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AN INEQUALITY FOR DETERMINANTS WITH REAL ENTRIES

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The aim of this note is to prove the following

THEOREM. *For every matrix $A = (a_{ij})_{i,j\leqslant n}$ with real entries we have the inequality*

$$(1)\qquad |\det A| \leqslant \prod_{i=1}^{n} \max\Big(\sum_{\substack{j=1\\ a_{ij}>0}}^{n} a_{ij}, -\sum_{\substack{j=1\\ a_{ij}<0}}^{n} a_{ij}\Big).$$

Proof. First we consider matrices A satisfying the condition

(2) each row of A contains at most one positive and at most one negative element.

Inequality (1) takes then the form

$$(3)\qquad |\det A| \leqslant \prod_{i=1}^{n} \max_{j} |a_{ij}|.$$

We prove the latter inequality by induction with respect to n. For $n = 1$ it is obvious. Assume that it is satisfied by all square matrices A satisfying (2) of degree less than n. If for some i_0 and j_0 we have $a_{i_0 j} = 0$ for all $j \neq j_0$, then

$$\det A = \pm a_{i_0 j_0} \det(a_{ij})_{\substack{i\neq i_0\\ j\neq j_0}},$$

and, by the inductive assumption,

$$|\det(a_{ij})_{\substack{i\neq i_0\\ j\neq j_0}}| \leqslant \prod_{i\neq i_0} \max_{j\neq j_0} |a_{ij}|$$

which implies (3). Therefore, suppose that for every $i \leqslant n$ and for some $k_i < l_i \leqslant n$ we have

$$a_{ik_i} a_{il_i} < 0, \quad a_{ij} = 0 \text{ for } j \neq k_i, l_i.$$

Without loss of generality we may assume that $k_1 = 1$, $l_1 = 2$, and

$$|a_{11}| \geqslant |a_{12}|. \tag{4}$$

We have

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \dfrac{a_{21}}{a_{11}} a_{12} & a_{23} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{n2} - \dfrac{a_{n1}}{a_{11}} a_{12} & a_{n3} & \dots & a_{nn} \end{vmatrix} = a_{11} \det B, \tag{5}$$

where

$$B = \begin{pmatrix} a_{22} - \dfrac{a_{21}}{a_{11}} a_{12} & a_{23} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n2} - \dfrac{a_{n1}}{a_{11}} a_{12} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

The matrix B satisfies condition (2). Indeed, if $i \geqslant 2$ and $k_i > 1$, then

$$a_{i2} - \frac{a_{i1}}{a_{11}} a_{12} = a_{i2} \tag{6}$$

and the $(i-1)$-st row of B contains exactly two non-zero elements, namely the numbers of opposite signs: a_{ik_i} and a_{il_i}. If $k_i = 1$ and $l_i > 2$, then the $(i-1)$-st row of B contains also exactly two non-zero elements of opposite signs, namely $a_{i2} - (a_{i1}/a_{11})a_{12}$ and a_{il_i}. Besides, by (4),

$$\left| a_{i2} - \frac{a_{i1}}{a_{11}} a_{12} \right| = \left| \frac{a_{i1}}{a_{11}} a_{12} \right| \leqslant |a_{i1}|. \tag{7}$$

Finally, if $k_i = 1$ and $l_i = 2$, then the $(i-1)$-st row of B contains only one non-zero element, namely $a_{i2} - (a_{i1}/a_{11}) a_{12}$. Since

$$a_{i2} - \frac{a_{i1}}{a_{11}} a_{12} > 0,$$

we have

$$\left| a_{i2} - \frac{a_{i1}}{a_{11}} a_{12} \right| < \max\left\{ |a_{i2}|, \left| \frac{a_{i1}}{a_{11}} a_{12} \right| \right\} \leqslant \max\{|a_{i2}|, |a_{i1}|\}. \tag{8}$$

By the inductive assumption, (6), (7), and (8), we have

$$|\det B| \leqslant \prod_{i=2}^{n} \max_j |a_{ij}|, \tag{9}$$

and (3) follows from (5) and (9). Thus (1) is true for all matrices A satisfying (2). In the general case we proceed by induction with respect to the number of non-zero elements of A.

If this number is 0, then $\det A = 0$ and (1) holds. Assume that (1) is true for all square matrices with less than N non-zero elements and consider a square matrix A with exactly N non-zero elements. If A satisfies (2), then (1) holds. If (2) is not fulfilled, then for a certain i_0 there exist j_1 and j_2 such that

$$j_1 \neq j_2, \quad a_{i_0 j_1} a_{i_0 j_2} > 0.$$

Assuming, without loss of generality, that $i_0 = 1$, $j_1 = 1$, and $j_2 = 2$, we have

$$\det A = \frac{a_{11}}{a_{11}+a_{12}} \begin{vmatrix} a_{11}+a_{12} & 0 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \frac{a_{12}}{a_{11}+a_{12}} \begin{vmatrix} 0 & a_{11}+a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

The inductive assumption applies to the determinants on the right-hand side, since the relevant matrices contain only $N-1$ non-zero elements. Hence

$$|\det A| \leqslant \left(\left| \frac{a_{11}}{a_{11}+a_{12}} \right| + \left| \frac{a_{12}}{a_{11}+a_{12}} \right| \right) \prod_{i=1}^{n} \max \Big(\sum_{\substack{i=1 \\ a_{ij}>0}}^{n} a_{ij}, \; - \sum_{\substack{j=1 \\ a_{ij}<0}}^{n} a_{ij} \Big)$$
$$= \prod_{i=1}^{n} \max \Big(\sum_{\substack{j=1 \\ a_{ij}>0}}^{n} a_{ij}, \; - \sum_{\substack{j=1 \\ a_{ij}<0}}^{n} a_{ij} \Big)$$

and the proof is complete.

Inequality (1) gives, in general, weaker bounds than Hadamard's inequality does. There are however cases in which the situation is reverse. Such cases are considered in [1] (proof of Theorem 2) and in [2] (Note at the end of the paper).

REFERENCES

[1] J. Browkin, B. Diviš and A. Schinzel, *Addition of sequences in general fields*, Monatshefte für Mathematik 82 (1976), p. 261-268.

[2] A. Schinzel, *Reducibility of lacunary polynomials III*, Acta Arithmetica 34 (1978), p. 227-266.

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