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ON A SUM OF VINOGRADOV

BY

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1. Introduction. Let m and n be positive integers with $m > 1$. The sum $f(m,n)$ given by

$$f(m,n) = \sum_{a=1}^{m-1} \frac{|\sin(\pi an/m)|}{\sin(\pi a/m)}$$

arises frequently in bounding certain exponential sums.

Estimates for $f(m,n)$ have been used in the study of the number of quadratic residues to a given prime modulus, p, on a given interval of integers (see Vinogradov [8]). The sum $f(m,n)$ also appears in the proof of the Pólya–Vinogradov inequality concerning the sum of characters, as can be seen for example in Apostol [1]. Expressions of the same kind are used in bounding exponential sums over primes (see Vaughan [7]).

Vinogradov proved in [8] that if $m > 60$ then

$$f(m,n) < m\log m - m .$$

Niederreiter [5] improved that result to get the following bound:

$$f(m,n) < \frac{2}{\pi} m\log m + \frac{2}{5} m + n .$$

Finally, Cochrane [2] obtained the more precise estimate that follows:

A) $\quad f(m,n) < \dfrac{4}{\pi^2} m\log m + 0.38m + 0.116\dfrac{d^2}{m} + O(1),$

B) $\quad m^{-1}\displaystyle\sum_{n=1}^{m} f(m,n) = \dfrac{4}{\pi^2} m\log m + \dfrac{4}{\pi^2}\left(\gamma - \log\dfrac{\pi}{2}\right) m + O(\log m \log\log m)$

where $d = (m,n)$ and γ stands for the Euler constant.

2. Statement of the results. From result B in Cochrane's paper it follows that the constant in the main term of result A is best possible, and the constant for the second term has to be greater than or equal to the corresponding constant in B, which is 0.05 approximately.

In this paper we improve the constant of the second term in A, the term in m, and we give an interpretation for the term $O(\log m \log\log m)$ in part B of Cochrane's result. We will prove the following result:

THEOREM 1. (a) *Let m and n be positive integers with $m > 1$. Then*

$$f(m,n) < \frac{4}{\pi^2} m \log m + \frac{4}{\pi^2}\left(\gamma - \log\frac{\pi}{4}\right) m + O(1).$$

(b) *The average of $f(m,n)$ over n satisfies*

$$m^{-1}\sum_{n=1}^{m} f(m,n) = \frac{4}{\pi^2} m \log m + \frac{4}{\pi^2}\left(\gamma - \log\frac{\pi}{2}\right) + \varphi(m)$$

where the last term satisfies for a positive C

$$-C(\log m \log\log m) \le \varphi(m) \le C(\log m).$$

Part (a) represents an improvement on part A of Cochrane's result, because the bound given there for the second term is

$$0.38m + 0.116\frac{d^2}{m}$$

while the corresponding term in Theorem 1 is

$$\frac{4}{\pi^2}\left(\gamma - \log\frac{\pi}{4}\right) m$$

which is approximately $0.332m$.

On the other hand, the proof contains an expression for $f(m,n)$ which will allow us to give a further improvement to $0.27m$.

Part (b) of Theorem 1 is the same as part B of Cochrane's result but giving a more precise meaning of the last term arising there. In fact, it is proved that the term $\log m \log\log m$ can appear only in a negative way and it comes from those m whose sum of divisors is "large", which by the average behaviour of the function $\sum_{d|m} d$ are relatively rare (see Hardy–Wright [4]).

In the following sections we prove some auxiliary lemmas, demonstrate Theorem 1 and indicate the ideas leading to the further improvement mentioned above. Finally, we calculate, as a consequence of a certain expression arising in the proof, the sums of some series involving cotangents and other trigonometric functions that seem to be new, at least they do not appear in the standard references such as Gradshteyn–Ryzhik [3] and Prudnikov–Brychkov–Marichev [6].

One of those series is given in

THEOREM 2. *Let m be an integer with $m \ge 1$. Then*

$$\sum_{k=1}^{\infty} \frac{1}{2k-1}\cot\frac{(2k-1)\pi}{2m} = \frac{\pi}{4}(m-1).$$

Incidentally, by taking $m = 2$, the preceding series gives the classical alternate series summing $\pi/4$, and by letting m tend to infinity, the value of $\zeta(2)$ can be found.

3. Auxiliary lemmas. Statement and proofs.

LEMMA 1. *Let m and n be positive integers with $m > 1$. Then $f(m,n)$ can be expressed as*

$$f(m,n) = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}S(m,n,k)$$

where

$$S(m,n,k) = \sum_{h=1}^{kn} kn\cot\frac{(2h-1)\pi}{2m}.$$

LEMMA 2. *For each m, n as before*

$$f(m,n) < \frac{4}{\pi}S\left(m,\left[\frac{m}{2}\right],1\right)$$

where $[n]$ means integer part.

LEMMA 3. *Let m be a positive integer. Then*

$$S\left(m,\left[\frac{m}{2}\right],1\right) = \frac{m\log m}{\pi} + \frac{m}{\pi}\left(\gamma - \log\frac{\pi}{4}\right) + O(1).$$

LEMMA 4. *Let m be a positive integer. Then*

$$\sum_{a=1}^{m-1}\frac{1}{\sin(\pi a/m)} = \frac{2m}{\pi}\left(\log m + \gamma - \log\frac{\pi}{2}\right) + O(1).$$

LEMMA 5.

$$\sum_{n=1}^{m} f(m,n)$$
$$= \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left\{\frac{m}{2}\sum_{a=1}^{m-1}\frac{1}{\sin(\pi a/m)} - \frac{m}{2}\sum_{\substack{a=1\\ ka=m}}\frac{1}{\sin(\pi a/m)}\right\}.$$

Proof of Lemma 1. By using the Fourier expansion

$$|\sin x| = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sin^2(kx)$$

it is seen that

$$f(m,n) = \sum_{a=1}^{m-1} \frac{1}{\sin(\pi a/m)} \left\{ \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{\sin^2(k\pi an/m)}{4k^2-1} \right\}$$
$$= \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} S(m,n,k)$$

where

$$S(m,n,k) = \sum_{a=1}^{m-1} \frac{\sin^2(k\pi an)}{\sin(\pi a/m)} .$$

Taking into account that

$$\frac{\sin^2(rx)}{\sin x} = \sin x + \sin 3x + \ldots + \sin(2r-1)x ,$$
$$\sum_{s=1}^{n} \sin sx = \frac{\cos \frac{1}{2}x - \cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x} ,$$

and interchanging the summation order we finally obtain

$$S(m,n,k) = \sum_{h=1}^{kn} \cot \frac{(2h-1)\pi}{2m} ,$$

which proves Lemma 1.

P r o o f o f L e m m a 2. Lemma 2 is a consequence of Lemma 1 and the fact that

$$\sum_{h=1}^{m} \cot \frac{(2h-1)\pi}{2m} = 0$$

so for each k

$$S(m,n,k) = \sum_{h=1}^{r(k)} \cot \frac{(2h-1)\pi}{2m}$$

where $r(k)$ is the residue of $kn \bmod m$. On the other hand, $\cot x > 0$ for $x \in (0, \pi/2)$ and $\cot x < 0$ for $x \in (\pi/2, \pi)$, so we have

$$S(m,n,k) \leq S(m,[m/2],1) .$$

Using this and the fact that

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}$$

we deduce Lemma 2.

Proof of Lemma 3. Substituting $x_r = (2r-1)\pi/(2m)$ in the well known identity

$$\cot x = \frac{1}{x} - 2x\sum_{h=1}^{\infty}\frac{1}{\pi^2h^2 - x^2}$$

we obtain

$$S\left(m, \left[\frac{m}{2}\right], 1\right) = \sum_{r=1}^{[m/2]} \cot\frac{(2r-1)\pi}{2m} = \frac{2m}{\pi}S_1 - \frac{4m}{\pi}S_2$$

where

$$S_1 = \sum_{r=1}^{[m/2]}\frac{1}{2r-1}, \qquad S_2 = \sum_{r=1}^{[m/2]}\sum_{h=1}^{\infty}\frac{2r-1}{(2mh)^2-(2r-1)^2}.$$

The sum S_1 can be estimated by

$$S_1 = \frac{1}{2}\log m + \frac{\gamma}{2} + \frac{\log 2}{2} + O\left(\frac{1}{m}\right)$$

using the identity $\sum_{k=1}^{m} 1/k = \log m + \gamma + O(1/m)$ where γ is the Euler constant.

The sum S_2 can be handled by changing the summation order and estimating the inner sum by a convenient Riemann sum, which gives

$$\begin{aligned} S_2 &= \sum_{h=1}^{\infty}\left\{\frac{1}{4}\log\frac{4h^2}{4h^2-1} + O\left(\frac{1}{h^2m}\right)\right\} \\ &= \frac{1}{4}\lim_{N\to\infty}\log\prod_{h=1}^{N}\frac{4h^2}{4h^2-1} + O\left(\frac{1}{m}\right), \end{aligned}$$

and using the Wallis formula:

$$S_2 = \frac{1}{4}\log\frac{\pi}{2} + O\left(\frac{1}{m}\right).$$

Finally by adding, we have

$$\begin{aligned} S\left(m, \left[\frac{m}{2}\right], 1\right) &= \frac{2m}{\pi}\left(\frac{1}{2}\log m + \frac{\gamma}{2} + \frac{\log 2}{2}\right) - \frac{4m}{\pi}\left(\frac{1}{4}\log\frac{\pi}{2}\right) + O(1) \\ &= \frac{m\log m}{\pi} + \frac{m}{\pi}\left(\gamma - \log\frac{\pi}{4}\right) + O(1). \end{aligned}$$

Proof of Lemma 4. The proof of Lemma 4 is similar to the proof of Lemma 3 using the simple fraction expansion of $(\sin x)^{-1}$ instead of that of $\cot x$.

Proof of Lemma 5. We make use of the formula we got in Lemma 1 for $f(m,n)$, and adding from $n=1$ to m we have

$$\begin{aligned}\sum_{n=1}^{m} f(m,n) &= \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left\{\sum_{n=1}^{m}\sum_{a=1}^{m-1}\frac{\sin^2(k\pi an/m)}{\sin(\pi a/m)}\right\}\\ &= \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sum_{a=1}^{m-1}\frac{1}{\sin(\pi a/m)}\cdot S_k(a,m)\end{aligned}$$

where

$$S_k(a,m)=\sum_{n=1}^{m-1}\sin^2\frac{k\pi an}{m}=\sum_{n=1}^{m-1}\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2k\pi an}{m}\right).$$

Using the fact that

$$\sum_{n=1}^{m}\cos\frac{2k\pi an}{m}=\begin{cases}0 & \text{if } ka\neq \dot{m},\\ m & \text{if } ka=\dot{m},\end{cases}$$

we obtain Lemma 5.

4. Proof of the theorem. (a) From Lemmas 1–3 we obtain

$$f(m,n)<\frac{4}{\pi}\left(\frac{m\log m}{\pi}+\frac{m}{\pi}\left(\gamma-\log\frac{\pi}{4}\right)\right)+O(1)$$

which is part (a) of the theorem; this already gives an improvement on Cochrane's result, but we will mention later on how one can obtain a further improvement.

(b) Using Lemmas 4 and 5 we have

$$\begin{aligned}\sum_{n=1}^{m} f(m,n) &= \frac{8}{\pi}\frac{m}{2}\frac{2m}{\pi}\left(\log m+\gamma-\log\frac{\pi}{2}\right)\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\\ &\quad-\frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\frac{m}{2}\sum_{\substack{a=1\\ ka=\dot{m}}}^{m-1}\frac{1}{\sin(\pi a/m)}+O(m)\\ &=\frac{4}{\pi^2}m^2\log m+\frac{4}{\pi^2}m^2\left(\gamma-\log\frac{\pi}{2}\right)+O(m)\\ &\quad-\frac{4m}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sum_{\substack{a=1\\ ka=\dot{m}}}^{m-1}\frac{1}{\sin(\pi a/m)}.\end{aligned}$$

Observe that the last term is negative. Now for each k such that $(k,m)=$

$d > 1$ we have

$$\sum_{\substack{a=1\\ ka=m}}^{m-1} \frac{1}{\sin(\pi a/m)} = \sum_{\lambda=1}^{d-1} \frac{1}{\sin(\pi\lambda/d)} = 2\sum_{\lambda=1}^{[d/2]} \frac{1}{\sin(\pi\lambda/d)} \le cd\log d\,.$$

We have to consider those k's such that $(k,m) = d > 1$, but this set is included in the set of those k's which are multiples of d, so the error for the k's such that $(k,m) = d$ is bounded in absolute value by

$$cm\sum_{k=d} \frac{1}{4k^2-1} d\log d = cmd\log d\sum_{h=1}^{\infty} \frac{1}{4h^2k^2-1} \le c'm\frac{d\log d}{d^2}$$

and we will have a bound of the total error if we consider all possible d's which are divisors of m, so we have

$$c'm\sum_{d|m} \frac{d\log d}{d^2} \le c'm\log m\sum_{d|m}\frac{1}{d}\,.$$

But

$$\sum_{d|m}\frac{1}{d} = \frac{1}{m}\sum_{d|m} d \le \frac{c}{m}\cdot m\log\log m\,,$$

which proves part (b) of the theorem.

5. An improvement of the theorem. If we observe that the residues of $kn \bmod m$ cannot be very near to $m/2$ simultaneously for "small" k's, and if we use the expression of Lemma 1 for $f(m,n)$, we can improve the result of part (a) of Theorem 1.

In fact, after some calculations, we can write the identity

$$\begin{aligned} f(m,n) &= \frac{4}{\pi^2}m\log m + \frac{4}{\pi^2}m\left(\gamma - \log\frac{\pi}{4}\right) \\ &\quad + \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\log\left|\sin\frac{(2kn-1)\pi}{2m}\right| + \text{lower order terms}\,, \end{aligned}$$

which is valid on certain conditions upon the relative size of n, and observing that the series is negative we can prove

$$f(m,n) < \frac{4}{\pi^2}m\log m + 0.27m + O(1)$$

by minimizing the series.

6. Some consequences. Some of the applications of $f(m,n)$ were indicated in the introduction. Here we want to mention another one. If we use the expression obtained in Lemma 1, we can prove the following

THEOREM 2. *Let m be a positive integer. Then*

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} \cot \frac{(2k-1)\pi}{2m} = \frac{\pi}{4}(m-1).$$

Proof. By taking $n = 1$ in Lemma 1 we have

$$m - 1 = f(m,1) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \left\{ \sum_{h=1}^{k} \cot \frac{(2h-1)\pi}{2m} \right\}$$

and writing

$$\frac{1}{4k^2-1} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1} \right)$$

it is not difficult to justify that

$$\begin{aligned} m - 1 &= \frac{4}{\pi} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \left(\sum_{h=1}^{k} \cot \frac{(2h-1)\pi}{2m} \right) \\ &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \cot \frac{(2k-1)\pi}{2m}. \end{aligned}$$

Some other sums of the same kind can be calculated using certain trigonometric identities and the preceding result, for example we can prove:

(a) $\displaystyle \sum_{k=1}^{\infty} \frac{1}{2k-1} \tan \frac{(2k-1)\pi}{2m} = \frac{\pi}{4}$ for any $m \in \mathbb{N}$,

(b) $\displaystyle \sum_{k=1}^{\infty} \frac{1}{2k-1} \frac{1}{\sin \frac{(2k-1)\pi}{2m}} = \frac{\pi}{4} m$ for any $m \in \mathbb{N}$.

Let us mention, finally, that the result of Theorem 2 gives, for $m = 2$, the classical alternate series for $\pi/4$, and by taking m tending to infinity gives the value of $\zeta(2)$. Most of the expansions used in this paper can be found in Zygmund [9].

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