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A GENERALIZATION OF THE DIRECT PRODUCT OF UNIVERSAL ALGEBRAS

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This note is an attempt to generalize the direct product of algebras so that it could be applied to algebras not necessarily similar. Such generalized product might be useful in constructing various algebras from known ones. It is in the spirit of the treatment of universal algebras in Marczewski's independence theory (see [3]), where the clone of all algebraic operations is essential rather than the fundamental operations themselves. Since the assumption of similarity of algebras is sacrificed, isomorphism and homomorphisms must be replaced by weak isomorphisms and weak homomorphisms in the sense introduced in [1]. The product constructed here will have the associative property. Also the projections of the product onto the factors are weak homomorphisms.

1. Notation. Let $\mathfrak{A}_t = (A_t, \Omega_t)$, $t \in T$, be a family of finitary algebras indexed by the elements of a set T. We shall denote by $\boldsymbol{A}_t = [\Omega_t]$ the set of all algebraic operations of the algebra $\mathfrak{A}_t$, i.e., all operations which can be obtained from Ω_t and the trivial operations e_j^n, such that

$$x_1 x_2 \ldots x_n e_j^n = x_j,$$

by compositions. The set of all algebraic n-ary operations is denoted by $\boldsymbol{A}_t^{(n)}$ or $[\Omega_t]^{(n)}$.

Elements of the cartesian product $A = \prod_{t\in T} A_t$ will be denoted by the letters $x, y, \ldots, a, b, \ldots$ and considered as mappings $T \to \bigcup_{t\in T} A_t$ such that $x(t) \in A_t$. The mapping $p_t: A \to A_t$ defined by $p_t(x) = x(t)$ is called the *t-th projection*. If E is a subset of $A = \prod_{t\in T} A_t$, then we use the notation

$$E(t) = p_t(E) = \{x(t) \mid x \in E\}.$$

Elements of the product $\boldsymbol{A}^{(n)} = \prod_{t\in T} \boldsymbol{A}_t^{(n)}$ are denoted by Greek letters, $\omega, \alpha, \beta, \ldots$ and considered as n-ary operations on $A = \prod_{t\in T} A_t$; namely, if $x_1, \ldots, x_n \in A$, then $x_1 \ldots x_n \omega$ is such an element of A that

$$[x_1 \ldots x_n \omega](t) = x_1(t) \ldots x_n(t) \omega(t) \tag{1}$$

for every t.

We denote by $\boldsymbol{A}$ the union of all $\boldsymbol{A}^{(n)}$:

$$\boldsymbol{A} = \bigcup_{n=0}^{\infty} \boldsymbol{A}^{(n)}.$$

The elements of A_t [respectively $\boldsymbol{A}_t$], unless they are projections $x(t)$ [$\omega(t)$] of an element of A [$\boldsymbol{A}$], are denoted by letters with a superscript t: x^t [ω^t].

In the case of two algebras $\mathfrak{A} = (A, \Omega)$, $\mathfrak{B} = (B, \overline{\Omega})$ we shall use the notation $A \times B$, elements of $A \times B$ are denoted by (x, y), elements of $[\Omega] \times [\overline{\Omega}]$ by $\omega \times \overline{\omega}$.

For a subset $U \subset T$ we denote by A_U, $\boldsymbol{A}_U^{(n)}$, $\boldsymbol{A}_U$ the corresponding partial products; more precisely:

$$A_U = \prod_{t \in U} A_t, \qquad \boldsymbol{A}_U^{(n)} = \prod_{t \in U} \boldsymbol{A}_t^{(n)}, \qquad \boldsymbol{A}_U = \bigcup_{n=0}^{\infty} \boldsymbol{A}_U^{(n)}. \tag{2}$$

Elements of A_U [respectively $\boldsymbol{A}_U$] are denoted by x^U [respectively ω^U], unless they are restrictions of an element x [ω] of A [$\boldsymbol{A}$] to U in which case we rather use the notation $x \mid U$ [$\omega \mid U$].

For any subset $\varrho \subset \boldsymbol{A}$ we denote by ϱ_U the restriction of ϱ to $\boldsymbol{A}_U$, i.e., $\varrho_U = \{\omega \mid U : \omega \in \varrho\}$.

Given a partition $\pi = \{U_s : s \in S\}$ of the set T into disjoint non-empty subsets, $T = \bigcup_s U_s$, we define the subset ϱ_π of $\prod_{s \in S} \boldsymbol{A}_{U_s}$ as

$$\varrho_\pi = \Big\{\overline{\omega} \in \prod_{s \in S} \boldsymbol{A}_{U_s} : \overline{\omega}(s) = \omega \mid U_s \text{ for some } \omega \in [\varrho]\Big\}, \tag{3}$$

where $[\varrho]$ is the set of all operations in A derived from the operations of ϱ and the trivial operations by composition.

2. Weak homomorphisms. We recall the notion of weak homomorphism.

Given two algebras $\mathfrak{B} = (B, \overline{\Omega})$ and $\mathfrak{A} = (A, \Omega)$, a mapping $h: B \to A$ is called a *weak homomorphism* if

(r) for every $\overline{\omega} \in \overline{\Omega}^{(n)}$ there exists an $\omega \in [\Omega]^{(n)}$ such that

$$h(y_1 \ldots y_n \overline{\omega}) = h(y_1) \ldots h(y_n) \omega, \tag{4}$$

and

(l) for every $\omega \in \Omega^{(n)}$ there exists an $\overline{\omega} \in [\overline{\Omega}]^{(n)}$ such that (4) holds.

We shall call mappings satisfying condition (r) *r-morphisms*, those satisfying (l) will be called *l-morphisms*. Thus, a weak homomorphism is a mapping which is simultaneously an r-morphism and l-morphism.

The class of all universal algebras with l-morphism forms a category which we shall call the *l-category* of algebras. The category of all universal algebras with r-morphism will be called the *r-category*. The category of all algebras with weak homomorphism as morphisms will be called the *lr-category* of algebras. Note that the isomorphisms in all three categories coincide; they are the weak isomorphisms of [1].

3. The ϱ-product. Let ϱ be a relation in (i.e., a subset of) $\prod_{t\in T} A_t$ which satisfies the following conditions:

(i) $\varrho \subset \boldsymbol{A}$, i.e., for every $\omega \in \varrho$ all $\omega(t)$ are of the same arity,

(ii) $[\varrho](t) \supset \Omega(t)$.

We define the *ϱ-product* of the family $\{\mathfrak{A}_t\}$ as the algebra

$$\mathfrak{A} = \prod_{t\in T}{}_{\varrho} \mathfrak{A}_t = (A, \varrho) = \Big(\prod_{t\in T} A_t, \varrho\Big). \tag{5}$$

The set of all algebraic operations of $\mathfrak{A}$ is $[\varrho]$ and the set of all algebraic n-ary operations is $[\varrho]^{(n)}$. We shall use the notation $\mathfrak{A}\times_{\varrho}\mathfrak{B}$ for the product of two algebras.

LEMMA 1. *If ϱ satisfies conditions* (i) *and* (ii), *then* $[\varrho](t) = \boldsymbol{A}_t$ *for every $t\in T$.*

Proof. Obviously, since $\varrho \subset \boldsymbol{A}$, $[\varrho](t) \subset \boldsymbol{A}_t$. It remains to prove the other inclusion. Let $\omega^t \in \boldsymbol{A}_t$. The operation ω^t is the result of a finite chain of compositions of trivial operations and operations of Ω_t. We call the number of operations involved the *length* of ω^t. If ω^t is non-trivial and of length 1, then $\omega^t \in \Omega_t$, and by (ii) there exists an $\omega \in [\varrho]$ such that $\omega(t) = \omega^t$. Suppose now that for every operation $a^t \in \boldsymbol{A}_t$ of length $\leqslant n$ there exists an operation $a \in [\varrho]$ such that $a(t) = a^t$. Let $\omega^t \in \boldsymbol{A}_t$ be of length $n+1$. Then we can find an operation $a^t \in \boldsymbol{A}_t^{(k)}$ and operations $a_1^t, a_2^t, \ldots, a_k^t$ of length $\leqslant n$, such that

$$\omega^t = a_1^t a_2^t \ldots a_k^t a^t.$$

By our assumption there are operations $a, a_1, \ldots, a_k \in [\varrho]$ such that $a^t = a(t)$. $a_1^t = a_1(t), \ldots, a_k^t = a_k(t)$. Set $\omega = a_1 \ldots a_k a$. Then $\omega \in [\varrho]$ and $\omega(t) = \omega^t$. Consequently, $\boldsymbol{A}_t \subset [\varrho](t)$.

THEOREM 1. *The projections $p_t\colon A \to A_t$ are weak homomorphisms of algebras.*

Proof. Let $\omega \in [\varrho]$ be an algebraic n-ary operation of $\mathfrak{A}_t$. By the lemma, $\omega(t) \in \boldsymbol{A}_t^{(n)}$ is an algebraic n-ary operation of $\mathfrak{A}_t$, and

$$p_t(x_1 \ldots x_n \omega) = x_1(t) \ldots x_n(t)\omega(t) = p_t(x_1) \ldots p_t(x_n)\omega(t). \tag{6}$$

Thus p_t is an r-morphism.

Given now an operation $\omega^t \in \boldsymbol{A}_t^{(n)}$, we can find (using Lemma 1) an operation $\omega \in \boldsymbol{A}^{(n)}$ such that $\omega^t = \omega(t)$ and (4) holds again. Thus p_t is an l-morphism too.

Two algebras $\mathfrak{B} = (B, \Omega)$ and $\mathfrak{B}^* = (B, \Omega^*)$ with the same set of elements are called *equivalent*, in symbols $\mathfrak{B} \sim \mathfrak{B}^*$, if $[\Omega] = [\Omega^*]$. Equivalence can also be expressed in terms of weak isomorphism by saying that $\mathfrak{B}$ and $\mathfrak{B}^*$ are equivalent if and only if the identity mapping $B \to B$ is a weak isomorphism.

It is easy to notice that if $\mathfrak{A}_t \sim \mathfrak{A}_t^*$ for every t, then

$$\prod_{t\in T}{}_{\varrho} \mathfrak{A}_t \sim \prod_{t\in T}{}_{\varrho} \mathfrak{A}_t^*.$$

If, moreover, $[\varrho] = [\varrho^*]$, then

$$\prod_{t\in T}{}_{\varrho} \mathfrak{A}_t \sim \prod_{t\in T}{}_{\varrho^*} \mathfrak{A}_t^*.$$

4. Associativity of the ϱ-product. Consider a subset $U \subset T$ and a partition $\pi = \{U_s : s \in S\}$ of T into disjoint non-empty sets.

LEMMA 2. *If ϱ satisfies conditions* (i) *and* (ii), *then ϱ_U satisfies the same conditions restricted to U; in other words, $\varrho_U \subset \boldsymbol{A}_U$, $[\varrho_U]_t \supset \boldsymbol{A}_t$ for $t \in U$.*

If $\omega^U \in \varrho_U$, then $\omega^U = \omega | U$ for some $\omega \in \boldsymbol{A}$. Therefore there exists an n such that $\omega \in \boldsymbol{A}^{(n)}$ and $\omega | U \in \boldsymbol{A}_U^{(n)} \subset \boldsymbol{A}_U$. Thus $\varrho_U \subset \boldsymbol{A}_U$.

The second condition is obviously satisfied since it is a particular case of (ii).

LEMMA 3. *If ϱ satisfies* (i) *and* (ii), *then ϱ_π satisfies the same conditions for the family of algebras $\{(A_{U_s}, \boldsymbol{A}_{U_s}) : s \in S\}$.*

Proof. Let $\overline{\omega} \in \varrho_\pi$. Then there exists an $\omega \in \varrho$ such that $\overline{\omega}(s) = \omega | U_s$. But $\omega \in \boldsymbol{A}^{(n)}$ for some n, thus $\omega | U_s = \overline{\omega}(s) \in \boldsymbol{A}_{U_s}^{(n)}$. Consequently, $\overline{\omega} \in \prod_{s\in S} \boldsymbol{A}_{U_s}^{(n)}$ which shows that

$$\varrho_\pi \subset \bigcup_{n=0}^{\infty} \prod_{s\in S} \boldsymbol{A}_{U_s}^{(n)}.$$

To prove the second property, take $\overline{\omega}^s \in \boldsymbol{A}_{U_s}^{(n)}$. There exists an $\omega \in [\varrho]^{(n)}$ such that $\overline{\omega}^s = \omega | U_s$. Let $\overline{\omega} \in \prod_{s\in S} \boldsymbol{A}_{U_s}^{(n)}$ be defined by $\overline{\omega}(s) = \omega | U_s$. Then $\overline{\omega} \in \varrho_\pi$ and, obviously, $\overline{\omega}(s) = \omega^s$. Thus $[\varrho_\pi](s) \supset \boldsymbol{A}_{U_s}$ for every $s \in S$.

LEMMA 4. $[\varrho_\pi] = \varrho_\pi$.

Proof. It suffices to prove that $[\varrho_\pi] \subset \varrho_\pi$. We shall prove the lemma by induction with respect to the length of the operation $\overline{\omega} \in [\varrho_\pi]$. If $\overline{\omega}$ is of length 1, then $\overline{\omega} \in \varrho_\pi$. Suppose now that all operations $\overline{a} \in [\varrho_\pi]$ of length $\leqslant n$ belong to ϱ_π and let $\overline{\omega} \in [\varrho_\pi]$ be of length $n+1$. Then $\overline{\omega} = \overline{a}_1, \ldots, \overline{a}_k \overline{a}$ for some $\overline{a} \in \varrho_\pi$ and $\overline{a}_1, \ldots, \overline{a}_k$ of length $\leqslant n$. Consequently, there exist operations $a, a_1, \ldots, a_k \in [\varrho]$ such that $\overline{a}_j(s) = a_j | U_s$ and $a(s) = a | U_s$. The operation $\omega = a_1 \ldots a_k a \in [\varrho]$ and $\omega | U_s = \omega(s)$. Consequently, $\overline{\omega} \in \varrho_\pi$.

THEOREM 2. *The canonical mapping* $\varphi\colon \prod_{t\in T} A_t \to \prod_{s\in S} A_{U_s}$ *defined by formula*

$$[\varphi(x)](s) = x \mid U_s \tag{7}$$

is a weak isomorphism of algebras

$$\prod_{t\in T}{}_{\varrho}\mathfrak{A}_t \to \prod_{s\in S}{}_{\varrho_\pi}\Big(\prod_{t\in U_s}{}_{\varrho U_s}\mathfrak{A}_t\Big).$$

Proof. For a given $\bar{x}\in\prod_{s\in S} A_{U_s}$ define $x\in A$ as follows:

$$x(t) = [\bar{x}(s)](t), \qquad t\in U_s,\ s\in S.$$

Then, obviously, $\bar{x} = \varphi(x)$ and so the mapping is onto. On the other hand, if $x, y\in A$ and $x \neq y$, then there exists a $t_0\in T$ such that $x(t_0) \neq y(t_0)$. Further, $t_0\in U_{s_0}$ for some s_0 and, consequently, $x\mid U_{s_0} \neq y\mid U_{s_0}$ or $[\varphi(x)](s_0) = [\varphi(y)](s_0)$. Thus $\varphi(x) \neq \varphi(y)$, and the mapping is one-to-one.

Let now $\omega\in[\varrho]^{(n)}$ and let $\bar{\omega}(s) = \omega\mid U_s$. Then $\bar{\omega}\in\varrho_\pi$ and satifies the condition

$$\varphi(x_1)\ldots\varphi(x_n)\bar{\omega} = \varphi(x_1\ldots x_n\omega)$$

showing that φ is an r-morphism. Indeed,

$$\begin{aligned}\big[[\varphi(x_1\ldots x_n\omega)](s)\big](t) &= [x_1\mid U_s\ldots x_n\mid U_s\omega\mid U_s](t)\\ &= \big[[\varphi(x_1)](s)\ldots[\varphi(x_n)](s)\big](t).\end{aligned}$$

In order to prove that φ is a weak isomorphism it suffices now to prove that $\omega\to\bar{\omega}$ is a one-to-one mapping of $[\varrho]$ onto $[\varrho_\pi] = \varrho_\pi$. That the mapping is onto follows immediately from definition (3) of ϱ_π. To prove that it is one-to-one take $\omega_1, \omega_2\in A$ and $\omega_1 \neq \omega_2$. Then for some $t_0\in T$ we have $\omega_1(t_0) \neq \omega_2(t_0)$. Let $t_0\in U_{s_0}$. Then we have $\omega_1\mid U_{s_0} \neq \omega_2\mid U_{s_0}$ but this means that $\bar{\omega}_1(s_0) \neq \bar{\omega}_2(s_0)$ and $\bar{\omega}_1 \neq \bar{\omega}_2$.

5. Examples. A. The direct product of similar algebras is a particular case of ϱ-products. In this case all Ω_t's coincide and the relation ϱ is the diagonal of the product $\prod_{t\in T}\Omega_t$.

The direct product is useful in construction of new algebras from the known ones. A factorization of an algebra into a direct product gives additional insight into the structure of the algebra. For example Świerczkowski's four-element algebra $\mathfrak{S}$ (see [3]) is the direct product of two Post algebras $\mathfrak{P}_*$.

We think that the notion of ϱ-products might have similar applications. The following example deals with the ϱ-product of trivial algebras, i.e., algebras with trivial operations only. The direct product of such algebras is always the trivial algebra.

B. Consider n trivial algebras $\mathfrak{A}_i = (A_i, \emptyset)$, $i = 1, 2, \ldots, n$. The sets $A_i^{(k)}$ consist of all trivial k-ary operations $e_j^{(k)}$ ($j = 1, 2, \ldots, k$). We use the same notation for the trivial operations in all algebras. Let ϱ consist of one non-trivial n-ary operation

$$\omega = (e_1^n, e_2^n, \ldots, e_n^n).$$

The ϱ-product

$$\prod_{i=1}^{n}{}_{\varrho} \mathfrak{A}_i = \Big(\prod_{i=1}^{n} A_i, \{\omega\}\Big)$$

is an n-dimensional diagonal algebra introduced by J. Płonka in [4]. The representation theorem proved by J. Płonka for diagonal algebras shows that every n-dimensional diagonal algebra admits this kind of factorization.

We shall return to some aspects of this example later.

C. Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra and $\mathfrak{B} = (B, \{\cup\})$ a semilattice. Denote by $\cup^n$ the n-ary operation $x_1 \ldots x_n \cup^n = x_1 \cup x_2 \cup \ldots \cup x_n$ of $\mathfrak{B}$. Let now $\varrho \in [\Omega] \times [\{\cup\}]$ consists of all pairs $[\omega, \cup^{n(\omega)}]$, where $\omega \in \Omega$ and $n(\omega)$ is the arity of ω. The ϱ-product $\mathfrak{A} \times_{\varrho} \mathfrak{B}$ is a particular case of the sum of a direct system of algebras intruced by J. Płonka in [5]. Here B serves as the set of indices, all algebras of the system are isomorphic and the homomorphisms φ_{ij} are isomorphisms.

D. Let $\mathfrak{B} = (B, \vee, \wedge, ', a)$ be a Boolean algebra with an additional constant a distinct from $0, I$. As shown in [2] among the derived operations of this algebra there are the binary operations

$$x \cup y = [a \wedge (x \vee y)] \vee [a' \wedge x \wedge y],$$

$$x \cap y = [a \vee x \vee y] \wedge [a'(x \wedge y)]$$

which are distributive with respect to $\vee$ and $\wedge$ and $(B, \cup, \cap, ')$ is a Boolean algebra with unity a and zero a'.

Therefore the algebras $\mathfrak{B}$ and $(B, \vee, \wedge, \cup, \cap, ')$ are equivalent. The latter algebra admits a ϱ-factorization into two Boolean algebras.

Let $\mathfrak{A}_1 = (A_1, \vee, \wedge, ')$ be the restriction of $(B, \vee, \wedge, ')$ to a, i.e., $A_1 = \{a \wedge x \colon x \in B\}$ and $\mathfrak{A}_2 = (A_2, \vee, \wedge, ')$ be the restriction to a', i.e. $A_2 = \{a' \wedge x \colon x \in B\}$. Thus A_1 and A_2 are subsets of A and for any two elements if $x_1 \in A_1$, $x_2 \in A_2$, then $x_1 \wedge x_2 = 0$. Let $\varrho = \{\vee \times \vee, \wedge \times \wedge, \vee \times \wedge, \wedge \times \vee, '\}$. Then the ϱ-product $\mathfrak{A}_1 \times_{\varrho} \mathfrak{A}_2$ is isomorphic to $(B, \vee, \wedge, \cup, \cap, ')$, the isomorphism being $\Phi\colon (x_1, x_2) \to x_1 \vee x_2$.

One sees immediately that $\Phi^{-1}(x) = (a \wedge x, a' \wedge x)$ for $x \in B$, thus Φ is one-to-one and onto. We shall check that Φ is a homomorphism for the operation $\cup$; other operations lead to similar computations. Let $(x_1, x_2), (y_1, y_2) \in A_1 \times A_2$. Then $\Phi[(x_1, x_2)(\vee \times \wedge)(y_1, y_2)] = \Phi(x_1 \vee y_1,$

$x_2 \wedge y_2) = x_1 \vee y_1 \vee (x_2 \wedge y_2)$. But since $x_1 < a, y_1 < a$ and $x_2 < a'$, $y_2 < a'$, we have $x_1 = a \wedge (x_1 \vee x_2)$, $y_1 = a \wedge (y_1 \vee y_2)$ and $x_2 = a' \wedge \wedge (x_1 \vee x_2)$, $y_2 = a' \wedge (y_1 \vee y_2)$. Consequently,

$$\begin{aligned}\Phi[(x_1, x_2)(\vee \times \wedge)(y_1, y_2)] \\ &= \big(a \wedge [(x_1 \vee x_2) \vee (y_1 \vee y_2)]\big) \vee [a' \wedge (x_1 \vee x_2) \wedge (y_1 \wedge y_2)] \\ &= \big[a \wedge \big(\Phi(x_1, x_2) \vee \Phi(y_1, y_2)\big)\big] \vee [a' \wedge \Phi(x_1, x_2) \wedge \Phi(y_1, y_2)] \\ &= \Phi(x_1, x_2) \cup \Phi(y_1, y_2).\end{aligned}$$

6. Maximal ϱ-products. We call a ϱ-product of algebras *maximal* if $[\varrho] = A$. It is obvious that any two maximal ϱ-products of a given family of algebras are equivalent. Maximal ϱ-products might be useful in investigating generating sets and independence.

THEOREM 3. *If $[\varrho] = A$ and if each of the algebras $\mathfrak{A}_t$ is generated by a finite set G_t of cardinality $\leqslant n$ (n independent of t), then the product $\prod_{t\in T} G_t$ generates the algebra $\prod_{t\in T}^{\varrho} \mathfrak{A}_t$.*

Proof. Let x be an arbitrary element of A. By our assumption about G_t there exists an n-ary operation ω^t such that

$$x(t) = x_1^t \dots x_n^t \omega^t, \qquad \text{where } x_i^t \in G_t.$$

Since $[\varrho] = A$, the operation $\omega \in A$ such that $\omega(t) = \omega^t$ is an algebraic operation of $\mathfrak{A}$. The elements $x_1, \dots, x_n$ defined by $x_1(t) = x_1^t, \dots, x_n(t) = x_n^t$ belong to $\prod_{t\in T} G_t$ and $x = x_1 \dots x_n \omega$. Thus $\prod_{t\in T} G_t$ generates $\mathfrak{A}$.

Note that if T is a finite set, we can drop in Theorem 3 the assumption that the cardinalities of the generating sets are bounded or even finite. We have the following:

THEOREM 4. *If $[\varrho] = A$ and* card $T < \aleph_0$, *and if for every t, G_t generates $\mathfrak{A}_t$, then $\prod_{t\in T} G_t$ generates the ϱ-product.*

Proof. Let $x \in A$ be an arbitrary element. Each of the projections $x(t)$ of this point is generated by a finite number n_t of elements of G^t. The number $n = \max_{t\in T} n_t$ is finite because the cardinality of T is finite. For every t there exists an n-ary operation ω^t and a system of elements $x_1^t, \dots, x_n^t \in A^t$ such that $x^t = x_1^t \dots x_n^t \omega^t$. Define now $x_1, \dots, x_n \in A$ and $\omega \in A^{(n)}$ by $x_i(t) = x_i^t$ $(i = 1, \dots, n)$ and $\omega(t) = \omega^t$. Obviously, $x_1, \dots, x_n \in \prod_{t\in T} G_t$, $\omega \in A = [\varrho]$, and $x_1 \dots x_n \omega = x$. Thus $\prod_{t\in T} G_t$ generates the product.

THEOREM 5. *In a maximal ϱ-product n distinct elements $a_1 \dots a_n$ are independent if and only if for every t the projections $a_1(t), \dots, a_n(t)$ are distinct and independent.*

Proof. 1. Suppose that the elements $a_1, \ldots, a_n$ are distinct and independent but for some $t_0 \epsilon T$ the elements $a_1(t_0), \ldots, a_n(t_0)$ are not. Then there exist two distinct n-ary algebraic operations $\omega_1^{t_0}, \omega_2^{t_0} \epsilon \boldsymbol{A}_{t_0}$ such that

(8) $$a_1(t_0) \ldots a_n(t_0)\omega_1^{t_0} = a_1(t_0) \ldots a_n(t_0)\omega_2^{t_0}.$$

Define two operations $\omega_1, \omega_2 \epsilon \boldsymbol{A}^{(n)}$ by formulae

$$\omega_1(t) = \begin{cases} \omega_1^{t_0} & \text{for } t = t_0, \\ e_1^n & \text{for } t \neq t_0, \end{cases} \qquad \omega_2(t) = \begin{cases} \omega_2^{t_0} & \text{for } t = t_0, \\ e_1^n & \text{for } t \neq t_0. \end{cases}$$

Since the algebra is maximal, $\omega_1, \omega_2 \epsilon [\varrho]$ and we have

$$a_1 \ldots a_n \omega_1 = a_1 \ldots a_n \omega_2,$$

because $\omega_1(t) = \omega_2(t)$ for $t \neq t_0$ and (8) holds for $t = t_0$. But this contradicts the assumption of independence of the elements $a_1, \ldots, a_n$.

2. Suppose now that $a_1(t), \ldots, a_n(t)$ are independent for every t and let $\omega_1, \omega_2 \epsilon [\varrho]^{(n)}$ be two operations such that

$$a_1 \ldots a_n \omega_1 = a_1 \ldots a_n \omega_2.$$

Then, for every t,

$$a_1(t) \ldots a_n(t)\omega_1(t) = a_1(t) \ldots a_n(t)\omega_2(t)$$

and by the assumption of independence of $a_1(t) \ldots a_n(t)$ we have $\omega_1(t) = \omega_2(t)$ for every t, or $\omega_1 = \omega_2$. But this means that the elements $a_1, \ldots, a_n$ are independent.

Remark. Note that the second part of the proof does not use the assumption that $[\varrho] = \boldsymbol{A}$. Thus if for every t the t-projections of n elements $a_1, \ldots, a_n$ of any ϱ-product are independent, then the elements themselves are independent.

The following theorem characterizes the maximal ϱ-product:

THEOREM 6. *The maximal ϱ-product of a family of algebras $\{\mathfrak{A}_t\}$ is the product of this family in the r-category of universal algebras.*

Proof. Let $\mathfrak{B} = (B, \overline{\Omega})$ be any universal algebra, and let $h_t: B \to A_t$ be an r-morphism for every $t \epsilon T$. Define $h: B \to \prod_{t \epsilon T} A_t$ in the obvious way, i.e. for $y \epsilon B$

$$[h(y)](t) = h_t(y).$$

Obviously, we have $h_t(y) = p_t(h(y))$ and h is determined uniquely by this property.

If all mappings h_t are r-morphism, then given $\overline{\omega} \epsilon [\overline{\Omega}]^{(n)}$ there exists in every algebra $\mathfrak{A}_t$ an operation $\omega^t \epsilon \boldsymbol{A}_t^{(n)}$ such that

(9) $$h_t(y_1 \ldots y_n \overline{\omega}) = h_t(y_1) \ldots h_t(y_n)\omega^t$$

for every n elements $y_1, \ldots, y_n \in B$. Let $\omega \in A^{(n)}$ be defined by $\omega(t) = \omega^t$. Since the ϱ-product is maximal, ω is an algebraic operation. This operation satisfies the condition

$$h(y_1 \ldots y_n \overline{\omega}) = h(y_1) \ldots h(y_n)\omega$$

since (9) is just another form of the identity

$$[h(y_1 \ldots y_n \overline{\omega})](t) = [h(y_1) \ldots h(y_n)\omega](t).$$

Consequently, h is an r-morphism, which completes the proof of the theorem.

The ϱ-product of example B of section 5 is a maximal ϱ-product of trivial algebras. Since all elements of a trivial algebra are independent, Theorem 5 enables us to find the greatest number of independent elements of a diagonal algebra knowing its ϱ-factorization. If the cardinalities of the trivial algebras are $m_1, m_2, \ldots, m_k$, then the cardinality of the ϱ-product is $m_1 m_2 \ldots m_k$ and the largest independent set of elements consists of $\min(m_1, m_2, \ldots, m_n)$ elements. On the other hand, if the factors are not one-element algebras, one can always find two distinct elements which are not independent, e.g., $(x_1, x_2, \ldots, x_n)$ and $(y_1, x_2, \ldots, x_n)$ with $x_1 \neq y_1$.

7. Quasi-maximal ϱ-products. Another look at the proof of Theorem 5 shows that the property of ϱ, which was essentially used in the proof, was that operations $\omega \in A^{(n)}$, such that $\omega(t) = e_1^n$ for all but one value t_0 and $\omega(t_0)$ is an arbitrary operation of $\mathfrak{A}_{t_0}$, belong to $[\varrho]$. This leads to the following definition of quasi-maximal ϱ-products for which Theorem 5 holds.

Denote by $A^{*(n)}$ the set

(10) $$A^{*(n)} = \{\omega \in A^{(n)} \colon \omega(t) = e_1^n \text{ for all but one value } t\}$$

and let $A^* = \bigcup_{n=1}^{\infty} A^{*(n)}$. A ϱ-product is called *quasi-maximal* if $[\varrho] \supset A^*$. Obviously, every maximal ϱ-product is quasi-maximal since $A^* \subset A$.

This definition implies immediately the following theorem:

THEOREM 7. *A quasi-maximal product of a finite family of algebras is maximal.*

Proof. Indeed, one can easily notice that $[A^*]^{(n)}$ consists of all n-ary operations $\omega \in A^{(n)}$ such that $\omega(t)$ is the same trivial operation for all but a finite number of values of t. If the family is finite, then $[A^*]^{(n)} = A^{(n)}$.

In the infinite case quasi-maximal ϱ-products are not necessarily maximal.

As we mentioned before, the following generalization of Theorem 5 holds:

THEOREM 8. *Distinct elements $a_1, \ldots, a_n$ of a quasi-maximal ϱ-product of a family of algebras $\{\mathfrak{A}_t\}$ are independent if and only if for every t the elements $a_1(t), \ldots, a_n(t)$ are independent.*

Among all quasi-maximal ϱ-products there is a distinguished one. This is the product $\prod_{T\epsilon t} {}_{A^*} \mathfrak{A}_t$.

To justify this definition, one has to check that A^* satisfies condition (ii) for the relation ϱ, but this is obvious in view of (10).

We call a ϱ-product *equivalent* to this one, i.e., such that $[\varrho] = [A^*]$ a special quasi-maximal ϱ-product. It is clear that if a special quasi-maximal product of a family $\{\mathfrak{A}_t\}$ is maximal, then every quasi-maximal product of this family is maximal.

Note also that if a ϱ-product is maximal (quasi-maximal) and π is a partition of the index set, then the partial products of section 4 are maximal (quasi-maximal) too.

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I have also been informed that the results of Theorems 3 and 5 have been previously obtained by W. Narkiewicz (in 1960, so far unpublished).

REFERENCES

[1] A. Goetz, *On weak isomorphisms and weak homomorphisms of abstract algebras*, Colloquium Mathematicum 16 (1966), p. 163–167.

[2] — *On various Boolean structures in a given Boolean algebra*, Publicationes Mathematicae (Debrecen), to appear.

[3] E. Marczewski, *Independence in abstract algebras. Results and problems*, Colloquium Mathematicum 14 (1966), p. 169–188.

[4] J. Płonka, *Diagonal algebras*, Fundamenta Mathematicae 58 (1966), p. 309–321.

[5] — *Sums of direct systems of abstract algebras*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 15 (1967), p. 133–135.

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