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LARGE ANALYTIC FUNCTIONS III

BY

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TO PROFESSOR ZYGMUND WITH RESPECT AND AFFECTION

We continue the study [2, 3] of $\mathfrak{A}$, the algebra of quotients f/g, where f, g are in H^∞ of the circle and g is outer. For each outer function k in H^2 form the weight function $w = |k|^2$, and the space $L^2(w)$ based on the measure $w d\sigma$ (where σ is normalized Lebesgue measure on the circle). The closure of analytic trigonometric polynomials in $L^2(w)$ is denoted by $H^2(w)$. Then $\mathfrak{A}$ is the union of all such spaces $H^2(w)$, and it is given a locally convex topology as the inductive limit of the spaces $H^2(w)$.

THEOREM 1. *Any countable set of functions (f_n) in $\mathfrak{A}$ is contained in a single space $H^2(w)$.*

Let t be a positive constant, and w any weight function bounded by 1. If t is sufficiently large, then $((\log w)+t)_- = \min((\log w)+t, 0)$ has integral as close to 0 as we please. Now for each n let w_n be a weight function such that f_n belongs to $H^2(w_n)$. The weight function can be truncated if necessary so that its values are less than 1. Then for suitable positive constants t_n the function

$$\log w = \sum((\log w_n) + t_n)_- \tag{1}$$

is summable. Since the sum is non-positive, $w \le 1$ as well. Thus w is a weight function.

Obviously $w \le c_n w_n$, where $c_n = \exp(t_n)$. Therefore f_n belongs to $H^2(w)$ for each n, and the proof is finished.

COROLLARY. *The algebra $\mathfrak{A}$ is complete in this sense: for any sequence (f_n) in $\mathfrak{A}$ there are positive numbers (b_n) such that the sum*

$$\sum_{n=1}^{\infty} b_n r_n f_n \tag{2}$$

converges in $\mathfrak{A}$ for each choice of $r_n = 1, -1$.

This notion of completeness was found to be useful in [1]. It is not a standard notion, and $\mathfrak{A}$ is not complete in other senses.

We may suppose that no f_n is 0. If w is a weight function such that all the f_n belong to $H^2(w)$, take $b_n = \|f_n\|^{-1}n^{-2}$, where the norm refers to $H^2(w)$. Then (2) converges in $H^2(w)$ for every choice of the r_n, and therefore also in $\mathfrak{A}$.

THEOREM 2. *Multiplication in $\mathfrak{A}$ is separately continuous.*

Fix h in $\mathfrak{A}$. Let T be the operator defined by $Tg = hg$ for g in $\mathfrak{A}$. Let V be a convex balanced neighborhood of 0 and U the set of all f such that Tf is in V. We show that U is open, and this will prove the theorem.

It is obvious that U is convex and balanced. Therefore U is open if its intersection with each subspace $H^2(w)$ is open. Write $w = |k|^2$ with k outer; then $H^2(w)$ can be identified with H^2/k, and the norm of g/k in $H^2(w)$ is the norm of g in H^2.

We fix k. Write $h = h_1/h_2$, where h_1 and h_2 are in H^∞ and h_2 is outer. Let f/k be an element of $H^2(w)$ such that $T(f/k)$ is in V. We want to show that $q(e) = T((f+e)/k)$ is in V if e is in H^2 of small enough norm.

Now $q(e) = h_1(f+e)/(h_2k)$, belonging to $H^2(|h_2k|^2)$. The intersection of V with this space is open. Since $q(0)$ belongs to V, a neighborhood (in the metric of $H^2(|h_2k|^2)$) of $q(0)$ is contained in V. Since h_1 is bounded, $q(e)$ is in V if $\|e\|_2$ is small enough. This completes the proof.

COROLLARY 1. *If h is outer, T is a homeomorphism of $\mathfrak{A}$ onto itself.*

Theorem 2 implies that $\mathfrak{A}$ acts by multiplication in its dual space $\mathfrak{A}^*$. Let F be an element of $\mathfrak{A}^*$, and let g belong to $\mathfrak{A}$. Define a functional gF by setting $(gF)(f) = F(gf)$ for f in $\mathfrak{A}$. By the theorem, gF is a continuous functional.

F is determined by a function f in H^2 by the formula

$$F(h) = \int \bar{f} h \, d\sigma, \tag{3}$$

where h is a trigonometric polynomial. Obviously gF is mediated by the analytic part of the function $\bar{g}f$, if this makes sense. Thus the Toeplitz operator $T_{\bar{g}}$ has been extended to have meaning for all g in $\mathfrak{A}$, with domain $\mathfrak{A}^*$.

It is easy to see that each space H^p is contained in $\mathfrak{A}$. The next theorem answers an obvious question.

THEOREM 3. *Each space H^p, $0 < p$, is continuously embedded in $\mathfrak{A}$.*

We shall show that every sequence (f_n) convergent to 0 in H^p contains a subsequence convergent to 0 in $H^2(w)$ for a well-chosen weight function w. This subsequence converges to 0 in $\mathfrak{A}$. It follows that the original sequence tends to 0 in $\mathfrak{A}$, because if some neighborhood V of 0 in $\mathfrak{A}$ fails to contain all but a finite number of f_n, we can apply the fact just mentioned to the subsequence of f_n not in V, leading to a contradiction. We may assume that no f_n is the null function.

Let S be a subsequence of positive integers such that

$$\sum_S |f_n|^p = h \tag{4}$$

is a summable function. Then also $\log h$ is summable, because each $|f_n|$ has summable logarithm, and $|f_n|^p \leq h$ a.e. for each n in S. This inequality implies that $|f_n|^2/h^{2/p} \leq 1$ a.e. Since f_n (n in S) tends pointwise to 0 by (4), the dominated convergence theorem implies that this subsequence of f_n tends to 0 in the space $H^2(w)$ where $w = \min(1, h^{-2/p})$. This concludes the proof.

John E. McCarthy has carried the study of $\mathfrak{A}$ much further [4]. He combines difficult results of N. Yanagihara [6, 7, 8] with new ideas to determine the dual space of $\mathfrak{A}$, and he shows that the topology of $\mathfrak{A}$ is induced by an invariant metric. Then it follows that multiplication is jointly continuous. Recently a new paper [5] extends and simplifies many of Yanagihara's results.

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