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# ON THE THÉORÈME FONDAMENTAL OF J. LERAY AND J. SCHAUDER

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**1. Introduction.** Let $E$ be a Banach space and let $\Omega$ be an open and bounded subset of $[0, 1] \times E$. Given $t \in [0, 1]$, we denote by

$$\Omega_t = \{x \in E;\ (t, x) \in \Omega\}$$

the *slice* of $\Omega$ at $t$. Let $h\colon \Omega \to E$ by any given map. Then by $h_t\colon \Omega_t \to E$ we denote the map $x \mapsto h(t, x)$.

The following continuation principle is contained in the well-known paper of Leray and Schauder [10]:

THÉORÈME FONDAMENTAL ([10], p. 63). *Let $\Omega$ be as above and let $k\colon \bar{\Omega} \to E$ be a compact map such that*

(i) $x \neq k(t, x)$ *for all* $(t, x) \in \partial\Omega$, *the boundary of* $\Omega$;

(ii) *there exists* $t_0 \in [0, 1]$ *such that the equation* $x = k(t_0, x)$ *has a finite number of solutions* $x_1, x_2, \ldots, x_n$;

(iii) *the Leray–Schauder degree* $\deg(I - k_{t_0}, \Omega_{t_0}, 0)$ *is different from zero.*

*Then there exists a continuum of solutions of the equation* $x = k(t, x)$ *connecting* $\Omega_0$ *with* $\Omega_1$.

The main purpose of this note is to extend this result in two directions. First, we obtain the existence of continua of solutions for a class of maps wider than that of compact perturbations of the identity, allowing also the parameter space to be $n$-dimensional, $n \geqslant 1$. Secondly, we give some information on the topological covering dimension of those continua of solutions.

To this aim let us recall first the definition of 0-epi (zero-epi) maps introduced in [4].

A continuous map $f\colon \bar{U} \to F$, defined on the closure $\bar{U}$ of an open bounded subset $U$ of $E$, taking values in a Banach space $F$, is said to be *zero-epi* (0-*epi*) if the equation $f(x) = h(x)$ is solvable in $U$ whenever $h\colon \bar{U} \to F$ is compact and $h(x) = 0$ for all $x \in \partial U$.

We list now the main properties of 0-epi maps, referring to [4] for further results.

1. Homotopy property. *Let $H$: $\bar{U}\times[0, 1]\to F$ be a continuous map such that*

(i) $H(\cdot, \cdot)-H(\cdot, 0)$ *is compact;*

(ii) $H(x, t)\neq 0$ *for all* $x\in\partial U$ *and all* $t\in[0, 1]$.

*Then either* $H(\cdot, 0)$ *and* $H(\cdot, 1)$ *are both 0-epi or both not 0-epi.*

2. Localization property. *Let $f$: $\bar{U}\to F$ be 0-epi. Assume that*

$$f^{-1}(0)\subset V\subset U.$$

*Then the restriction* $f|_{\bar{V}}$ *is 0-epi.*

An important class of 0-epi maps is given by the following:

*Let $k$: $\bar{U}\to E$ be a compact map. Assume that the Leray–Schauder degree* $\deg(I-k, U, 0)$ *is defined and different from zero. Then $I-k$ is 0-epi.*

We remark that the converse need not be true even in the finite dimensional case (see [4]).

The following example shows that the Théorème fondamental of Leray and Schauder does not admit a direct extension to the context of 0-epi maps.

Example 1.1. Let $\Omega$ be the open subset of $[0, 1]\times[-2, 2]$ shaded in Fig. 1 and let $f$: $\bar{\Omega}\to \boldsymbol{R}$ be the map whose graph is shown in Fig. 2. The set of zeroes of $f$, indicated in Fig. 1 with fat line, does not contain any continuum joining $f_0^{-1}(0)$ with $f_1^{-1}(0)$.

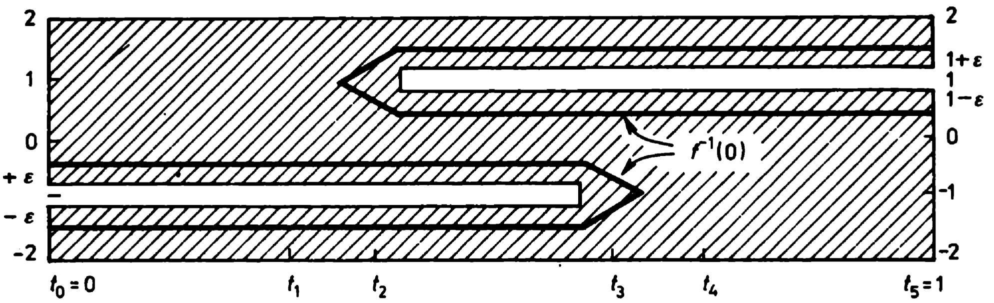


Fig. 1

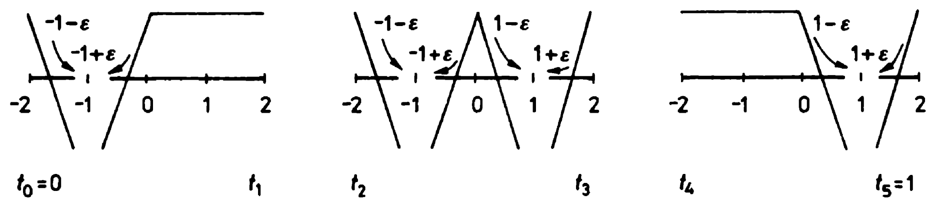


Fig. 2

Some comments and remarks regarding the Théorème fondamental and the above example are in order.

We mention first the fact that the result of Leray and Schauder holds if assumption (ii) is removed. This has been shown by Browder [2].

Further, notice that assumptions (i) and (ii) of the Théorème fondamental imply, by the homotopy property of the degree, that

$$\deg(I-k_t, \Omega_t, 0) \neq 0 \quad \text{for all } t \in [0, 1].$$

Now, Example 1.1 is such that $f_t: \bar{\Omega}_t \to F$ is 0-epi for any $t \in [0, 1]$. Therefore, assumptions of this type are not sufficient to ensure the existence of a continuum like that obtained in the Théorème fondamental.

Finally, observe that the same assumptions (i) and (iii) imply, via, e.g., the Leray–Schauder reduction lemma for the degree, that

$$\deg(G, \Omega', 0) \neq 0,$$

where

$$\Omega' = \{(t, x) \in \Omega: 0 < t < 1\}$$

and $G: \Omega' \to \boldsymbol{R} \times E$ is the map defined by

$$G(t, x) = \big(t - t_0, x - k(t, x)\big).$$

These last are the type of assumptions we are going to work with in order to obtain an extension of the Théorème fondamental (see Theorem 2.1 below).

We add in passing that with a slight modification of Example 1.1 we can see that the generalized homotopy principle does not hold in the general context of 0-epi maps. To show this it suffices to modify the domain $\Omega$ by filling up the upper strip. This way one joins the 0-epi map $f_0$ with $f_1$ which is not 0-epi.

**2. Definitions, notation and preliminary results.** Let $I^n = [-1, 1]^n$ be the unit cube in $\boldsymbol{R}^n$ and let $U \subset I^n \times E$ be open and bounded. Let $f: \bar{U} \to F$ be proper and such that $0 \notin f(\partial_r U)$, where $\partial_r U$ stands for the boundary of $U$ relative to $\mathring{I}^n \times E$, and $\mathring{I}^n$ is the interior of $I^n$. Consider the open subset

$$V = U \cap (\mathring{I}^n \times E)$$

of $\boldsymbol{R}^n \times E$ and define the map $G: \bar{V} \to \boldsymbol{R}^n \times F$ by

$$G(\lambda, x) = \big(\pi(\lambda, x), f(\lambda, x)\big),$$

where $\pi: \boldsymbol{R}^n \times E \to \boldsymbol{R}^n$ is the natural projection. Notice that $G(\lambda, x) \neq (0, 0)$ for all $(\lambda, x) \in \partial V$ since

$$\partial V \subset \big(\bar{U} \cap \pi^{-1}(\partial I^n)\big) \cup \partial_r U.$$

We will assume that $G$ is 0-epi (this last assumption means that $f$ is zero-regularizable by $\pi$ in the sense of Furi and Pera [5]).

The following result shows that assumptions of the above type are quite natural in order to obtain an extension of the Théorème fondamental to the context of 0-epi maps (recall Example 1.1).

PROPOSITION 2.1. *Let $f$ and $G$ be as above. Then the map $f_\lambda$: $U_\lambda \to F$ defined by $f_\lambda(x) = f(\lambda, x)$ is 0-epi for any $\lambda \in \mathring{I}^n$. Moreover, if $\lambda_0 \in \partial I^n$, then the set*

$$\{(\lambda_0, x) \in \bar{U} : f(\lambda_0, x) = 0\}$$

*is nonempty.*

Proof. Let $\lambda_0 \in \mathring{I}^n$ be given (notice that for such $\lambda_0$ we have $U_{\lambda_0} = V_{\lambda_0}$). We shall show first that the map $(\pi - \lambda_0, f)$ is 0-epi. To this aim consider the homotopy

$$H: \bar{V} \times [0, 1] \to \mathbf{R}^n \times F$$

defined by

$$H(\lambda, x; t) = \big(\pi(\lambda, x) - t\lambda_0, f(\lambda, x)\big).$$

Clearly, $H(\cdot, \cdot; \cdot) - H(\cdot, \cdot; 0)$ is compact and $H(\lambda, x; t) \neq 0$ for all $(\lambda, x) \in \partial V$ and all $t \in [0, 1]$. Hence the map

$$H(\cdot, \cdot; 1) = \big(\pi(\cdot, \cdot) - \lambda_0, f(\cdot, \cdot)\big)$$

is 0-epi. We prove now that $f_{\lambda_0}$ is 0-epi. Let $h$: $\bar{U}_{\lambda_0} \to F$ be a continuous and compact map such that $h(x) = 0$ for all $x \in \partial U_{\lambda_0}$. We have to show that the equation $f_{\lambda_0}(x) = h(x)$ is solvable in $U_{\lambda_0}$. Consider the continuous and compact map

$$\tilde{h}: \bar{U}_{\lambda_0} \cup \partial V \to F$$

defined by

$$\tilde{h}(\lambda, x) = \begin{cases} h(x) & \text{if } x \in \bar{U}_{\lambda_0}, \\ 0 & \text{if } (\lambda, x) \in \partial V. \end{cases}$$

Let $\bar{h}$: $\bar{V} \to F$ be any continuous and compact extension of $\tilde{h}$ to $\bar{V}$. The equation

$$\big(\pi(\lambda, x) - \lambda_0, f(\lambda, x) - \bar{h}(\lambda, x)\big) = (0, 0)$$

is solvable in $V$ since the map $\bar{h}$ vanishes on $\partial V$. This implies that for some $x_0 \in U_{\lambda_0}$ we have

$$f_{\lambda_0}(x_0) = \bar{h}(\lambda_0, x_0) = h(x_0)$$

since $\bar{h}$ agrees with $h$ on $\bar{U}_{\lambda_0}$. This shows that $f_{\lambda_0}$ is 0-epi.

Finally, let $\lambda_0 \in \partial I^n$. If the set

$$K = \{(\lambda_0, x) \in \bar{V}: f(\lambda_0, x) = 0\}$$

is empty, then the homotopy $H$: $\bar{V} \times [0, 1] \to \mathbf{R}^n \times F$ defined above is admissible. In particular, the map $(\pi - \lambda_0, f)$ is 0-epi, and hence $f(\lambda_0, x) = 0$ admits a solution, which contradicts the emptiness of the set $K$.

Remark 2.1. Regarding the assumptions of Proposition 2.1 notice that even if $f_\lambda = f$ for all $\lambda \in [0, 1]$ and $\Omega = [0, 1] \times U$, where $U$ is an open and bounded subset of $E$, then the map $(\lambda, f)$ need not be 0-epi. To construct such an example one can use the Hopf map as it has been done in [8].

In order to state our main result we need some preliminaries.

DEFINITION 2.1 (see, e.g., [3]). A normal topological space $X$ has *covering dimension equal to n* provided that $n$ is the smallest integer with the property that, whenever $\mathscr{U}$ is an open covering of $X$, there exists a refinement $\mathscr{U}'$ of $\mathscr{U}$, which also covers $X$, and no more than $n+1$ members of $\mathscr{U}'$ have nonempty intersection.

DEFINITION 2.2 (see [1]). A continuous map $g: X \to I^n$ is called *A–H-essential* if any map $h: X \to I^n$ satisfying $g(x) = h(x)$ for all $x \in g^{-1}(\partial I^n)$ is onto, i.e., $h(X) = I^n$.

A–H-essential maps play an important rôle in describing the topological structure of a space. In this context we recall the following two results:

PROPOSITION 2.2 (see [1]). *A compact metrizable space $X$ has covering dimension at least $n$ if and only if there exists an A–H-essential map of $X$ onto $I^n$.*

In fact, Aleksandrov proved Proposition 2.2 for the small inductive dimension, $\operatorname{ind} X$, of $X$. But, in the case of metrizable spaces, the covering dimension of $X$ and $\operatorname{ind} X$ coincide (cf., e.g., [3]).

DEFINITION 2.3 (see [9]). A continuous map $g: X \to I^n$ is called *weakly confluent* if each continuum $C \subset I^n$ has the property that some component of $g^{-1}(C)$ is mapped onto $C$ by $g$.

PROPOSITION 2.3 (see [7]). *If $g: X \to I^n$ is an A–H-essential map of a compact Hausdorff space $X$ onto $I^n$, then $g$ is weakly confluent.*

Finally, we say that a set $S \subset I^n \times X$ *well-covers* $I^n$ if the projection $\pi: S \to I^n$ is weakly confluent.

The following is the main result of this note:

THEOREM 2.1. *Let $U \subset I^n \times E$ be open and bounded and let $f: \bar{U} \to F$ be proper. Assume that the following holds:*

(i) $0 \notin f(\partial_r U)$;

(ii) *the map $G: \bar{V} \to \mathbf{R}^n \times F$ defined by*

$$G(\lambda, x) = \big(\pi(\lambda, x), f(\lambda, x)\big) \quad \text{for } (\lambda, x) \in \bar{V} = U \cap \pi^{-1}(I^n)$$

*is 0-epi.*

*Then there exists a continuum $\Sigma$ of $S = f^{-1}(0)$ with covering dimension at least $n$. Moreover, $\Sigma$ well-covers $I^n$.*

Theorem 2.1 will be proved by means of the following lemmata.

LEMMA 2.1. *The map $\pi: S \to I^n$ is A–H-essential.*

Proof. Observe first that, by Proposition 2.1, we have $\pi(S) = I^n$. We put

$$\dot{S} = S \cap \pi^{-1}(\partial I^n).$$

Let $g: S \to I^n$ be a continuous map such that $g(\lambda, x) = \pi(\lambda, x)$ for all $(\lambda, x) \in \dot{S}$ and $g(S) \subsetneq I^n$. Let $r: g(S) \to \partial I^n$ be a retraction of $g(S)$ onto $\partial I^n$. Then the map

$$r \circ g: S \to \partial I^n$$

is such that

$$rog(\lambda, x) = g(\lambda, x) = \pi(\lambda, x) \quad \text{for all } (\lambda, x) \in \dot{S}.$$

Since $\partial I^n$ is an ANR, $rog$ can be extended to a map $\tilde{g}$ defined on an open neighbourhood $W$ of $S$ such that $W \subset V$. By the localization property of 0-epi maps, the restriction of the map $G = (\pi, f)$ to $\bar{W}$ is 0-epi. Consider the homotopy

$$H: \bar{W} \times [0, 1] \to \mathbf{R}^n \times F$$

defined by

$$H(\lambda, x; t) = \big((1-t)\pi(\lambda, x) + t\tilde{g}(\lambda, x), f(\lambda, x)\big).$$

Let us show that $H$ is admissible. In fact, the difference

$$H(\cdot, \cdot; t) - H(\cdot, \cdot; 0) = \big(-t\pi(\cdot, \cdot) + t\tilde{g}(\cdot, \cdot), 0\big)$$

is compact for any $t \in [0, 1]$ and uniformly continuous in $t$ with respect to bounded subsets of $\bar{W}$. Moreover, if

$$(\bar{\lambda}, \bar{x}) \in \partial W \quad \text{and} \quad H(\bar{\lambda}, \bar{x}; \bar{t}) = (0, 0) \text{ for some } \bar{t} \in [0, 1],$$

then $f(\bar{\lambda}, \bar{x}) = 0$, which is impossible unless $\bar{\lambda} \in \partial I^n$, i.e., $(\bar{\lambda}, \bar{x}) \in \dot{S}$, but then

$$(1-\bar{t})\pi(\bar{\lambda}, \bar{x}) + \bar{t}\tilde{g}(\bar{\lambda}, \bar{x}) = \pi(\bar{\lambda}, \bar{x}) \neq 0$$

since $\tilde{g}$ agrees with $\pi$ on $\dot{S}$. Thus,

$$H(\cdot, \cdot; 1) = \big(\tilde{g}(\cdot, \cdot), f(\cdot, \cdot)\big)$$

is 0-epi. On the other hand, by the construction of $\tilde{g}$ we have

$$(\tilde{g}, f)^{-1}(0, 0) = \emptyset.$$

This contradicts the fact that $(\tilde{g}, f)$ is 0-epi. Hence $\pi$ is A–H-essential on $S$.

Remark 2.2. By Proposition 2.2 the covering dimension of $S$ is at least $n$.

Remark 2.3. From the properness of $f$ it follows that $S$ is compact, and thus, by Proposition 2.3, the restriction $\pi|_S$ is weakly confluent. In particular, there exists a connected subset $C$ of $S$ such that $\pi(C) = I^n$. The existence of such a set $C$ can be obtained also by using the covering results due to Furi and Pera [6].

LEMMA 2.2. *There exists a connected subset $\Sigma$ of $S$ such that the restriction of $\pi$ to $\Sigma$ is A–H-essential.*

Proof. Let $\mathscr{C}$ be the family of all closed subsets $C$ of $S$ for which $\pi|_C$ is A–H-essential. By Lemma 2.1 we have $\mathscr{C} \neq \emptyset$, since $S \in \mathscr{C}$.

Consider in $\mathscr{C}$ the partial ordering induced by inclusion. Now, let $\mathscr{C}'$ be a chain in $\mathscr{C}$. We shall show that

$$\Sigma = \bigcap_{C \in \mathscr{C}'} C$$

is a lower bound for $\mathscr{C}'$ (notice that $\Sigma \neq \emptyset$ since $S$ is compact). To this aim let us show first that $\pi(\Sigma) = I^n$. In fact, assume that $\pi(\Sigma) \subsetneqq I^n$ and let $r\colon \pi(\Sigma) \to \partial I^n$ be a retraction of $\pi(\Sigma)$ onto $\partial I^n$. Then the map $r\circ\pi\colon \Sigma \to \partial I^n$ is such that

$$r\circ\pi(\lambda, x) = \pi(\lambda, x) \quad \text{if } (\lambda, x) \in \dot{\Sigma} = \Sigma \cap \pi^{-1}(\partial I^n).$$

Let $g\colon \Sigma \cup \dot{S} \to \partial I^n$ be a continuous map defined by

$$g(\lambda, x) = \begin{cases} r\circ\pi(\lambda, x) & \text{if } (\lambda, x) \in \Sigma, \\ \pi(\lambda, x) & \text{if } (\lambda, x) \in \dot{S}. \end{cases}$$

Since $\partial I^n$ is an ANR, the map $g$ can be extended to a continuous map $\tilde{g}\colon W \to \partial I^n$, where $W$ is an open neighbourhood of $\Sigma \cup \dot{S}$. Moreover, by the compactness of $S$ and the definition of $\Sigma$ there exists $C \in \mathscr{C}'$ such that $C \subset W$. Furthermore, since $\dot{C} \subset \dot{S}$, by the construction of $\tilde{g}$ we have $\tilde{g}|_{\dot{C}} = \pi|_{\dot{C}}$, and $\tilde{g}|_C$ takes values on $\partial I^n$. This contradicts the fact that $\pi|_C$ is A–H-essential. Therefore, $\pi(\Sigma) = I^n$.

Now, assume that $\Sigma \notin \mathscr{C}$, i.e., $\pi|_\Sigma$ is A–H-inessential. This means that there exists a continuous map $g\colon \Sigma \to I^n$ such that

$$g|_{\dot{\Sigma}} = \pi|_{\dot{\Sigma}} \quad \text{and} \quad g(\Sigma) \subsetneqq I^n.$$

Let $r\colon g(\Sigma) \to \partial I^n$ be a retraction of $g(\Sigma)$ onto $\partial I^n$. Then the map $r\circ g\colon \Sigma \to \partial I^n$ agrees with $\pi$ on $\dot{\Sigma}$. Let $g_1\colon \Sigma \cup \dot{S} \to \partial I^n$ be a continuous map defined by

$$g_1(\lambda, x) = \begin{cases} r\circ g(\lambda, x) & \text{if } (\lambda, x) \in \Sigma, \\ \pi(\lambda, x) & \text{if } (\lambda, x) \in \dot{S}. \end{cases}$$

Since $\partial I^n$ is an ANR, there exist an open neighbourhood $W$ of $\Sigma \cup \dot{S}$ and a continuous map $\tilde{g}\colon W \to \partial I^n$ which extends $g_1$. Then, as above, we can find an element $C$ of $\mathscr{C}'$ such that $C \subset W$ and $\pi|_C$ is A–H-essential. But $\tilde{g}|_C\colon C \to \partial I^n$ agrees on $\dot{C}$ with $\pi$. This is a contradiction, and therefore $\Sigma \in \mathscr{C}$. By Zorn's Lemma there exists a minimal element, still denoted by $\Sigma$, belonging to $\mathscr{C}$.

The remainder of the proof of Lemma 2.2 will be devoted to show that $\Sigma$ is connected. Assume that $\Sigma = A_1 \cup A_2$, where $A_1$ and $A_2$ are closed and satisfying $A_1 \cap A_2 = \emptyset$. We shall show that either $\pi|_{A_1}$ or $\pi|_{A_2}$ is A–H-essential. Indeed, suppose that $\pi|_{A_i}$, $i = 1, 2$, are A–H-inessential. Then there exist $g_i\colon A_i \to I^n$ such that

$$g_i(A_i) \subsetneqq I^n \quad \text{and} \quad g_i|_{\dot{A}_i} = \pi|_{\dot{A}_i}, \quad i = 1, 2.$$

Let $r_i\colon g_i(A_i) \to \partial I^n$, $i = 1, 2$, be continuous retractions. Consider the maps $r_i\circ g_i\colon A_i \to \partial I^n$, $i = 1, 2$. Then

$$r_i\circ g_i|_{\dot{A}_i} = \pi|_{\dot{A}_i}, \quad i = 1, 2.$$

Let $g\colon \Sigma \to \partial I^n$ be a continuous map defined by

$$g(\lambda, x) = \begin{cases} r_1\circ g_1(\lambda, x) & \text{if } (\lambda, x) \in A_1, \\ r_2\circ g_2(\lambda, x) & \text{if } (\lambda, x) \in A_2. \end{cases}$$

Since $\dot{\Sigma} = \dot{A}_1 \cup \dot{A}_2$, we have $g|_{\dot{\Sigma}} = \pi|_{\dot{\Sigma}}$. This contradicts the fact that $\pi|_{\Sigma}$ is A–H-essential, since $g$ is not onto. Therefore, either $\pi|_{A_1}$ or $\pi|_{A_2}$ is A–H-essential. The minimality of $\Sigma$ yields $A_1 = \Sigma$. Thus $A_2 = \emptyset$.

Proof of Theorem 2.1. Apply Propositions 2.2 and 2.3 to the minimal connected set $\Sigma$ obtained in Lemma 2.2.

Remark 2.4. In the context of compact perturbations of the identity, Theorem 2.1 is contained in [11].

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