TERMINAL SUBSETS OF CONVEX SETS
IN FINITE DIMENSIONAL REAL NORMED SPACES

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Generalizing the notion of terminal point in metric spaces we introduce, by analogy to the classical definition of extreme subset in linear spaces, a notion of terminal subset. We present some properties of terminal subsets in metric spaces and characterize terminal subsets of convex sets in finite dimensional real normed spaces.

1. Terminal subsets in metric spaces. Let M be a metric space with metric d. As in [1], p. 33, we say that y is *metrically between x and z* and we write xyz if $d(x, y)+d(y, z) = d(x, z)$ and $x \neq y \neq z$.

A point t of a set $S \subset M$ is called a *terminal subset* of S if it is not metrically between any two points of S (see [1], p. 53).

We can generalize this notion as follows. We call a subset T of a set $S \subset M$ a *terminal subset* of S if any two points of S, such that a point of T is metrically between them, belong to T, i.e. if

(1) xyz does not hold for any $x \in S$, $y \in T$, $z \in S \setminus T$.

Obviously, t is a terminal subset of S if and only if $\{t\}$ is a terminal subset of S.

The notion of terminal subset is an analog of the notion of extreme subset in a real linear space L. A subset E of a set $S \subset L$ is called *extreme in S* if any two points x, z of S, such that a point of E lies on the segment

$$\text{segm}\,\{x, z\} = \{(1-\alpha)x+\alpha z;\ 0 < \alpha < 1\},$$

belong to E. In other words, a subset E of S is extreme if

(2) $y \notin \text{segm}\,\{x, z\}$ for any $x \in S$, $y \in E$, $z \in S \setminus E$.

A point e of $S \subset L$ is called *extreme* if $\{e\}$ is an extreme subset of S.

A set $D \subset M$ is called *d-convex* (or *convex relative M*, cf. [3], p. 285) if for any x, $z \in D$ from xyz it follows that $y \in D$.

Properties of terminal subsets formulated in Theorem 1 below are analogous to well known properties [in brackets] of extreme subsets. The proof is similar to the classical one.

THEOREM 1. *In an arbitrary metric space [in an arbitrary real linear space] the following properties hold.*

1. *Any union of terminal [extreme] subsets of a set S is also a terminal [extreme] subset of S.*

2. *Any intersection of terminal [extreme] subsets of a set S is also a terminal [extreme] subset of S.*

3. *If T is a terminal [extreme] subset of S and S is a terminal [extreme] subset of P, then T is a terminal [extreme] subset of P.*

4. *If T ⊂ S ⊂ P and if T is a terminal [extreme] subset of P, then T is a terminal [extreme] subset of S.*

5. *The empty set ∅ and the set S are terminal [extreme] subsets of S.*

6. *A point t of a d-convex [convex] set D is terminal [extreme] if and only if the set D\{t} is d-convex [convex].*

Remark. We can also consider a common generalization of the notions of terminal subset and of extreme subset. Denote by $\mathcal{P}(X)$ the power set of an arbitrary set X. Let $\mathcal{G} \subset \mathcal{P}(X)$ and let $\Phi: \mathcal{G} \to \mathcal{P}(X)$ be a function. If $A \subset B \subset X$ and if

$$A \cap \Phi(K) \subset \bigcup \{\Phi(M); \ M \in \mathcal{G} \ \text{and} \ M \subset A \cap K\}$$

for any $K \in \mathcal{D}$ such that $K \subset B$, then we call A a Φ-extreme subset of B. If X is a metric space [real linear space] and Φ maps any set $\{x, z\}$ onto the set $\{y; xyz\}$ [onto segm $\{x, z\}$], then we get the notion of terminal [extreme] subset. Parts 1-5 of Theorem 1 can be generalized for Φ-extreme subsets (without extra restrictions, part 2 holds only for finite intersections).

2. **Some lemmas.** Let R^n denote a Minkowski-Banach space, i.e. an n-dimensional real linear space with metric d induced by a norm: $d(x, y) = \|y - x\|$. Below, in the definition of a face and in Lemmas 1-3 only the linear structure of R^n is needed.

Let S be a convex set in R^n and let $b \in S$. Sets of the form

$$F_b(S) = \{b\} \cup \{a; a \in S \ \text{and} \ b \in \text{segm}\{a, c\} \ \text{for a point} \ c \in S\}$$

and the empty set are called *faces of S* (comp. [2], part II, § 4, exercise 4). By *proper* faces of S we understand the faces of S different from $\emptyset$ and S.

By aff K denote the affine hull of a given set $K \subset R^n$.

LEMMA 1 ([2], part II, § 4, exercise 4). *For any convex set $S \subset R^n$ and any $b \in S$ we have*

1. $F_b(S)$ *is convex.*

2. $S \cap \text{aff} F_b(S) = F_b(S)$.

3. *If $a \in F_b(S)$, then $F_a(S) \subset F_b(S)$.*

4. If $F_a(S) \subset F_b(S)$ and $F_a(S) \neq F_b(S)$, then $\dim F_a(S) < \dim F_b(S)$.

LEMMA 2. If S is a convex set and $b \in S$, then

1. $F_b(S)$ is the smallest extreme subset of S which contains b.

2. E is an extreme subset of S if and only if E is the union of some faces of S [moreover, E is the union of all maximal (with respect to inclusion) members of the family of faces of S being subsets of E].

3. If $\text{segm}\{v, x\}$ is a subset of the union of some proper faces of S, then at least one of them contains both v and x.

Proof. 1. Let x, $z \in S$ and let a point $y \in F_b(S)$ lie in $\text{segm}\{x, z\}$. By the definition of $F_y(S)$ we have x, $z \in F_y(S)$. Moreover, from part 3 of Lemma 1 it results that $F_y(S) \subset F_b(S)$. Hence x, $z \in F_b(S)$. Since the points x, z are arbitrary points of S, we get that $F_b(S)$ is extreme in S. From the definitions of $F_b(S)$ and of an extreme subset it follows that $F_b(S)$ is a subset of any extreme subset of S which contains b. Therefore $F_b(S)$ is the smallest extreme subset of S which contains b.

2. Let E be an extreme subset of S. If $y \in E$, then by the first part we have $F_y(S) \subset E$. Since $y \in F_y(S)$, E is the union of all faces $F_y(S)$ with $y \in E$. The inverse implication results immediately from part 1 of Theorem 1.

Now, the part in brackets is a consequence of part 4 of Lemma 1 and of the finiteness of the dimension of R^n.

3. Let $\text{segm}\{v, x\}$ be a subset of the union of some proper faces of S. Thus the point $u = \frac{1}{2}v + \frac{1}{2}x$ belongs to a proper face $F_w(S)$ of the union. From part 3 of Lemma 1 we infer that $F_u(S) \subset F_w(S)$. Obviously, x, $v \in F_u(S)$. Hence x, $v \in F_w(S)$.

Lemma 2 is proved.

A face F of a convex set S can be equivalently defined (cf. e.g. [5], p. 162) as a convex extreme subset of S. Applying our Theorems 2–4 to a face F of S we need a point y such that $F = F_y(S)$. It is known that $F = F_y(S)$ if and only if y belongs to the relative interior of F (cf. [2], part II, § 4, exercise 4b).

The set

$$\text{cone}_y S = \bigcup_{x \in S} \{y + \lambda(x - y); \ \lambda > 0\},$$

i.e. the smallest cone with vertex y which contains a given set S is called the induced cone of S with vertex y. We need only the case where $y \in S$ (then, obviously, $y \in \text{cone}_y S$) and where S is convex (then $\text{cone}_y S$ is convex, cf. [5], p. 14).

LEMMA 3. For any convex set S and any $y \in S$ we have

$$F_y(\text{cone}_y S) = \text{cone}_y F_y(S) = \text{aff} F_y(S).$$

Proof. $F_y(\text{cone}_y S)$ consists of y and all points $x \in \text{cone}_y S$ such that $y \in \text{segm}\{x, z\}$ for a point $z \in \text{cone}_y S$. Hence $F_y(\text{cone}_y S)$ consists of y and all

$x \in \mathrm{cone}_y \{x_1\}$, where x_1 is a point of S such that $y \in \mathrm{segm} \{x_1, z_1\}$ for a point $z_1 \in S$. Thus $F_y(\mathrm{cone}_y S) = \{x; \ x \in \mathrm{cone}_y \{x_1\}$ for a point $x_1 \in F_y(S)\}$ $= \mathrm{cone}_y F_y(S)$.

From the definition of $F_y(S)$ it follows that $\mathrm{cone}_y F_y(S)$ is symmetric with respect to y. Moreover, $\mathrm{cone}_y F_y(S)$ is convex because (by part 1 of Lemma 1) the face $F_y(S)$ is convex. From [2], part II, § 1, Corollary 2, we get that · $\mathrm{cone}_y F_y(S)$ is a plane. Moreover, we have $F_y(S) \subset \mathrm{cone}_y F_y(S)$. Thus we obtain $\mathrm{aff} F_y(S) \subset \mathrm{cone}_y F_y(S)$. Since $\mathrm{aff} F_y(S)$ is a cone with vertex y, we have $\mathrm{cone}_y F_y(S) \subset \mathrm{aff} F_y(S)$. Thus $\mathrm{cone}_y F_y(S) = \mathrm{aff} F_y(S)$.

Denote by $\Sigma(y, \delta)$ the sphere $\{w; \ d(y, w) = \delta\}$ and by $B(y, \delta)$ the ball $\{w; \ d(y, w) \leqslant \delta\}$. It is known that $B(y, \delta)$ is closed, n-dimensional, convex and symmetric with respect to y. Two proper faces of $B(y, \delta)$ which are symmetric with respect to y will be called *opposite faces of* $B(y, \delta)$.

LEMMA 4 ([4], p. 115). *Let* $x, z \in \Sigma(y, \delta)$ *and let* v *be the point symmetric to* z *with respect to* y. *Then* xyz *holds if and only if* $\mathrm{segm} \{v, x\} \subset \Sigma(y, \delta)$.

Since $\Sigma(y, \delta)$ is the union of all proper faces of $B(y, \delta)$, from part 3 of Lemma 2 and from Lemma 4 we get

LEMMA 5. *Let* $x, z \in \Sigma(y, \delta)$. *Then* xyz *holds if and only if* x *and* z *are points of opposite faces of* $B(y, \delta)$ [*equivalently: of opposite maximal faces of* $B(y, \delta)$].

By the *ray* with the initial point y passing through a point x different from y we understand the set $\{y + \lambda(x - y); \ \lambda \geqslant 0\}$.

LEMMA 6 ([4], p. 114). *If* xyz, *then* $x_1 y z_1$ *for every* x_1 *lying on the ray with the initial point* y *passing through* x *and for every* z_1 *lying on the ray with the initial point* y *passing through* z.

From Lemmas 5 and 6 we get

LEMMA 7. *The relation* xyz *holds if and only if the rays with the initial point* y *passing through* x *and* z, *respectively, intersect two opposite faces of the ball* $B(y, 1)$ [*equivalently: two opposite maximal faces of* $B(y, 1)$].

LEMMA 8. *Any terminal subset of arbitrary set* $S \subset R^n$ *is extreme. In particular, terminal points are extreme.*

The lemma is true because $y \in \mathrm{segm} \{x, z\}$ implies xyz.

3. Terminal subsets of convex sets in Minkowski-Banach spaces. In Theorem 5 at the end of this paper we show that the only terminal subsets of a convex set $S \subset R^n$ are unions of terminal faces of S. Theorems 2-4 facilitate the verification which faces of S are terminal.

THEOREM 2. *A face* $F_y(S)$ *of a convex set* S *is terminal if and only if any two points of* S, *such that* y *is metrically between them, belong to* $F_y(S)$, *i.e. if and only if*

(3) xyz *does not hold for any* $x \in S$ *and* $z \in S \setminus F_y(S)$.

Proof. If $F_y(S)$ is terminal in S then (3) follows immediately from (1) and from $y \in F_y(S)$.

Assume that $F_y(S)$ is not terminal in S. Due to (1) there exist points $x_1 \in S$, $y_1 \in F_y(S)$ and $z_1 \in S \setminus F_y(S)$ such that $x_1 y_1 z_1$. Since $y_1 \in F_y(S)$, in virtue of the definition of $F_y(S)$, there exists $u \in S$, such that $y \in \text{segm} \{y_1, u\}$. Obviously, $u \in F_y(S)$. Moreover, $y = (1-\alpha)u + \alpha y_1$, where $0 < \alpha < 1$. Put $x = (1-\alpha)u + \alpha x_1$ and $z = (1-\alpha)u + \alpha z_1$. Since d is induced by a norm and since $x_1 y_1 z_1$, we have

$$d(x, y) + d(y, z)$$

$$= d\big((1-\alpha)u + \alpha x_1, (1-\alpha)u + \alpha y_1\big) + d\big((1-\alpha)u + \alpha y_1, (1-\alpha)u + \alpha z_1\big)$$

$$= d(\alpha x_1, \alpha y_1) + d(\alpha y_1, \alpha z_1) = \alpha d(x_1, y_1) + \alpha d(y_1, z_1)$$

$$= \alpha [d(x_1, y_1) + d(y_1, z_1)] = \alpha d(x_1, z_1)$$

$$= d(\alpha x_1, \alpha z_1) = d\big((1-\alpha)u + \alpha x_1, (1-\alpha)u + \alpha z_1\big) = d(x, z).$$

Consequently, xyz.

From the convexity of S and from x_1, z_1, $u \in S$ we get x, $z \in S$.

Since $F_y(S)$ is (by part 1 of Lemma 2) an extreme subset of S and $u \in F_y(S)$, $z_1 \in S \setminus F_y(S)$, $z \in \text{segm} \{u, z_1\}$, we have $z \notin F_y(S)$.

We have shown that $x \in S$, $z \in S \setminus F_y(S)$ and xyz. Therefore (3) is not satisfied.

THEOREM 3. *A face $F_y(S)$ of a convex set S is terminal if and only if for any two opposite faces F_1, F_2 [equivalently: for any two opposite maximal faces F_1, F_2] of any ball $B(y, \delta)$ such that $S \cap F_1 \neq \emptyset$ we have $(S \setminus F_y(S)) \cap F_2 = \emptyset$.*

Proof. Assume that $F_y(S)$ is terminal in S. Let F_1, F_2 be two opposite faces of a ball $B(y, \delta)$. Let $x \in S \cap F_1$. If there exists $z \in (S \setminus F_y(S)) \cap F_2$, then by Lemma 5 we have xyz, a contradiction with (3). Thus $(S \setminus F_y(S)) \cap F_2 = \emptyset$.

Now, assume $F_y(S)$ is not terminal in S. From (3) we get that there exist $v \in S$ and $w \in S \setminus F_y(S)$ for which vyw. Take a sphere $\Sigma(y, \delta)$ with $0 < \delta < \min \{d(y, v), d(y, w)\}$. Denote by x and z the common points of $\Sigma(y, \delta)$ with $\text{segm} \{y, v\}$ and $\text{segm} \{y, w\}$, respectively. Since S is convex, x, $z \in S$. From vyw and Lemma 6 we have xyz. Because $F_y(S)$ is extreme (comp. part 1 of Lemma 2), $z \in \text{segm} \{y, w\}$, $y \in F_y(S)$ and $w \in S \setminus F_y(S)$, we have $z \notin F_v(S)$.

Thus $x \in S$ and $z \in S \setminus F_y(S)$. Moreover, from $x, z \in \Sigma(y, \delta)$, xyz and Lemma 5 it follows that x and z lie in some two maximal opposite faces F_1, F_2 of $B(y, \delta)$. Consequently, the sets $S \cap F_1$ and $(S \setminus F_y(S)) \cap F_2$ are non-empty, which ends the proof.

Note that in Theorem 3 it is sufficient to consider only balls with small δ (with any δ smaller than a fixed positive ε).

THEOREM 4. *A face $F_y(S)$ of a convex set S is terminal if and only if the plane $\operatorname{aff} F_y(S)$ is terminal in the induced cone $\operatorname{cone}_y S$.*

Proof. Assume $F_y(S)$ is not terminal in S. By (3) there exist $x \in S$ and $z \in S \setminus F_y(S)$ such that xyz. In virtue of part 2 of Lemma 1 from $z \in S$ and $z \notin F_y(S)$ we get $z \notin \operatorname{aff} F_y(S)$. Hence $z \in \operatorname{cone}_y S \setminus \operatorname{aff} F_y(S)$. Moreover, $x \in \operatorname{cone}_y S$ and $y \in \operatorname{aff} F_y(S)$. By (1) the plane $\operatorname{aff} F_y(S)$ is not terminal in $\operatorname{cone}_y S$.

Now, assume $\operatorname{aff} F_y(S)$ is not terminal in $\operatorname{cone}_y S$. By Lemma 3 we have $\operatorname{aff} F_y(S) = F_y(\operatorname{cone}_y S)$. In virtue of (3) there exist $x_1 \in \operatorname{cone}_y S$ and $z_1 \in \operatorname{cone}_y S \setminus \operatorname{aff} F_y(S)$ such that $x_1 y z_1$. Consequently, there exist points $x_2 \in S \cap \operatorname{segm}\{y, x_1\}$ and $z_2 \in S \cap \operatorname{segm}\{y, z_1\}$. Obviously, $z_2 \notin \operatorname{aff} F_y(S)$. Hence $z_2 \notin F_y(S)$. From $x_1 y z_1$ and Lemma 6 we get $x_2 y z_2$. Moreover, $x_2 \in S$ and $z_2 \in S \setminus F_y(S)$. Therefore (3) is not satisfied. Consequently, $F_y(S)$ is not terminal in S.

The proof is complete.

Note that to verify with the help of Theorem 3 whether $\operatorname{aff} F_y(S)$ is terminal in $\operatorname{cone}_y S$ it is sufficient to consider only one ball, for instance $B(y, 1)$.

Example 1. In R^2 with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ are only terminal points of the disk $x_1^2 + x_2^2 \leqslant 1$. This follows from the form of the balls of the space and from Theorems 3 and 4.

Example 2. Consider Minkowski–Banach space R^3 with the unit ball in the form of the convex hull of two circles: $x_1^2 + x_3^2 = 1$, $x_2 = 0$ and $x_2^2 + x_3^2 = 1$, $x_1 = 0$. From Theorem 3 it follows that the only terminal faces of the cube $|x_1 + x_2| \leqslant 1$, $|x_1 + x_3| \leqslant 1$, $|x_2 + x_3| \leqslant 1$ are two two-dimensional faces parallel to the plane $x_3 = 0$, four one-dimensional faces perpendicular to the plane $x_3 = 0$ and all eight vertices of the cube.

THEOREM 5. *A subset T of a convex set $S \subset R^n$ is terminal in S if and only if T is the union of some terminal faces of S.*

Proof. In virtue of part 1 of Theorem 1 any union of terminal faces of S is a terminal subset of S. Thus it is sufficient to show that any terminal subset T of S is the union of some terminal faces of S.

By Lemma 8 the set T is an extreme subset of S. By part 2 of Lemma 2 the set T is the union of the family $\mathcal{F}_T$ of all maximal members of the family of faces of S being subsets of T. Below we show that any face $F_y(S)$ from $\mathcal{F}_T$ is terminal in S.

Suppose $F_y(S)$ is not terminal in S. By Theorem 2 there exist $x \in S$ and $z \in S \setminus F_y(S)$ such that xyz. Put $z_1 = \frac{1}{2}y + \frac{1}{2}z$. Since $y \in F_y(S)$, $z \in S \setminus F_y(S)$, $z_1 \in \operatorname{segm}\{y, z\}$ and $F_y(S)$ is extreme (see part 1 of Lemma 2), we have $z_1 \notin F_y(S)$. Because y, $z \in S$ and $z_1 \in \operatorname{segm}\{y, z\}$, we get that $y \in F_{z_1}(S)$. From part 3 of Lemma 1 we get the inclusion $F_y(S) \subset F_{z_1}(S)$. From $z_1 \notin F_y(S)$ it follows that $F_y(S)$ is a proper subset of $F_{z_1}(S)$.

Now, if $z_1 \in T$, then by part 1 of Lemma 2 we have $F_{z_1}(S) \subset T$ which, in view of $F_y(S) \subsetneqq F_{z_1}(S)$, contradicts $F_y(S) \in \mathscr{F}_T$. So $z_1 \notin T$. From xyz, $z_1 = \frac{1}{2}y + \frac{1}{2}z$ and Lemma 6 we have xyz_1. From $x \in S$, $y \in T$, $z_1 \in S \setminus T$ and (1) we get that T is not terminal in S. The contradiction shows that our supposition is false. Thus $F_y(S)$ is terminal in S.

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