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A MULTIDIMENSIONAL TAYLOR'S THEOREM WITH MINIMAL HYPOTHESIS

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1. Introduction. Let $f : \mathbf{R} \to \mathbf{R}$ be given and fix $x \in \mathbf{R}$. Denote the ith derivative of f by $f^{(i)}$; in particular, $f^{(0)}(x) = f(x)$. As usual, $f(h) = o(h^n)$ means $\lim_{h\to 0} f(h)/h^n = 0$. The following theorems are well known.

(1) TAYLOR'S THEOREM. *If $f^{(n+1)}$ exists on an open interval and if $f^{(n)}$ is continuous on its closure, then for x and $x+h$ in the closure*

$$f(x+h) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!}h^i + \frac{f^{n+1}(\xi)}{(n+1)!}h^{n+1},$$

where ξ is strictly between x and $x+h$.

Remark. In particular, if $f^{(n+1)}$ exists on the *closed* interval, the conclusion of Theorem 1 follows.

(2) PEANO'S THEOREM. *If $f^{(n)}$ exists at x, $n \geq 1$, then*

$$f(x+h) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!}h^i + o(h^n).$$

Remarks. The formula of Theorem (1) is an extension of the mean value formula. The estimate of Theorem (2) is an extension of the definition of differentiability at a point. In many texts the name "Taylor's Theorem" refers to both the *formula* of Theorem (1) and the *estimate* of Theorem (2). In order to make the distinction explicit here, we use the names "Taylor's" and "Peano's" respectively. See also Remark 2 of Section 4 below.

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Obviously the hypothesis in (2) cannot be relaxed since $f^{(n)}(x)$ appears in the estimate. This version of (1) can be found in many places (e.g., pp. 286–287 of [9], p. 99 of [16]).

Peano's Theorem (2) is much less well known and is usually absent from current textbooks. It is curious that this theorem was first stated in (an annotation written by Peano to) a calculus textbook ([8], [12], p. 60) and only after 5 years had passed did its proof appear in a research journal ([12], pp. 95–96, [14]). It can also be found in expository articles ([3], [4]) and in some textbooks. (E.g., pp. 289–290 of [9] and pp. 114–115 of [16].) Again, the usual textbook hypothesis that $f^{(n)}$ be continuous at x implies, but is not implied by, the hypothesis of Theorem (2). The proof of (2) follows from (1) with a very short careful argument. (Of course the conclusion of (2) is immediate from Taylor's theorem (1) if one makes the stronger hypothesis that $f^{(n)}$ is continuous at x.) Incidentally, Peano's Theorem led Peano to define a generalized higher order derivative now known as the Peano derivative ([12], pp. 204–209, [13]). The Peano derivative $f_n(x)$ is defined as a number satisfying $f(x+h)=\sum_{i=0}^{n}\frac{f_i(x)}{i!}h^i+o(h^n)$, so that Peano's Theorem asserts that the nth Peano derivative is a direct generalization of $f^{(n)}(x)$. Clearly $f_0(x)=f^{(0)}(x)$, $f_1(x)=f^{(1)}(x)$, but for $n\geq 2$ the Peano derivative is strictly more general than the ordinary one since the function $g(x)=x^3\sin 1/x$, $x\neq 0$, $g(0)=0$, has $g_2(0)=0$ and $g^{(2)}(0)$ not existing. See [2], [6], and [11] for more about the Peano derivative.

2. Theorems. Passing now to d dimensions we first establish some notation. For $x=(x_1,\ldots,x_d)$ in $\mathbf{R}^d$, write $|x|:=\sqrt{\sum x_i^2}$. For $r>0$, $B_r(x)$ is the open ball of radius r centered at x. For a multiindex $\alpha=(\alpha_1,\ldots,\alpha_d)$ with each α_j a non-negative integer, write $|\alpha|:=\sum\alpha_i$. By $\partial^\alpha f$ we mean $(\partial/\partial x_1)^{\alpha_1}[(\partial/\partial x_2)^{\alpha_2}\ldots[(\partial/\partial x_d)^{\alpha_d}f]\ldots]$. (The order is significant here since, for example, $(\partial/\partial x_1)(\partial g/\partial x_2)(0,0)\neq(\partial/\partial x_2)(\partial g/\partial x_1)(0,0)$ when

$$g(x_1,x_2)=\frac{x_1^2-x_2^2}{x_1^2+x_2^2},\quad x_1^2+x_2^2\neq 0,$$

$g(0,0)=0$ [7]). Let $h^\alpha:=h_1^{\alpha_1}\ldots h_d^{\alpha_d}$ and $\alpha!:=\alpha_1!\ldots\alpha_d!$. Also for $f:\mathbf{R}^d\to\mathbf{R}$, $f(h)=o(|h|^n)$ means $\lim_{h\to 0}f(h)/|h|^n=0$. We say that f is differentiable at x if all its first order partial derivatives exist and satisfy

$$f(x+h)=f(x)+\sum_{i=1}^{d}\left(\frac{\partial f}{\partial x_i}\right)(x)h_i+o(|h|).$$

(3) TAYLOR'S THEOREM. *If $\partial^\alpha f$, $|\alpha|=n+1$, exist on an open ball, and if for each j, $(\partial/\partial x_j)^n f$ is a continuous function of x_j on the closure*

of that ball, $j = 1, \ldots, d$, then for x and $x + h$ in the closure of that ball

$$f(x+h) = \sum_{|\alpha| \le n} \frac{\partial^\alpha f(x)}{\alpha!} h^\alpha + \sum_{|\alpha| = n+1} \frac{\partial^\alpha f(\xi_\alpha)}{\alpha!} h^\alpha ,$$

where each ξ_α belongs to $B_{|h|}(x) \setminus \{x\}$.

COROLLARY. *In particular, if the partial derivatives of order $n+1$ exist on a* closed *ball, the conclusion of Theorem* 3 *follows.* (This is the version of Theorem (3) alluded to in the abstract.)

Taylor's Theorem (3) enables us to prove the d-dimensional version of Peano's Theorem (2).

(4) PEANO'S THEOREM. *If for every α, $|\alpha| = n - 1 \ge 0$, $\partial^\alpha f$ is differentiable at x, then*

$$f(x+h) = \sum_{|\alpha| \le n} \frac{\partial^\alpha f(x)}{\alpha!} h^\alpha + o(|h|^n) .$$

Remark. Direct proofs of Theorem 4 are known. For example, see [5] or [1]. But any version of Taylor's theorem with one of the standard remainder terms cannot lead to Theorem (4) if the dimension is not 1; because if $d \ge 2$ the hypothesis of such a version of Taylor's theorem is necessarily stronger than that of Theorem (4). Notice, however, that the hypotheses of Theorems (3) and (4) in the case of dimension $d = 1$ are, respectively, those of Theorems (1) and (2). Again the hypotheses are quite sharp. For example, one might hope to replace the hypothesis of Theorem (4) with the weaker hypothesis of the mere existence of all $\partial^\alpha f(x)$, $|\alpha| = n$, since this statement has the same specialization if $d = 1$. However, such a strong Peano's Theorem cannot hold. For example, in $\mathbf{R}^2$ let χ be the characteristic function of the union of the coordinate axes ($\chi(x_1, x_2) = 1$ if $x_1 x_2 = 0$, $\chi(x_1, x_2) = 0$ otherwise). Then $\chi(0,0) = 1$, $(\partial\chi/\partial x_1)(0,0) = (\partial\chi/\partial x_2)(0,0) = 0$, but $\chi(h,h) = 0 \ne 1 + o(\sqrt{h^2 + h^2})$ so the conclusion of Peano's Theorem with $n = 1$ fails.

Taylor's Theorem (3) with $n = 0$ appears in the textbooks of Hunt [10] and Smith([15], pp. 230–231). For $d \ge 2$ and higher values of n to the best of our knowledge such a strong theorem has not appeared in print before. Peano's Theorem (4) with $n = 1$ is true by definition of differentiability.

The conclusion of (4) follows immediately from (3) if we demand that every $\partial^\alpha f$, $|\alpha| = n$, be continuous at x. We will see in the next section that with slightly more effort, we can get by with the actual hypothesis.

3. Proofs.

Proof of Taylor's Theorem (3). Let $e := (0,\dots,0,1)$ be the unit vector in the direction of the dth coordinate axis. To a vector $h = (h_1,\dots,h_{d-1},h_d)$ in $\mathbf{R}^d$ let there correspond its projection $h' := (h_1,\dots,h_{d-1},0)$, so that $h = h' + h_d e$.

We induct on the dimension d. If $d = 1$, then Taylor's Theorem (3) coincides with the well known Taylor's Theorem (1). Assume now that $d \geq 2$ and that (3) holds for all dimensions $\leq d-1$. A translation allows us to assume $x = 0$.

We write

$$f(h) = f(h') + [f(h) - f(h')]. \tag{5}$$

Applying the induction hypothesis to the first summand in (5) gives

$$f(h') = \sum_{\substack{|\alpha|\leq n \\ \alpha_d=0}} \frac{\partial^\alpha f(0)}{\alpha!} h^\alpha + \sum_{\substack{|\alpha|=n+1 \\ \alpha_d=0}} \frac{\partial^\alpha f(\xi_\alpha)}{\alpha!} h^\alpha \tag{6}$$

where $\xi_\alpha \in \mathbf{R}^{d-1} \times \{0\}$ and $0 < |\xi_\alpha| < |h'|$. Fixing the first $d-1$ coordinates and thinking of the bracketed term in (5) as a function of a single real variable (the dth coordinate) allows us to apply the induction hypothesis to get

$$f(h) - f(h') = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_d}\right)^i f(h') \frac{h_d^i}{i!} + \partial^\alpha f(\xi_\alpha) \frac{h^\alpha}{\alpha!} \tag{7}$$

where $\alpha = (0,\dots,0,n+1)$ and $\xi_\alpha = \xi_d e$ with ξ_d strictly between 0 and h_d. Here the terms $(\partial/\partial x_d)^i f(h')$ are not evaluted at 0, but for each fixed i, $1 \leq i \leq n$, we may apply the induction hypothesis again:

$$\begin{aligned} \frac{h_d^i}{i!}\left[\left(\frac{\partial}{\partial x_d}\right)^i f(h')\right] &= \frac{h_d^i}{i!}\Bigg[\sum_{\substack{|\beta|\leq n-i \\ \beta_d=0}} \partial^\beta \left(\frac{\partial}{\partial x_d}\right)^i f(0) \frac{h^\beta}{\beta!} \\ &\qquad + \sum_{\substack{|\beta|=n-i+1 \\ \beta_d=0}} \partial^\beta \left(\frac{\partial}{\partial x_d}\right)^i f(\eta_\beta) \frac{h^\beta}{\beta!}\Bigg] \\ &= \sum_{\substack{|\alpha|\leq n \\ \alpha_d=i}} \frac{\partial^\alpha f(0)}{\alpha!} h^\alpha + \sum_{\substack{|\alpha|=n+1 \\ \alpha_d=i}} \frac{\partial^\alpha f(\xi_\alpha)}{\alpha!} h^\alpha \end{aligned}$$

with $\eta_\beta \in \mathbf{R}^{d-1} \times \{0\}$ and $0 < |\eta_\beta| < |h'|$ and where $\xi_\alpha := \eta_\beta$ for the unique β whose first $d-1$ coordinates agree with those of α. Observe that $\beta_d = 0$ and $\alpha_d = i$ so that $\alpha! = i!\beta!$. Putting these n expansions (one for each i) into (7) and then putting this and (6) into (5) completes the proof.

Proof of Theorem (4). Fix n and x. Let

$$g(h) := f(x+h) - \sum_{|\alpha|\leq n} \frac{\partial^\alpha f(x)}{\alpha!} h^\alpha .$$

We must prove $g(h) = o(|h|^n)$. For $|h|$ small g has partial derivatives of order $\leq n-1$ on $B_{|h|}(0)$, and $\partial^\alpha g(0) = 0$ for all $|\alpha| \leq n-2$. Hence we may use Taylor's Theorem (3) to expand g to order $n-2$ with remainder terms of order $n-1$:

$$(8) \qquad g(h) = \sum_{|\alpha|=n-1} \frac{\partial^\alpha g(\xi_\alpha)}{\alpha!} h^\alpha , \quad 0 < |\xi_\alpha| < |h| .$$

Since $\partial^\alpha g$ is differentiable at 0 for each α with $|\alpha| = n-1$, by the definition of differentiability and facts that $\partial^\alpha g(0) = 0$ and that $(\partial/\partial x_i)\partial^\alpha g(0) = 0$ for all i we have $\partial^\alpha g(\xi_\alpha) = o(|\xi_\alpha|) = o(|h|)$. We complete the proof by substituting these relations into equation (8) and observing that for $|\alpha| = n-1$, $o(|h|)h^\alpha = o(|h|^n)$.

4. Remarks. 1. A typical textbook version of Taylor's Theorem (3) starts with the stronger hypothesis of f having all partial derivatives of order $\leq n+1$ continuous on an open ball containing x and $x+h$ and has a remainder term of the form $\sum_{|\alpha|=n+1} \partial^\alpha f(x+\theta h)h^\alpha/\alpha!$ for some $\theta \in (0,1)$. Thus *all* of the $(n+1)$st order partial derivatives are evaluated at the *same* point. This point is on the interior of the line segment joining x and $x+h$. We will call this version of Taylor's Theorem the *Lagrange–Taylor Theorem.* (The lightest hypothesis for the Lagrange–Taylor Theorem that we have seen is on p. 285 of [15].)

Let $P := \{(x_1, 0, 0, \ldots, 0) : x_1 \text{ lies between } 0 \text{ and } h_1\} \cup \{(h_1, x_2, 0, \ldots, 0) : x_2 \text{ lies between } 0 \text{ and } h_2\} \cup \ldots \cup \{(h_1, \ldots, h_{d-1}, x_d) : x_d \text{ lies between } 0 \text{ and } h_d\}$ be the rectangular path associated with the proof of Theorem (3). In our proof each $(n+1)$st partial derivative is evaluated at a different point. In fact, it is clear from that in our version of Taylor's Theorem each ξ_α belongs to the (one-dimensional) interior of one of the segments of P.

2. Here is an example where Taylor's theorem (3) applies even though Peano's Theorem (4) and the Lagrange–Taylor Theorem do not. It also highlights the necessity of evaluating each partial derivative of the remainder in Taylor's Theorem (3) at a different point. The example is a C^∞ "fattening" of the characteristic function χ considered in Section 2. Let $\varphi(t) : \mathbf{R} \to [0, \infty)$ be a C^∞ bump. More exactly, we demand that

$$\varphi(t) = \begin{cases} 1 & \text{for } |t| < .1 , \\ 0 & \text{for } |t| > .9 , \end{cases}$$

and that φ be infinitely differentiable on $\{t : .1 \le |t| \le .9\}$. Then let

$$\phi(x,y) := \begin{cases} \varphi(y/x) + \varphi(x/y) & \text{if } xy \neq 0\,, \\ 1 & \text{if } xy = 0\,. \end{cases}$$

Then $\phi : \mathbf{R}^2 \to [0,\infty)$ has infinitely many derivatives at all points (in particular $\partial^\alpha \phi(0,0) = 0$ for all $\alpha \neq 0$), so that Taylor's Theorem (3) holds for every n. However, $\phi(0,0) = 1$ and $\phi(t,t) = 0$ for all $t \neq 0$ so that Peano's Theorem (4) with $x = (0,0)$ and $h = (1,1)$ fails for every $n \geq 1$. Also note that the conclusion of the Lagrange–Taylor Theorem fails for $x = (0,0)$, $h = (1,1)$ and each n.

3. Our formulation of Theorem (3) is "democratic" in the sense that x and $x+h$ may be any pair of point in the closed ball. Frequently x is a distinguished point, the center of a ball on which the goodness of the function f is prescribed. In this case the hypothesis in Theorem (3) can easily be lightened by examining the proof; in fact, only the behavior of the function on the rectangular path P defined in Remark 1 is relevant. It seems overly technical to belabor this point in the absence of a specific application.

4. Since the hypothesis of Theorem (3) does not force the partial derivatives to commute, there are $d!$ Taylor Theorems similar to our Theorem (3). We vary first the first coordinate, next the second coordinate, and so on, in forming the polygonal path P. One could make the order of the selection of coordinates vary according to any permutation of $\{1,2,\ldots,d\}$; then one would have to carefully redefine $\partial^\alpha f$ to take this ordering into account.

5. Our proof of Taylor's Theorem (3) does not use the chain rule; in fact, it does not require directional derivatives in non-coordinate axes directions. (This explains why we can lighten the hypothesis.) This means that certain pedagogical advantages come with the proof: for example, the chain rule can be preceded by Taylor's Theorem or can be dispensed with altogether in an applied course.

6. The conditions on the nth order partial derivatives in the hypothesis of Taylor's Theorem (3) are "sparse" in two senses. First, there are $\binom{n+d}{d}$ ($\simeq d^n$ as $n \to \infty$) distinct partial derivatives appearing in the term $\sum_{|\alpha|=n} \partial^\alpha f(x) h^\alpha/\alpha!$ but only d of them (the unmixed ones) need to have any continuity properties whatsoever. Second, the continuity required of each of these d is only in a single direction.

5. Applications of Taylor's Theorem (3). We mention two major types of applications. The first are those which arise as a consequence of Peano's Theorem (4) and include most applications given in advanced calculus books. The most common of these is the derivation of sufficient conditions for a critical value of a function to be a relative extremum. For example, in [16], pp. 212–220, Peano's Theorem (4) is derived from scratch (as it must be, since Peano's Theorem (4) does not follow from the Lagrange–Taylor Theorem mentioned in Remark 1) and used to develop sufficient conditions for a relative extremum.

A second and more direct use of the Lagrange–Taylor Theorem occurs when the function to be expanded is presumed to have all partial derivatives of a certain order bounded. In such situations Theorem (3) produces exactly the same estimates as the Lagrange–Taylor Theorem from a definitely weaker hypothesis. We give an example of this type of application from the area of numerical solutions of partial differential equations. Let $d = 2$ and let $M := \sup_{|\alpha|=4, |\xi|\le|h|, |\eta|\le|h|}\{|\partial^\alpha f(x+\xi, y+\eta)|\}$ and assume *only* that M is finite. Use Taylor's Theorem (3) to expand each of the four terms $f(x \pm h, y \pm h)$ to order 4 and substitute to produce the formulas

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2}{h^2}\left\{\frac{1}{4}[f(x+h,y+h) + f(x-h,y+h) + f(x+h,y-h) + f(x-h,y-h)] - f(x,y)\right\} + E_1$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{1}{4h^2}[f(x+h,y+h) - f(x-h,y+h) - f(x+h,y-h) + f(x-h,y-h)] + E_2\,,$$

where E_1 and E_2 are linear combinations of fourth order partial derivatives. The triangle inequality followed by simple majorization of each $|\partial^\alpha f|, |\alpha| = 4$, by M produces the well-known estimates $|E_1| \le \frac{4}{3}Mh^2$ and $|E_2| \le \frac{2}{3}Mh^2$.

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