subset Z - \{p\}$. Since no point of A_j remains fixed under f, we have $f(C_j^j) \subset Z - A_j$. Consequently, since $S_j^j \times \{0\} = S_j^j$ is a retract of $Z - (A_j \cup \{p\})$, there is a retraction mapping C_j^j onto $S_j^j \times \{0\}$. Then, as in the proof of Proposition 1, we obtain a contradiction.

A similar argument can be used to show that if X denotes the cone over the set $\{a_0\} \cup \bigcup_{k=1}^{\infty} S_k^n$, then X is a contractible $(n+1)$-dimensional compactum which is LC^{n-1} but which is not an AFS-space.

Remark. The statements found in Proposition 1 and Theorems 2 and 3 also hold for the class Q of metric or separable metric spaces.

REFERENCES

[1] K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44, Warszawa 1967.

[2] J. R. Martin, ***On absolute fixed-point sets***, Colloquium Mathematicum 35 (1976), p. 61-71.

THE UNIVERSITY OF SASKATCHEWAN
SASKATOON, CANADA

Reçu par la Rédaction le 24. 4. 1976